1 Reading

1. Hayashi, Chapters 4 and 5.

2 Hayahsi Exercises


3 Weak Instruments

Consider the instrumental variables regression set-up of the previous problem under the weak instrument asymptotic framework of Staiger and Stock (1997, Etca) which assumes that \( \pi \) in (??) is local-to-zero:

\[
\pi = \pi_n = \frac{1}{\sqrt{n}} \cdot g, \tag{1}
\]

\[ g = \text{fixed vector of constants.} \]

1. Let \( \hat{\pi} \) denote the OLS estimate of \( \pi \) in (??). Under (1), show that

\[
\sqrt{n} \hat{\pi} \xrightarrow{d} N(g, \sigma^2 \Sigma^{-1}) \Rightarrow \hat{\pi} \xrightarrow{A} N \left( \frac{g}{\sqrt{n}}, \frac{1}{n} \sigma^2 \Sigma^{-1} \right).
\]

2. Show that the asymptotic distribution of the Wald statistic for testing \( H_0 : \pi = 0 \) in (??) is \( \chi^2(k, d) \) where \( d = g' \Sigma_{xx} g / \sigma^2 \). Hint: Use the following result. Let the \( n \times 1 \) vector \( Y \) have a multivariate normal distribution: \( Y \sim N(\mu, \Sigma) \). Then \( (Y - \mu)' \Sigma^{-1} (Y - \mu) \sim \chi^2(n) \) and \( Y' \Sigma^{-1} Y \sim \chi^2(n, d) \) where \( d = \mu' \Sigma^{-1} \mu \)
is the non-centrality parameter. Here $\chi^2(n, d)$ is called a non-central chi-square distribution with $n$ degrees of freedom and non-centrality parameter $d$. The quantity $d = g'\Sigma_x g' / \sigma^2_v$ is a unit-free measure of instrument quality called the concentration parameter. The closer $d$ is to zero, the weaker are the instruments.

3. Consider testing the null hypothesis $H_0 : \delta = \delta_0$ vs. $H_1 : \delta \neq \delta_0$. The Anderson-Rubin (AR) statistic is computed as the F-statistic for testing $H_0 : \psi = 0$ from the regression

$$y_i - z_i\delta_0 = x_i'\psi + w_i,$$

where $\psi = \pi(\delta - \delta_0)$ and $w_i = v_i(\delta - \delta_0) + \varepsilon_i$. Show that

$$\text{AR}(\delta_0) = \frac{(y - z\delta_0)'P_X(y - z\delta_0)/k}{(y - z\delta_0)'Q_X(y - z\delta_0)/(n - k)},$$

where $P_X = X(X'X)^{-1}X'$ and $Q_X = I_n - P_X$.

4. Under (1) and $H_0 : \delta = \delta_0$, show that $k \cdot \text{AR}(\delta_0) \xrightarrow{d} \chi^2(k)$.

5. $\text{AR}(\delta_0)$ is closely related to the CU-GMM J-statistic (under conditional homoskedasticity) evaluated at $\delta = \delta_0$:

$$J(\delta_0, S^{-1}(\delta_0)) = \frac{(y - z\delta_0)'P_X(y - z\delta_0)}{(y - z\delta_0)'(y - z\delta_0)/n}.$$

Under (1) and $H_0 : \delta = \delta_0$, show that $J(\delta_0, S^{-1}(\delta_0)) \xrightarrow{d} \chi^2(k)$.

6. Under (1) and $H_0 : \delta = \delta_0$, show that $k \cdot \text{AR}(\delta_0) - J(\delta_0, S^{-1}(\delta_0)) \xrightarrow{p} 0$ so that $k \cdot \text{AR}(\delta_0)$ and $J(\delta_0, S^{-1}(\delta_0))$ are asymptotically equivalent. This result shows that AR statistic is really a joint test of $H_0 : \delta = \delta_0$ and the validity of the overidentifying restrictions $E[x_i\varepsilon_i] = 0$. 

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