A. Cross Relaxation and the Nuclear Overhauser Effect

- In the last lecture we discussed how the relaxation of longitudinal magnetization follows the decay equation:

\[
\frac{dM_z}{dt} = -\frac{M_z - M_0}{T_1}
\]

which has the solution \( M_z(t) = M_0 \left(1 - e^{-t/T_1}\right) \)

- This equation accurately describes the restoration of longitudinal magnetization for dilute spins...spins that are separated from neighboring spins such that they interact solely with the “lattice”.

- Suppose however that spin ½ nuclei are closer together such that their magnetic dipole moments interact directly through space. In this case other pathways for relaxation occur and the relaxation is more complex. For argument’s sake, assume two spins I and S are close enough that their spin magnetic dipole moments interact. This usually requires that the spins be within 5 Angstroms of each other. In this case the spin-lattice relaxation equations become:

\[
\frac{dM_z^I}{dt} = -\rho_I \left(M_z^I - M_0^I\right) - \sigma_{IS} \left(M_z^S - M_0^S\right)
\]

\[
\frac{dM_z^S}{dt} = -\rho_S \left(M_z^S - M_0^S\right) - \sigma_{IS} \left(M_z^I - M_0^I\right)
\]

- Because the I and S spin magnetizations appear in both equations, these relaxation equations are coupled. In each case \( \rho_{I,S} = \frac{1}{T_{I,S}} \), so in each case the first terms refer to relaxation of spins I and S directly into the lattice.

- But the second terms are new. The term \( \sigma_{IS} \) are cross relaxation rates, the rate at which energy is exchanged between magnetic dipole moments of I and S.

  - Suppose spin I is irradiated weakly until spin I populations are equilibrated, i.e. the transitions are saturated. Then, the spin I magnetization is zero...\( M_z^I = 0 \).

  - Because the magnetic dipoles of I and S are close in space, energy put into the I spin system is transferred in part into the S spin system Meanwhile the S spin system also relaxes on its own so if I is irradiated for a long time a steady state can be reached where \( \frac{dM_z^S}{dt} = 0 \).
o With the conditions \( M_z^I = 0 \) and \( \frac{dM_z^S}{dt} = 0 \), the S spin relaxation equation becomes:
\[
\frac{dM_z^S}{dt} = -\rho_S \left( M_z^S - M_0^S \right) - \sigma_{IS} \left( 0 - M_0^I \right)
\]

o Rearrange
\[
\frac{M_z^S}{M_0^S} = 1 + \frac{\sigma_{IS} M_0^I}{\rho_S M_0^S}
\]

- Longitudinal magnetization is proportional to the magnetogyric ratio \( \gamma \). So the magnetization ratio on the right hand side of the equation above can be replaced by a ratio of these constants…
\[
\frac{M_z^S}{M_0^S} = 1 + \frac{\sigma_{IS} M_0^I}{\rho_S M_0^S} = 1 + \frac{\sigma_{IS} \gamma_I}{\rho_S \gamma_S} = 1 + \eta_{IS}
\]

- Note that the ratio \( \frac{M_z^S}{M_0^S} \) is different from 1 due to the presence of cross relaxation. The deviation of this ration from 1 is called the Nuclear Overhauser Effect (NOE). The parameter \( \eta_{IS} \) is called the NOE factor or NOF. For small molecules that tumble quickly, the maximum value of the NOF is \( \eta_{IS} = \frac{\sigma_{IS} \gamma_I}{\rho_S \gamma_S} = \frac{\gamma_I}{2\gamma_S} \) or 0.5 for two protons. This means the ratio
\[
\frac{M_z^S}{M_0^S} = 1 + \frac{\gamma_I}{2\gamma_S} \approx \frac{3}{2}
\]

- If instead we have a \(^1H-^13C\) pair instead of a \(^1H-^1H\) pair, using the fact that \( \gamma_H \approx 4\gamma_C \)
\[
\frac{M_z^C}{M_0^C} = 1 + \frac{\gamma_H}{2\gamma_C} \approx 1 + \frac{4\gamma_C}{2\gamma_C} = 3
\]

- For \( \eta_{IS} > 0 \) the NOE is positive.
- In the slow tumbling limit \( \eta_{IS} < 0 \), which is characteristic of macromolecules. The NOE is said to be negative.
- In macromolecules, if the NOE is observed prior to the onset of the steady state, the intensity of the NOE between two protons varies with time and is proportion to \( 1/t^6 \).