Financial Econometrics and Quantitative Risk Management Return Properties

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Lecture Outline

- Course introduction
- Return definitions
- Empirical properties of returns
Reading

- FRF chapter 1
- FMUND chapter 1 and chapter 2
- SDAFE chapter 2 and chapter 4
Discrete Returns

Simple Net Return

\[ R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \% \Delta P_t \]

Gross Return

\[ 1 + R_t = \frac{P_t}{P_{t-1}} \]
2—Period Return

\[ R_t(2) = \frac{P_t - P_{t-2}}{P_{t-2}} = \frac{P_t}{P_{t-2}} - 1 \]

\[ R_t(2) = \frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} - 1 \]

\[ = (1 + R_t)(1 + R_{t-1}) - 1. \]

\[ k-\text{ Period Return} \]

\[ 1 + R_t(k) = (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}) \]

\[ = \prod_{j=0}^{k-1} (1 + R_{t-j}). \]
Adjusting for Dividends (Total Returns)

\[ R_t = \frac{P_t + D_t - P_{t-1}}{P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}} + \frac{D_t}{P_{t-1}} \]

Adjusting for Inflation (Real Returns)

\[ 1 + R_t^{\text{Real}} = \frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t} \]
Portfolio Return

\[ P_{p,t} = \sum_{i=1}^{n} w_i P_{i,t}, \sum_{i=1}^{n} w_i = 1 \]

\[ R_{p,t} = \sum_{i=1}^{n} w_i R_{i,t} \]

Excess Returns

\[ Z_t = R_t - R_{ft} \]

\[ R_{ft} = \text{T-bill rate or LIBOR rate} \]
Continuously Compounded Returns

\[ r_t = \ln(1 + R_t) = \ln \left( \frac{P_t}{P_{t-1}} \right) \]
\[ = \ln(P_t) - \ln(P_{t-1}) \]
\[ = P_t - P_{t-1} \]
\[ e^{r_t} = 1 + R_t = \frac{P_t}{P_{t-1}} \]
\[ \implies P_t = P_{t-1} e^{r_t} \]

Note:

\[ R_t = e^{r_t} - 1 \]
2-period return

\[ r_t(2) = \ln(1 + R_t(2)) = \ln \left( \frac{P_t}{P_{t-2}} \right) = p_t - p_{t-2} \]

\[ = \ln \left( \frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} \right) \]

\[ = \ln \left( \frac{P_t}{P_{t-1}} \right) + \ln \left( \frac{P_{t-1}}{P_{t-2}} \right) \]

\[ = r_t + r_{t-1}. \]
$k$–period return

\[ r_t(k) = \ln(1 + R_t(k)) = \ln\left(\frac{P_t}{P_{t-k}}\right) = p_t - p_{t-k} \]

\[ = \sum_{j=0}^{k-1} r_{t-j} \]
Adjusting for Dividends (Total Returns)

\[ r_t = \ln(1 + R_t) = \ln \left( \frac{P_t + D_t}{P_{t-1}} \right) \]
\[ = \ln(P_t + D_t) - \ln(P_{t-1}) \]

Adjusting for Inflation (Real Returns)

\[ r_t^{\text{Real}} = \ln(1 + R_t^{\text{Real}}) = \ln \left( \frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t} \right) \]
\[ = r_t - \pi_t \]
Portfolio Return

\[ r_{t,p} = \ln(1 + R_{t,p}) \]
\[ = \ln \left( 1 + \sum_{i=1}^{n} w_i R_{i,t} \right) \]
\[ \neq \sum_{i=1}^{n} w_i r_{i,t} \]

But

\[ r_{t,p} \approx \sum_{i=1}^{n} w_i r_{i,t} \text{ if } R_{i,t} \text{ is small} \]
Excess Returns

\[ Z_t = R_t - R_{ft} \]
\[ z_t = \ln(Z_t) = \ln(R_t - R_{ft}) \neq r_t - r_{ft} \]

But if \( Z_t \) is small then

\[ z_t \approx r_t - r_{ft} \]
Stylized Facts of Asset Return Distributions

- Fat tails
- Asymmetry
- Aggregated normality
- Absence of serial correlation
- Volatility clustering
- Time-varying cross correlation
Shape Characteristics

Let $\tilde{\rho}$ be a random variable with pdf $f$

$$
\mu = E[r] \text{ : center}
$$
$$
\sigma^2 = \text{var}(r) = E[(r - \mu)^2] \text{ : spread}
$$
$$
\text{skew}(r) = E\left[\frac{(r - \mu)^3}{\sigma^3}\right] \text{ : symmetry}
$$
$$
\text{kurt}(r) = E\left[\frac{(r - \mu)^4}{\sigma^4}\right] \text{ : tail thickness}
$$

Note: The $k^{th}$ moment and central moment of $\tilde{\rho}$ is

$$
m'_k = E[\tilde{\rho}^k]
$$
$$
m_k = E[(\tilde{\rho} - \mu)^k]
$$
Normal Distribution

\( X \sim N(\mu, \sigma^2) \)

\[
f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right), \quad -\infty \leq x \leq \infty
\]

\[
E[X] = \mu \\
\text{var}(X) = \sigma^2 \\
\text{skew}(X) = 0 \\
\text{kurt}(X) = 3
\]

\[
m_k = 0 \text{ for } k \text{ odd}
\]
Sample moments

Let \( \{r_t, \ldots, r_T\} \) denote a random sample of size \( T \) where \( r_t \) is a realization of the random variable \( \tilde{r} \).

\[
\hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} r_t, \quad \hat{\sigma}^2 = \frac{1}{T-1} \sum_{t=1}^{T} (r_t - \hat{\mu})^2 = \hat{m}_2
\]

\[
\text{skew} = \frac{\hat{m}_3}{\hat{\sigma}^3}, \quad \text{kurt} = \frac{\hat{m}_4}{\hat{\sigma}^3}
\]

\[
\hat{m}_k = \frac{1}{T-1} \sum_{t=1}^{T} (r_t - \hat{\mu})^k,
\]

Note: we divide by \( T - 1 \) to get unbiased estimates. Check software to see how moments are computed.
Testing for Normality

- **QQ-plot**: plot standardized empirical quantiles vs. theoretical quantiles from specified distribution. Note: Shapiro-Wilks (SW) test for normality: correlation coefficient between values used in QQ-plot

- **Jarque-Bera (JB) test for normality**

\[
JB = \frac{T}{6} \left( \hat{\text{skew}}^2 + \frac{(\hat{\text{kurt}} - 3)^2}{4} \right)
\]

\[
\overset{\chi^2(2)}{\sim}
\]

Note: if \( \tilde{r} \sim N(\mu, \sigma^2) \) then

\[
\sqrt{T} \hat{\text{skew}} \overset{\text{A}}{\sim} N(0, 6), \quad \sqrt{T} (\hat{\text{kurt}} - 3) \overset{\text{A}}{\sim} N(0, 24)
\]
• Kolmogorov-Smirnov (KS) test compares the empirical CDF of returns with the CDF of the normal distribution (or any other assumed distribution)

  - Sort returns: \( r_1 \leq \cdots \leq r_T \) and compute empirical CDF \( \hat{F}_r(r_t) = \frac{t}{T} \)

  - Evaluate normal CDF: \( \Phi \left( \frac{r_t - \hat{\mu}}{\hat{\sigma}} \right) \)

  - Compute KS statistic: \( KS = \sup_t \left| \Phi \left( \frac{r_t - \hat{\mu}}{\hat{\sigma}} \right) - \frac{t}{T} \right| \)
Student’s-t distribution

Let \( Z \sim N(0, 1) \), \( W \sim \chi^2(v) \) such that \( Z \) and \( W \) are independent. Then

\[
X = \frac{Z}{\sqrt{W/v}} \sim t_v
\]

where \( t_v \) denotes a (standardized) Student’s t distribution with \( v \) degrees of freedom. Note:

\[
E[X] = 0, \ \text{var}(X) = \frac{v}{v-2}, \ v > 2
\]

\[
\text{skew} = 0, \ \text{kurt} - 3 = \frac{6}{v-4}, \ v > 4
\]

Existence of moments depends on degrees of freedom (df) parameter \( \nu \). Cauchy = Student’s-t with 1 df. Only density exists.
If $X \sim t_v$ then

$$Y = \mu + \frac{\sigma X}{\sqrt{v/(v - 2)}}$$

has moments

$$E[Y] = \mu, \quad \text{var}(Y) = \sigma^2$$
Density function

\[ f(x; \nu) = \left[ \frac{\Gamma\{(\nu + 1)/2\}}{(\pi \nu)^{1/2} \Gamma(\nu/2)} \right] \frac{1}{\{1 + (x^2/\nu)\}^{(\nu+1)/2}} \]

\[ \Gamma(t) = \int_{0}^{\infty} x^{t-1} \exp(-x) \, dx = \text{gamma function} \]

The d.f. parameter \( \nu \) can be estimated by MLE.

Note: A simple method of moments estimator for \( \nu \) is based on kurtosis:

\[ \text{kurt} - 3 = \frac{6}{\nu - 4} \Rightarrow \nu = 6/(\text{kurt} - 3) + 4 \]
Skew Normal Distribution

Azzalini and Capitanio (2002) define $Z \sim SN(\xi, \omega, \alpha)$ as a skew-normal random variable with density

$$f_Z(z) = 2\phi(z - \xi)\Phi(\omega^{-1}(z - \xi))$$

$$\phi(z) = (2\pi)^{-1/2} \exp\left(-\frac{z^2}{2}\right), \quad \Phi(z) = \int_{-\infty}^{z} \phi(x)dx$$

$\xi =$ location parameter, $-\infty < \xi < \infty$

$\omega =$ scale parameter, $\omega > 0$

$\alpha =$ shape (skew) parameter, $-\infty < \alpha < \infty$

Note: Estimation and simulation functions in R package sn
Remarks

- $\alpha = 0, \ Z \sim N(\xi, \omega^2)$

- $\alpha > 0 \Rightarrow$ positive skewness

- $\alpha < 0 \Rightarrow$ negative skewness
Skew-t Distribution

Azzalini and Capitanio (2002) define \( Y \sim St(\xi, \omega, \alpha, v) \) as a skew-t random variable using the transformation

\[
Y = \xi + V^{-1/2}Z \\
Z \sim SN(\xi, \omega, \alpha) \\
V \sim \chi^2(v)/v
\]

The parameters \( \xi, \omega \) and \( \alpha \) have the same interpretation as in the skew-normal and

\[
v = \text{degrees of freedom parameter, } v > 0
\]

Note: Estimation and simulation functions in R package sn
Defn: The stochastic process \{\tilde{r}_t\} is covariance stationary if

\[ E[\tilde{r}_t] = \mu \text{ for all } t \]
\[ \text{cov}(\tilde{r}_t, \tilde{r}_{t-j}) = E[(\tilde{r}_t - \mu)(\tilde{r}_{t-j} - \mu)] = \gamma_j \text{ for all } t \text{ and any } j \]

The parameter \( \gamma_j \) is called the \( j^{th} \) order or lag \( j \) autocovariance of \{\tilde{r}_t\}

The autocorrelations of \{\tilde{r}_t\} are defined by

\[ \rho_j = \frac{\text{cov}(\tilde{r}_t, \tilde{r}_{t-j})}{\sqrt{\text{var}(\tilde{r}_t)\text{var}(\tilde{r}_{t-j})}} = \frac{\gamma_j}{\gamma_0} \]

and a plot of \( \rho_j \) against \( j \) is called the autocorrelation function (ACF)
The lag \( j \) sample autocovariance and lag \( j \) sample autocorrelation are defined as

\[
\hat{\gamma}_j = \frac{1}{T} \sum_{t=j+1}^{T} (r_t - \bar{r})(r_{t-j} - \bar{r})
\]

\[
\hat{\rho}_j = \frac{\hat{\gamma}_j}{\hat{\gamma}_0}
\]

where \( \bar{r} = \frac{1}{T} \sum_{t=1}^{T} r_t \) is the sample mean.

The sample ACF (SACF) is a plot of \( \hat{\rho}_j \) against \( j \).
Example: White noise (GWN) processes

Perhaps the most simple stationary time series is the independent Gaussian white noise process \( \{ \tilde{\rho}_t \} \sim iid N(0, \sigma^2) \equiv GWN(0, \sigma^2) \). This process has \( \mu = \gamma_j = \rho_j = 0 \ (j \neq 0) \).

Two slightly more general processes are the independent white noise (IWN) process, \( \{ \tilde{\rho}_t \} \sim IWN(0, \sigma^2) \), and the white noise (WN) process, \( \{ \tilde{\rho}_t \} \sim WN(0, \sigma^2) \).

Both processes have mean zero and variance \( \sigma^2 \), but the IWN process has independent increments, whereas the WN process has uncorrelated increments.
The SACF is typically shown with 95% confidence limits about zero. These limits are based on the result that if \( \{\tilde{r}_t\} \sim iid (0, \sigma^2) \) then

\[
\hat{\rho}_j \overset{A}{\sim} N \left( 0, \frac{1}{T} \right), \ j > 0.
\]

The notation \( \hat{\rho}_j \overset{A}{\sim} N \left( 0, \frac{1}{T} \right) \) means that the distribution of \( \hat{\rho}_j \) is approximated by normal distribution with mean 0 and variance \( \frac{1}{T} \) and is based on the central limit theorem result \( \sqrt{T} \hat{\rho}_j \overset{d}{\to} N (0, 1) \). The 95% limits about zero are then \( \pm \frac{1.96}{\sqrt{T}} \).
Testing for White Noise

Consider testing the null hypothesis

\[ H_0 : \{ \tilde{r}_t \} \sim WN(0, \sigma^2) \]

Under the null, all of the autocorrelations \( \rho_j \) for \( j > 0 \) are zero. To test this null, Box and Pierce (1970) suggested the \( Q \)-statistic

\[ Q(k) = T \sum_{j=1}^{k} \hat{\rho}_j^2 \]

Under the null, \( Q(k) \) is asymptotically distributed \( \chi^2(k) \). In a finite sample, the \( Q \)-statistic may not be well approximated by the \( \chi^2(k) \). Ljung and Box (1978) suggested the \textit{modified} \( Q \)-statistic

\[ MQ(k) = T(T + 2) \sum_{j=1}^{k} \frac{\hat{\rho}_j^2}{T - j} \]