Question 1. Let $\mathcal{R}$ denote the simple daily return on an asset and assume that $\mathcal{R} \sim \mathcal{N}(0.01, 0.10)$. For the initial wealth $V_0 = 100$, the profit and loss are random variables defined as $\Pi = V_0 \mathcal{R}$ and $L = -\Pi$. Let $\alpha \in (0.1)$ denote the confidence level for the daily VaR.

1. Derive the normal distributions for $\Pi$ and $L$.

2. Give mathematical expressions for $\mathbb{E}_{\mathcal{R}}^{\alpha}$ based on the normal distributions for $\mathcal{R}$, $\Pi$ and $L$.

3. Using the expressions from the previous question, compute $\mathbb{E}_{\mathcal{R}}^{\alpha}$ for $\alpha = 0.95$ and 0.99. The values based on $\mathcal{R}$, $\Pi$ and $L$ should all be equivalent.

Question 2. Let $\mathbf{R} = (R_1, \ldots, R_n)'$ denote an $n \times 1$ vector of asset returns with $E[\mathbf{R}] = \mathbf{\mu}$ and $\text{var}(\mathbf{R}) = \mathbf{\Sigma}$. Let $\mathbf{w} = (w_1, \ldots, w_n)'$ denote an $n \times 1$ vector of portfolio weights satisfying $\sum_{i=1}^{n} w_i = 1$. Let $R_p = \mathbf{w}' \mathbf{R}$ denote the portfolio return. Three standard portfolio risk measures are: (1) $\sigma_p(\mathbf{w}) = \sqrt{\text{var}(R_p)}$; (2) $VaR_\alpha(\mathbf{w})$ defined as $\Pr(R_p \leq VaR_\alpha(\mathbf{w})) = \alpha$; (3) $ES_\alpha(\mathbf{w}) = E[R_p | R_p \leq VaR_\alpha(\mathbf{w})]$.

1. Show that $\sigma_p(\mathbf{w})$ is a linearly homogenous function of the portfolio weights $\mathbf{w}$.

2. Suppose $R_p \sim N(\mathbf{\mu}, \mathbf{\Sigma})$. Give analytic expressions for $VaR_\alpha(\mathbf{w})$ and $ES_\alpha(\mathbf{w})$.

3. Using the above results, show that $VaR_\alpha(\mathbf{w})$ and $ES_\alpha(\mathbf{w})$ are linearly homogeneous functions of the portfolio weights $\mathbf{w}$.

4. Give an analytic expression for the $n \times 1$ vector of asset marginal contributions to $\sigma_p(\mathbf{w})$. That is, compute $MCR^\sigma_p = \partial \sigma_p(\mathbf{w}) / \partial \mathbf{w}$.

5. Show that the asset specific percent contribution to risk

$$PCR^\sigma_i = \frac{w_i MCR^\sigma_i}{\sigma_p(\mathbf{w})}, \quad i = 1, \ldots, n$$
can be expressed as

\[ PCR_i^p = w_i \beta_i \]

where

\[ \beta_i = \frac{\text{cov}(R_i, R_p)}{\text{var}(R_p)}. \]

Question 3. Let \( R \) denote the simple daily return on an asset and assume that \( R \sim N(\mu, \sigma^2) \), where \( \mu \) and \( \sigma^2 \) are unknown and must be estimated from an observed sample of size \( T \). A natural estimate for daily \( VaR_\alpha \) is

\[ \hat{VaR}_\alpha = -V_0 \hat{q}_{1-\alpha}^R \]

where

\[ \hat{q}_{1-\alpha}^R = \hat{\mu} + \hat{\sigma} \times q_{1-\alpha}^{Z} \]  

\[ \hat{\mu} \] is the sample mean, \( \hat{\sigma} \) is the sample standard deviation and and \( q_{1-\alpha}^{Z} \) the \( 1 - \alpha \) lower quantile of \( Z \sim N(0, 1) \). The Central Limit Theorem gives the result

\[ \left( \begin{array}{c} \hat{\mu} \\ \hat{\sigma} \end{array} \right) \sim N \left( \left( \begin{array}{cc} \mu \\ \sigma \end{array} \right), \left( \begin{array}{cc} \sigma^2/T & 0 \\ 0 & \sigma^2/T \end{array} \right) \right) \]  

which implies that, for large enough \( T \), \( \hat{\mu} \sim N \left( \mu, \frac{\sigma^2}{T} \right) \), \( \hat{\sigma} \sim N \left( \sigma, \frac{\sigma^2}{T} \right) \) and that \( \hat{\mu} \) and \( \hat{\sigma} \) are independent.

1. Use (1) - (3) to derive mathematical expressions for \( \text{var}(\hat{VaR}_\alpha) \) and \( SE(\hat{VaR}_\alpha) = \sqrt{\text{var}(\hat{VaR}_\alpha)}. \)

2. Assuming \( \sigma = 0.10 \) and \( V_0 = 100 \), plot \( SE(\hat{VaR}_\alpha) \) for a grid of 25 \( \alpha \) values between \( \alpha = 0.90 \) and \( \alpha = 0.995 \) for \( T = 25, 50 \) and 100. How well is \( VaR_\alpha \) estimated for \( \alpha \) values close to 1?

Question 4. In this question you will use the PerformanceAnalytics package to estimate daily historical, normal and modified (Cornish-Fisher) \( VaR \) for Microsoft and the S&P 500 index. First, download daily adjusted closing prices on Microsoft (ticker MSFT) and the S&P 500 (ticker ^GSPC) over the period 2000-01-03 to 2012-04-03. Compute simple daily returns from both sets of prices. For automatically downloading data in R, you can use the \texttt{getSymbols()} function from the quantmod package or the \texttt{get.hist.quote()} function from the tseries package. For calculating returns you can use the \texttt{CalculateReturns()} function from the PerformanceAnalytics package.

1. Plot the daily returns on MSFT and the S&P 500. Note any of the stylized facts we discussed in class. Using the PerformanceAnalytics function \texttt{chart.Histogram()} plot the histograms with a normal curve overlaid, and
using the PerformanceAnalytics `chart.QQPlot()` plot the normal QQ-plots. Does the normal distribution look appropriate for these two assets? Use the PerformanceAnalytics function `table.Stats()` to compute descriptive statistics for the returns on the two assets. Note the sample values of skewness and excess kurtosis.

2. Using the PerformanceAnalytics function `VaR()`, estimate daily 95% and 99% VaR for the two assets based on the empirical distribution (i.e., historical VaR), the normal distribution and the Cornish-Fisher distribution (i.e., modified VaR). The `VaR()` function calculates VaR based on the distribution of returns and gives $VaR_\alpha$ as the lower $1 - \alpha$-quantile of the return distribution. Summarize the results nicely in a table and comment. In addition, for each asset create a plot showing the returns together with horizontal lines indicating the 99% VaR values for the three methods. Note: be sure to read the online help for the function `VaR()`.

3. Using the PerformanceAnalytics function `ES()`, estimate daily 95% and 99% ES for the two assets based on the empirical distribution (i.e., historical ES), the normal distribution and the Cornish-Fisher distribution (i.e., modified ES). The `ES()` function calculates ES based on the distribution of returns and gives $ES_\alpha$ as the mean return less than the lower $1 - \alpha$-quantile of the return distribution. Summarize the results nicely in a table and comment. In addition, for each asset create a plot showing the returns together with horizontal lines indicating the 99% ES values for the three methods. Note: be sure to read the online help for the function `ES()`.

Question 5. In this question you will use the PerformanceAnalytics functions `StdDev`, `VaR` and `ES` to estimate risk budgets based volatility, VaR and ES for an equally weighted portfolio of Microsoft and the S&P 500 index.

1. Using the PerformanceAnalytics function `StdDev()`, decompose the volatility of an equally weighted portfolio of MSFT and S&P 500 into the individual asset components. Which assets contributes most to the volatility of the portfolio?

2. Using the PerformanceAnalytics function `VaR()`, decompose the 95% Gaussian, historical and modified VaR of an equally weighted portfolio of MSFT and S&P 500 into the individual asset components. Summarize the results nicely in a table and comment. Which assets contributes most to the 95% VaR of the portfolio? Are the results similar to the decomposition based on the volatility?

3. Using the PerformanceAnalytics function `ES()`, decompose the 95% Gaussian, historical and modified ES of an equally weighted portfolio of MSFT and S&P 500 into the individual asset components. Summarize the results nicely in a table and comment. Which assets contributes most to the 95% ES of the portfolio? Are the results similar to the decomposition based on the VaR?