Asset Risk Measures

Let $L_1, L_2, \ldots$ are iid random variables representing risks or losses associated with some investment with CDF $F$

Example 1. Let $R_t$ denote the daily simple return on an asset and let $W_0$ denote the initial value of the investment. Then the daily dollar return is

$$ R_t W_0 $$

If $R_t > 0$ then there is a profit and if $R_t < 0$ then there is a loss. If, by convention, losses are reported as positive amounts then the loss distribution $F$ is the distribution of

- dollar losses : $L_t = -R_t W_0$
- percent losses : $L_t = -R_t$
Example 2: Let \( r_t = \ln(1 + R_t) \) denote the daily continuously compounded return on an asset and let \( W_0 \) denote the initial value of the investment. Then the daily dollar return is

\[
W_0 \left( \exp(r_t) - 1 \right) = R_t W_0
\]

The loss distribution \( F \) is the distribution of

- dollar loss : \( L_t = -W_0(\exp(r_t) - 1) = -W_0 R_t \)
- percent loss : \( L_t = -(\exp(r_t) - 1) = -R_t \)
Value-at-Risk (VaR). For $0.95 \leq q < 1$, say, $VaR_q$ is the $q$th quantile of the distribution $F$

$$VaR_q = F^{-1}(q)$$

where $F^{-1}$ is the inverse of $F$.

Expected Shortfall (ES). $ES_q$ is the expected loss size, given that $VaR_q$ is exceeded:

$$ES_q = E[L|L > VaR_q]$$

Note: Writing $L = VaR_q + L - VaR_q$, $ES_q$ is related to $VaR_q$ via

$$ES_q = VaR_q + E[L - VaR_q|L > VaR_q]$$
Remark:

For positive losses, $q$ is the probability level associated with the upper quantile of $F$. Sometime, VaR and ES are stated in terms of the loss loss probability $\alpha = 1 - q$ for $0 < \alpha \leq 0.05$. Then

$$VaR_\alpha = F^{-1}(1 - \alpha)$$

$$ES_\alpha = E[L|L > VaR_\alpha]$$
Example: VaR and ES for normal distribution: $L \sim N(\mu, \sigma^2)$

VaR:

$$VaR^N_q = \mu + \sigma \cdot z_q, \quad 0.95 \leq q < 1$$
$$VaR^N_\alpha = -\mu - \sigma \cdot z_\alpha, \quad 0 < \alpha < 0.05$$

$$z_q = q \cdot 100\% \text{ upper quantile for } N(0, 1)$$
$$z_\alpha = \alpha \cdot 100\% \text{ lower quantile for } N(0, 1)$$

ES:

$$ES^N_q = \mu + \sigma \cdot \frac{\phi(z)}{1 - \Phi(z)} = \mu + \sigma \cdot \frac{\phi(z_q)}{1 - q}$$
$$ES^N_\alpha = -\mu + \sigma \cdot \frac{\phi(z_\alpha)}{\alpha}$$

$$z = (VaR_q - \mu)/\sigma$$
Nonparametric VaR and ES (Historical Simulation)

\[
\{L_1, \ldots, L_T\} = \text{sample of losses}
\]

\[
VaR_{q}^{HS} = \hat{F}^{-1}(q) = \text{empirical quantile}
\]

\[
ES_{q}^{HS} = \frac{1}{(1 - q)T} \sum_{t+1}^{T} L_t \times 1 \left( L_t > VaR_{q}^{HS} \right)
\]

\[
1 \left( L_t > VaR_{q}^{HS} \right) = \begin{cases} 
1 & \text{if } L_t > VaR_{q}^{HS} \\
0 & \text{otherwise}
\end{cases}
\]
VaR for location-scale Return Distributions

Assume that $L_t$ can be represented as

$$L_t = \mu + \sigma u_t, \quad u_t = (L - \mu)/\sigma$$

$$E[L_t] = \mu, \quad \text{var}(L_t) = \sigma^2$$

$$u_t \sim iid (0, 1) \text{ with CDF } F_u$$

Then

$$VaR_q = F^{-1}(q) = \mu + \sigma \cdot F_u^{-1}(q)$$

normal VaR : $F_u^{-1}(q) = N(0,1)$ quantile

Student’s t VaR : $F_u^{-1}(q) = \text{Student’s t quantile}$

Cornish-Fisher (modified) VaR : $F_u^{-1}(q) = \text{Cornish-Fisher quantile}$

EVT VaR : $F_u^{-1}(q) = \text{GPD quantile}$
Cornish-Fisher quantile estimate

Idea: Approximate unknown CDF $F_u$ using 2 term Edgeworth expansion around normal CDF $\Phi(\cdot)$ and invert expansion to get quantile estimate:

$$F_{u,CF}^{-1}(q) = z_q + \frac{1}{6}(z_q^2 - 1) \times skew_u + \frac{1}{24}(z_q^3 - 3z_q) \times kurt_u$$

$$- \frac{1}{36}(2z_q^3 - 5z_q) \times skew_u$$

$$z_q = \Phi^{-1}(q)$$

Reference:


R package PerformanceAnalytics