Lecture Outline

- Backtesting terminology
- Backtesting VaR
- Backtesting ES
Backtesting Terminology

Q: How does a VaR model’s forecasts perform over an historical period?

A: Compare ex ante VaR forecast to ex post realized return

Let $t = 1, \ldots, T$ denote the sample size.

**Definition 1** *(Estimation window)* $W_E = \# \text{ of observations used to estimate risk model}

**Definition 2** *(Testing window)* $W_T = \text{data sample over which risk is forecast}

Note: $W_E + W_T = T$

Backtesting VaR Models

- Define the VaR violation (“Hit”) indicator

$$H_t = 1(r_t < VaR_{\alpha,t}) = \begin{cases} 1 & r_t < VaR_{\alpha,t} \\ 0 & r_t \geq VaR_{\alpha,t} \end{cases}$$

$$VaR_{\alpha,t} = q_{1-\alpha,t}^T$$

Let $p = 1 - \alpha$.

- VaR forecasts are efficient wrt $I_t$ if

$$E[H_t | I_{t-1}] = 1 - \alpha = p \Rightarrow H_t | I_{t-1} \sim Bernoulli(p), t = 1, \ldots, T$$

- $n_1 = \text{number of sample VaR violations}, n_0 = T - n_1$. Note: $\hat{p}_{mle} = n_1/T$
Test of Unconditional Coverage

- Hypothesis to be tested
  \[ H_0 : E[H_t] = p \text{ vs. } H_1 : E[H_t] \neq p \]

- Bernoulli likelihood
  \[ f(p|H_1, \ldots, H_T) = p^{n_1}(1 - p)^{T-n_1} \]

- LR test for unconditional coverage
  \[ LR_{uc} = 2[\ln f(\hat{p}_{mle}|H_1, \ldots, H_T) - \ln f(p|H_1, \ldots, H_T)] \sim \chi^2(1) \]
  Reject \( H_0 : E[H_t] = p \) at 5% level if \( LR_{uc} > \chi^2_{.05}(1) = 3.84 \)

Test of Independence

- VaR forecasts that do not take temporal volatility dependence into account may be correct on average, but will produce violation clusters

- A test of independence is a test of no violation clusters (no dependence in VaR violations)

- Christoffersen (1998) models \( H_t \) as a binary first order Markov chain with transition matrix
  \[ \Pi = \begin{bmatrix} 1 - \pi_{01} & \pi_{01} \\ 1 - \pi_{11} & \pi_{11} \end{bmatrix} \]
  \( \pi_{ij} = \Pr(H_t = j|H_{t-1} = i) \)
• Approximate joint likelihood conditional on first observation is

\[ L(\Pi|H_2, \ldots, H_T) = (1 - \pi_{01})^{n_{00}}\pi_{01}^{n_{01}}(1 - \pi_{11})^{n_{10}}\pi_{11}^{n_{11}} \]

\[ n_{ij} = \sum_{t=2}^{T} 1(H_t = i|H_{t-1} = j) \]

• MLEs of transition probabilities

\[ \hat{\pi}_{01,mle} = \frac{n_{01}}{n_{00} + n_{01}}, \quad \hat{\pi}_{11,mle} = \frac{n_{11}}{n_{10} + n_{11}} \]

• Under null of independence, \( \pi_{01} = \pi_{11} \equiv \pi_0 \) and

\[ L(\pi_0|H_2, \ldots, H_T) = (1 - \pi_{01})^{(n_{00}+n_{10})}\pi_{01}^{n_{01}+n_{11}} \]

\[ \hat{\pi}_0 = \hat{\pi}_{mle} = n_1/T \]

• LR test for independence of VaR violations is

\[ LR_{ind} = 2 \left[ \ln L(\hat{\Pi}_{mle}|H_2, \ldots, H_T) - \ln L(\hat{\pi}_0|H_2, \ldots, H_T) \right] \sim \chi^2 (1) \]

Reject \( H_0 : \pi_{01} = \pi_{11} \equiv \pi_0 \) if \( LR_{ind} > \chi^2_{0.95} (1) = 3.84 \)
Test of Conditional Coverage

- Because $\hat{\pi}_0$ is unconstrained, the LR test for independence does not take correct coverage into account.

- To test correct conditional coverage $E[H_t|I_t-1] = \alpha$ Christoffersen suggests using

$$LR_{cc} = 2 \left[ \ln L(\hat{\pi}_0|H_2, \ldots, H_T) - \ln f(p|H_2, \ldots, H_T) \right]$$

$$= LR_{uc} + LR_{ind} \sim \chi^2(2)$$

which provides a means to check in which regard the violation series $\{H_t\}$ fails the correct conditional coverage property.

Backtesting ES

Problem: Harder to backtest ES than VaR because ES is an expectation rather than a single quantile

Method to Backtest Shortfall

Consider the normalized shortfall when $r_t \leq VaR_\alpha$

$$NS_t = \frac{r_t}{ES_{\alpha,t}}$$

From the definition of ES, we have

$$\frac{E[r_t|r_t < VaR_\alpha]}{ES_{\alpha,t}} = 1$$

Hence, in a correctly specified model we should have

$$E[NS_t] = 1$$