Financial Econometrics and Volatility Models
Stochastic Volatility

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Outline

- Stochastic Volatility and Stylized Facts for Returns
- Log-Normal Stochastic Volatility (SV) Model
- SV Model with Student-t Errors
- Asymmetric SV Model
- Multivariate SV Model
Reading

- APDVP, chapters 8 and 11
- FMUND, chapter 4 (section 7)
Stochastic Volatility and Stylized Facts for Returns

Assume daily cc returns can be described as

\[ r_t = \mu + \sigma_t u_t \]

where

1. \( \sigma_t \) is a positive random variable s.t. \( \text{var}(\sigma_t | r_{t-1}, r_{t-2}, \ldots) > 0 \)

2. \( \{\sigma_t\} \) is stationary, \( E[\sigma_t^4] < \infty \) and \( \rho_{\tau,\sigma^2} = \text{corr}(\sigma_t^2, \sigma_{t+\tau}^2) > 0 \) for all \( \tau \)

3. \( u_t \sim iid (0, 1) \)

4. \( \{u_t\} \) and \( \{\sigma_t\} \) are independent
SV vs. ARCH

The ARCH model is expressed as

\[ r_t = \mu + \sigma_t u_t \]
\[ \sigma_t^2 = a_0 + a_1 r_{t-1}^2 \]

However,

\[
\text{var}(\sigma_t^2 | r_{t-1}, r_{t-2}, \ldots, \cdot) = E[\sigma_t^4 | r_{t-1}, r_{t-2} \ldots] - E[\sigma_t^2 | r_{t-1}, r_{t-2} \ldots]^2 \\
= E[(a_0 + a_1 r_{t-1}^2)^2 | r_{t-1}, r_{t-2} \ldots] - E[a_0 + a_1 r_{t-1}^2 | r_{t-1}, r_{t-2} \ldots]^2 \\
= (a_0 + a_1 r_{t-1}^2)^2 - (a_0 + a_1 r_{t-1}^2)^2 = 0
\]

so that there is no unpredictable volatility component.
SV vs ARCH

- SV specification can be motivated by economic theory

- Discrete-time SV specification has continuous-time diffusion representation

- SV fits nicely into continuous-time finance theory
Properties of Returns in SV Model

Key result: Because \( \{u_t\} \) and \( \{\sigma_t\} \) are independent, for any functions \( f_1 \) and \( f_2 \) we have

\[
E[f_1(\sigma_t, \sigma_{t-1}, \ldots, )f_2(u_t, u_{t-1}, \ldots, )] = E[f_1(\sigma_t, \sigma_{t-1}, \ldots, )]E[f_2(u_t, u_{t-1}, \ldots, )]
\]

Moments

\[
E[r_t - \mu] = E[\sigma_t u_t] = E[\sigma_t]E[u_t] = 0
\]
\[
\text{var}(r_t) = E[(r_t - \mu)^2] = E[\sigma_t^2 u_t^2] = E[\sigma_t^2]E[u_t^2] = E[\sigma_t^2]
\]
Moments continued

\[
\]

\[
\text{kurt}(r_t) = \frac{E[(r_t - \mu)^4]}{E[\sigma_t^2]^2} = \frac{\text{kurt}(u_t)E[\sigma_t^4]}{E[\sigma_t^2]^2}
\]

\[
= \text{kurt}(u_t) \left(1 + \frac{\text{var}(\sigma_t^2)}{E[\sigma_t^2]^2}\right) > \text{kurt}(u_t)
\]

Autocovariances and Autocorrelations

\[
\gamma_{\tau,r} = \text{cov}(r_t, r_{t+\tau}) = \text{cov}(\sigma_t u_t, \sigma_{t+\tau} u_{t+\tau})
\]

\[
= E[\sigma_t u_t \sigma_{t+\tau} u_{t+\tau}] - E[\sigma_t u_t]E[\sigma_{t+\tau} u_{t+\tau}]
\]

\[
= E[\sigma_t \sigma_{t+\tau}]E[u_t u_{t+\tau}] - E[\sigma_t]E[u_t]E[\sigma_{t+\tau}]E[u_{t+\tau}] = 0
\]
Define $s_t = (r_t - \mu)^2 = \sigma_t^2 u_t^2$. Then

$$
\gamma_{\tau,s} = \text{cov}(s_t, s_{t+\tau}) = \text{cov}(\sigma_t^2 u_t^2, \sigma_{t+\tau}^2 u_{t+\tau}^2)
$$

$$
= E[\sigma_t^2 u_t^2 \sigma_{t+\tau}^2 u_{t+\tau}^2] - E[\sigma_t^2 u_t^2]E[\sigma_{t+\tau}^2 u_{t+\tau}^2]
$$

$$
= E[\sigma_t^2 \sigma_{t+\tau}^2]E[u_t^2]E[u_{t+\tau}^2] - E[\sigma_t^2]E[u_t^2]E[\sigma_{t+\tau}^2]E[u_{t+\tau}^2]
$$

$$
= E[\sigma_t^2 \sigma_{t+\tau}^2] - E[\sigma_t^2]E[\sigma_{t+\tau}^2]
$$

$$
= \text{cov}(\sigma_t^2, \sigma_{t+\tau}^2) = \gamma_{\tau,\sigma^2} > 0
$$

Positive dependence in squared returns result from positive dependence in $\sigma_t^2$

Note:

$$
\rho_{\tau,s} = \frac{\text{cov}(s_t, s_{t+\tau})}{\text{var}(s_t)} = \frac{\text{cov}(\sigma_t, \sigma_{t+\tau})}{\text{var}(\sigma_t)} \frac{\text{var}(\sigma_t^2)}{\text{var}(s_t)} = \rho_{\tau,\sigma^2} \left[ \frac{\text{var}(\sigma_t^2)}{\text{var}(s_t)} \right]
$$
Define

\[ a_t = |r_t - \mu| = \sigma_t |u_t| \]

Then for \( \tau > 0 \)

\[
E[a_t^p a_{t+\tau}^p] = E[\sigma_t^p \sigma_{t+\tau}^p |u_t|^p |u_{t+\tau}|^p]
\]

\[
= E[\sigma_t^p \sigma_{t+\tau}^p] E[|u_t|^p]^2
\]

\[
E[a_t^p] E[a_{t+\tau}^p] = E[\sigma_t^p] E[\sigma_{t+\tau}^p] E[|u_t|^p]^2
\]

and

\[
\text{cov}(a_t^p, a_{t+\tau}^p) = E[a_t^p a_{t+\tau}^p] - E[a_t^p] E[a_{t+\tau}^p]
\]

\[
= \left( E[\sigma_t^p \sigma_{t+\tau}^p] - E[\sigma_t^p] E[\sigma_{t+\tau}^p] \right) E[|u_t|^p]^2
\]

\[
= \gamma_{\tau, \sigma^p} E[|u_t|^p]^2 \text{ for } \tau > 0
\]
Note:

$$\text{var}(a_t^p) = E[a_t^{2p}] - E[a_t^p]^2$$

$$= E[\sigma_t^{2p}|u_t|^{2p}] - E[\sigma_t^p|u_t|^p]^2$$

$$= E[\sigma_t^{2p}] E[|u_t|^{2p}] - E[\sigma_t^p]^2 E[|u_t|^p]^2$$
Autocorrelations of $|a_t|^p$

Define

$$A(p) = \frac{E[\sigma_t^{2p}]}{E[\sigma_t^p]^2} \quad \text{and} \quad B(p) = \frac{E[|u_t|^{2p}]}{E[|u_t|^p]^2}$$

Taylor (1994) derived the following result

$$\rho_{\tau, a^p} = \text{corr}(a_t^p, a_{t+\tau}^p) = \frac{\text{cov}(a_t^p, a_{t+\tau}^p)}{\text{var}(a_t^p)} = \frac{\gamma_{\tau, \sigma^p} E[|u_t|^p]^2}{E[\sigma_t^{2p}] E[|u_t|^{2p}] - E[\sigma_t^p]^2 E[|u_t|^p]^2} = C(p) \rho_{\tau, \sigma^p}$$

$$C(p) = \frac{A(p) - 1}{A(p) B(p) - 1} \leq \frac{1}{B(p)}$$
Result: If $u_t \sim iid \ N(0, 1)$ then

$$E[|u_t|^p] = 2^{p/2} \pi^{-1/2} \Gamma((p + 1)/2)$$

$$\Gamma(u) = \int_0^\infty x^{u-1} e^{-x} dx, \ u > 0$$

$$\Gamma(1/2) = \sqrt{\pi}, \ \Gamma(1) = 1, \ \Gamma(u + 1) = u\Gamma(u)$$

$$\Gamma(n) = (n - 1)! \text{ if } n \text{ is an integer}$$

If $u_t \sim iid$ Student’s $t$ with $v$ df then

$$E[|u_t|] = \frac{2\sqrt{v-2}\Gamma((v+1)/2)}{\sqrt{\pi}(v-1)\Gamma(v/2)}$$
The Log-Normal AR(1) Stochastic Volatility Model

\[ r_t = \mu + \sigma_t u_t \]
\[ \ln(\sigma_t) - \alpha = \phi(\ln(\sigma_{t-1} - \alpha) + \eta_t), |\phi| < 1 \]
\[ \begin{pmatrix} u_t \\ \eta_t \end{pmatrix} \sim iid N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \sigma^2_\eta \end{pmatrix} \right) \]

Note

\[ \ln(\sigma_t) \sim N(\alpha, \beta^2) \]
\[ \beta^2 = \frac{\sigma^2_\eta}{1 - \phi^2} \Rightarrow \sigma^2_\eta = \beta^2(1 - \phi^2) \]
Log Normal Distribution

Definition: If $\ln(Y) \sim N(\mu, \sigma^2)$ then $Y \sim LN(\mu, \sigma^2)$ such that

$$f(y|\mu, \sigma^2) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{\ln(y) - \mu}{\sigma}\right)^2\right), \quad y > 0$$

$$E[Y^n] = \exp\left(n\mu + \frac{1}{2}n^2\sigma^2\right)$$

$$E[Y] = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

$$\text{var}(Y) = \exp\left(2\mu + \sigma^2\right) \left(\exp(\sigma^2) - 1\right)$$

In the Log-Normal AR(1) SV model

$$\sigma_t \sim LN(\alpha, \beta^2)$$
Alternative Parameterization

Some authors specify the log-Normal SV model as

\[ r_t = \mu + \exp(\omega/2) u_t \]
\[ w_t - \omega = \phi(w_{t-1} - \omega) + \eta_{w,t}, \quad \eta_{w,t} \sim iid \ N(0, \sigma_{\eta_w}^2) \]

Here

\[ w_t = \ln(\sigma_t^2) = 2 \ln(\sigma_t) \quad \text{and} \quad \sigma_t = \exp(\omega_t/2) \]

It follows that

\[ \omega = E[w_t] = 2E[\ln(\sigma_t)] = 2\alpha \]
\[ \beta_w^2 = \text{var}(w_t) = \text{var}(2 \ln(\sigma_t)) = 4\text{var}(\ln(\sigma_t)) = 4\beta^2 \]
\[ \sigma_{\eta_w}^2 = \beta_w^2(1 - \phi^2) = 4\beta^2(1 - \phi^2) \]
Basic Properties

- \( \{r_t\} \) is strictly stationary
- All moments of \( r_t \) are finite
- \( \text{kurt}(r_t) = 3 \exp(4\beta^2) \)
- \( \text{cov}(r_t, r_{t+\tau}) = 0 \) (\( \{r_t - \mu, I_t\} \) is a MDS)
- \( \text{cov}(s_t, s_{t+\tau}) > 0 \) when \( \phi > 0 \), \( s_t = (r_t - \mu)^2 \)
- ACF of \( a_t^p = |r_t - \mu|^p \) behaves like ACF of \( s_t \)
Extensions of Standard SV Model

- Fat tailed distribution for $u_t$ (e.g. Student’s t)

- Dependence between $u_t$ and $\eta_t$ to capture leverage effect

- Long memory behavior for $\ln(\sigma_t)$

- Multivariate formulation
Density and Moments

\[ r_t - \mu = \sigma_t u_t = \text{log-Normal } \times \text{ Normal} \]
\[ \Rightarrow \text{ no closed form expression for density} \]

Derivation of Moments

Exploit independence between \( \{\sigma_t\} \) and \( \{u_t\} \)

Utilize moments of log-Normal distribution

Absolute Moments

\[ E[|r_t - \mu|^p] = E[\sigma_t^p] = E[\sigma_t^p u_t^p] = E[\sigma_t^p] E[|u_t|^p] \]
Now

\[ \ln(\sigma^p_t) = p \ln(\sigma_t) \sim N(p\mu, p^2 \beta^2) \]
\[ \Rightarrow \sigma^p_t \sim LN(p\mu, p^2 \beta^2) \]

Hence

\[ E[\sigma^p_t] = \exp \left( p\alpha + \frac{1}{2} p^2 \beta^2 \right) \]

Furthermore, recall for \( u_t \sim iid N(0, 1) \)

\[ E[|u_t|^p] = 2^{p/2} \pi^{-1/2} \Gamma((p + 1)/2) \]

Therefore,

\[ E[|r_t - \mu|^p] = E[\sigma^p_t] E[|u_t|^p] = \exp \left( p\alpha + \frac{1}{2} p^2 \beta^2 \right) 2^{p/2} \pi^{-1/2} \Gamma \left( \frac{p + 1}{2} \right) \]
For $p = 1$ and $p = 2$

$$E[|r_t - \mu|] = \exp\left(\alpha + \frac{1}{2} \beta^2\right) \sqrt{2/\pi}$$

$$E[|r_t - \mu|^2] = \text{var}(r_t) = \exp\left(2\alpha + 2\beta^2\right) \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$$

$$= \exp\left(2\alpha + 2\beta^2\right)$$

because

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \sqrt{\pi}$$

Straightforward algebra gives

$$\text{kurt}(r_t) = \frac{E[|r_t - \mu|^4]}{\text{var}(r_t)^2} = 3 \exp\left(4\beta^2\right)$$
Autocorrelations

• \( \{r_t - \mu\} = \{\sigma_t u_t\} \) is a MDS \( \Rightarrow \{r_t\} \) is an uncorrelated processes

• \( \{\sigma_t\} \) is autocorrelated because \( \ln(\sigma_t) \) follows an AR(1) process

• \( a_t = |r_t - \mu| = \sigma_t |u_t|, \quad l_t = \ln a_t = \ln(\sigma_t) + \ln(|u_t|) \) and \( s_t = (r_t - \mu)^2 \) are autocorrelated and behave similarly to \( \{\sigma_t\} \)
Autocorrelations of $l_t$, $\sigma_t$, $a_t$, and $s_t$

Autocorrelations of $l_t = \ln(a_t) = \ln(\sigma_t) + \ln(|u_t|)$

$$
\text{cov}(l_t, l_{t+\tau}) = \text{cov}(\ln(\sigma_t) + \ln(|u_t|), \ln(\sigma_{t+\tau}) + \ln(|u_{t+\tau}|))
= \text{cov}(\ln(\sigma_t), \ln(\sigma_{t+\tau})) \quad \text{(b/c $u_t$ is iid)}
= \phi^\tau \beta^2 \quad \text{(b/c $\ln(\sigma_t)$ follows an AR(1))}
$$

Then

$$
\rho_{\tau,l} = \text{corr}(l_t, l_{t+\tau}) = \frac{\text{cov}(l_t, l_{t+\tau})}{\text{var}(l_t)} = \frac{\phi^\tau \beta^2}{\text{var}(\ln(\sigma_t) + \ln(|u_t|))}
$$
Now

$$\text{var}(\ln(\sigma_t)) = \beta^2, \text{ var}(\ln(|u_t|)) = \pi^2/8$$

Hence

$$\rho_{\tau,l} = \frac{\phi^\tau \beta^2}{\beta^2 + \pi^2/8} = \frac{8\phi^\tau \beta^2}{8\beta^2 + \pi^2} = C(0, \beta)$$

$$\text{sign}(\rho_{\tau,l}) = \text{sign}(\phi)$$
Autocorrelations of $\sigma^p_t$

As $\ln(\sigma^p_t)$ is a Gaussian AR(1) process, with mean $\alpha p$, variance $p^2 \beta^2$ and autoregressive coefficient $\phi$ it can be shown that

$$\ln(\sigma^p_t) + \ln(\sigma^p_{t+\tau}) = \ln(\sigma^p_t \sigma^p_{t+\tau}) \sim N(2p\alpha, 2(1 + \phi|\tau|)p^2 \beta^2)$$

This follows since

$$E[\ln(\sigma^p_t) + \ln(\sigma^p_{t+\tau})] = p\alpha + p\alpha = 2p\alpha$$

$$\text{var}(\ln(\sigma^p_t) + \ln(\sigma^p_{t+\tau})) = \text{var}(\ln(\sigma^p_t)) + \text{var}(\ln(\sigma^p_{t+\tau})) + 2\text{cov}(\ln(\sigma^p_t), \ln(\sigma^p_{t+\tau}))$$

$$= p^2 \beta^2 + p^2 \beta^2 + 2p^2 \beta^2 \phi|\tau|$$

$$= 2(1 + \phi|\tau|)p^2 \beta^2$$
Hence

\[ \sigma_t^p \sigma_{t+\tau}^p \sim LN(2p\alpha, 2(1 + \phi |\tau|)p^2 \beta^2) \]

It follows that

\[
E[\sigma_t^p \sigma_{t+\tau}^p] = \exp \left( 2p\alpha + (1 + \phi^\tau)p^2 \beta^2 \right)
\]

\[
\rho_{\tau, \sigma^p} = \frac{\exp(p^2 \beta^2 \phi^\tau) - 1}{\exp(p^2 \beta^2) - 1}
\]
Autocorrelations of $a_t^p = |r_t - \mu|^p$

Previous we stated that

\[
\rho_{\tau, a^p} = \text{corr}(a_t^p, a_{t+\tau}^p) = C(p) \rho_{\tau, \sigma^p}
\]

\[
C(p) = \frac{A(p) - 1}{A(p)B(p) - 1}, \quad A(p) = \frac{E[\sigma_t^{2p}]}{E[\sigma_t^p]^2} \quad \text{and} \quad B(p) = \frac{E[|u_t|^{2p}]}{E[|u_t|^p]^2}
\]

Hence, it can be shown that

\[
\rho_{\tau, a^p} = \frac{\exp(p^2 \beta^2 \phi^\tau) - 1}{B(p) \exp(p^2 \beta^2) - 1}
\]

When $p = 2$, we have

\[
\rho_{\tau, s} = \frac{\exp(4 \beta^2 \phi^\tau) - 1}{3 \exp(4 \beta^2) - 1}
\]
Log-Normal AR(1) SV Model with Student-t Errors

\[ r_t = \mu + \sigma_t u_t, \ u_t \sim iid \ St(v), \ v > 2 \]
\[ \ln(\sigma_t) - \alpha = \phi(\ln(\sigma_{t-1} - \alpha) + \eta_t, \ \eta_t \sim iid \ N(0, \sigma^2_\eta) \]

\( u_t \) is independent of \( \eta_t \) for all \( t \)

Here

\[ f(u|v) = c(v) \left[ 1 + \frac{u^2}{v - 2} \right]^{-(v+1)/2}, \ v > 2 \]
\[ c(v) = \frac{\Gamma \left( \frac{v+1}{2} \right)}{\Gamma \left( \frac{v}{2} \right) \sqrt{\pi(v - 2)}} \]

\[ E[u] = 0, \ \text{var}(u) = 1, \ \text{kurt}(u) = \frac{3(v - 2)}{v - 4} \]
Note: by definition

\[ u_t = v_t \sqrt{w_t} \]

\[ v_t \sim iid \ N(0, 1) \]

\[ (v - 2)w_t^{-1} \sim x_v^2 \]

Then we can write

\[ r_t - \mu = \sigma_t u_t = \sigma_t \sqrt{w_t} v_t = \sigma^*_t v_t \]

\[ \sigma^*_t = \sigma_t \sqrt{w_t} \]

\[ \ln \sigma^*_t = \ln \sigma_t + \frac{1}{2} \ln w_t \]

\[ = AR(1) + WN(0, \sigma^2_{\ln w}) \]

\[ = ARMA(1, 1) \]
Moments

\[ a_t = |r_t - \mu| = \sigma_t |u_t| \]

\[ u_t \sim \text{iid} St(v) \]

Then

\[
E[a_t^p] = E[\sigma_t^p] E[|u_t|^p] \\
= \exp \left( p\alpha + \frac{1}{2}p^2\beta^2 \right) E[|u_t|^p] < \infty \text{ for } p < v
\]

Example

\[
E[a_t] = \exp \left( \alpha + \frac{1}{2}\beta^2 \right) \frac{2(v - 2)\Gamma \left( \frac{v+1}{2} \right)}{\sqrt{\pi}(v - 1)\Gamma \left( \frac{v}{2} \right)}, \quad v > 1
\]

\[
E[a_t^2] = \exp(2\alpha + 2\beta^2), \quad v > 2
\]

\[
E[a_t^4] = \exp \left( 4\alpha + 8\beta^2 \right) \frac{3(v - 2)}{v - 4}, \quad v > 4
\]
Note: Moment existence depending on $v$ causes problems for GMM estimation of $v$. 