Econ 424
Time Series Concepts

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Time Series Processes

Stochastic (Random) Process

\[ \{ \ldots, Y_1, Y_2, \ldots, Y_t, Y_{t+1}, \ldots \} = \{ Y_t \}^{\infty}_{t=-\infty} \]

sequence of random variables indexed by time

Observed time series of length \( T \)

\[ \{ Y_1 = y_1, Y_2 = y_2, \ldots, Y_T = y_T \} = \{ y_t \}^{T}_{t=1} \]
Stationary Processes

- Intuition: \( \{Y_t\} \) is stationary if all aspects of its behavior are unchanged by shifts in time

- A stochastic process \( \{Y_t\}_{t=1}^\infty \) is *strictly stationary* if, for any given finite integer \( r \) and for any set of subscripts \( t_1, t_2, \ldots, t_r \) the joint distribution of

\[
(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_r})
\]

depends only on \( t_1 - t, t_2 - t, \ldots, t_r - t \) but not on \( t \).
Remarks

1. For example, the distribution of \((Y_1, Y_5)\) is the same as the distribution of \((Y_{12}, Y_{16})\).

2. For a strictly stationary process, \(Y_t\) has the same mean, variance (moments) for all \(t\).

3. Any function/transformation \(g(\cdot)\) of a strictly stationary process, \(\{g(Y_t)\}\) is also strictly stationary. E.g., if \(\{Y_t\}\) is strictly then \(\{Y_t^2\}\) is strictly stationary.
Covariance (Weakly) Stationary Processes \{Y_t\} :

- \( E[Y_t] = \mu \) for all \( t \)

- \( \text{var}(Y_t) = \sigma^2 \) for all \( t \)

- \( \text{cov}(Y_t, Y_{t-j}) = \gamma_j \) depends on \( j \) and not on \( t \)

Note 1: \( \text{cov}(Y_t, Y_{t-j}) = \gamma_j \) is called the j-lag autocovariance and measures the direction of linear time dependence

Note 2: A stationary process is covariance stationary if \( \text{var}(Y_t) < \infty \) and \( \text{cov}(Y_t, Y_{t-j}) < \infty \)
Autocorrelations

\[ \text{corr}(Y_t, Y_{t-j}) = \rho_j = \frac{\text{cov}(Y_t, Y_{t-j})}{\sqrt{\text{var}(Y_t)\text{var}(Y_{t-j})}} = \frac{\gamma_j}{\sigma^2} \]

Note 1: \( \text{corr}(Y_t, Y_{t-j}) = \rho_j \) is called the j-lag autocorrelation and measures the direction and strength of linear time dependence.

Note 2: By stationarity \( \text{var}(Y_t) = \text{var}(Y_{t-j}) = \sigma^2 \).

Autocorrelation Function (ACF): Plot of \( \rho_j \) against \( j \)
Example: Gaussian White Noise Process

\[ Y_t \sim \text{iid } N(0, \sigma^2) \text{ or } Y_t \sim GWN(0, \sigma^2) \]

\[ E[Y_t] = 0, \, \text{var}(Y_t) = \sigma^2 \]

\( Y_t \) independent of \( Y_s \) for \( t \neq s \)

\[ \Rightarrow \text{cov}(Y_t, Y_{t-s}) = 0 \text{ for } t \neq s \]

Note: “iid” = “independent and identically distributed”.

Here, \( \{Y_t\} \) represents random draws from the same \( N(0, \sigma^2) \) distribution
Example: Independent White Noise Process

\[ Y_t \sim \text{iid } (0, \sigma^2) \text{ or } Y_t \sim IWN(0, \sigma^2) \]

\[ E[Y_t] = 0, \quad \text{var}(Y_t) = \sigma^2 \]

\( Y_t \) independent of \( Y_s \) for \( t \neq s \)

Here, \( \{Y_t\} \) represents random draws from the same distribution. However, we don’t specify exactly what the distribution is - only that it has mean zero and variance \( \sigma^2 \). For example, \( Y_t \) could be iid Student’s t with variance equal to \( \sigma^2 \). This is like GWN but with fatter tails (i.e., more extreme observations).
Example: Weak White Noise Process

\[ Y_t \sim WN(0, \sigma^2) \]
\[ E[Y_t] = 0, \quad \text{var}(Y_t) = \sigma^2 \]
\[ \text{cov}(Y_t, Y_s) = 0 \text{ for } t \neq s \]

Here, \( \{Y_t\} \) represents an uncorrelated stochastic process with mean zero and variance \( \sigma^2 \). Recall, the uncorrelated assumption does not imply independence. Hence, \( Y_t \) and \( Y_s \) can exhibit non-linear dependence (e.g. \( Y_t^2 \) can be correlated with \( Y_s^2 \)).
Nonstationary Processes

Defn: A nonstationary stochastic process is a stochastic process that is not covariance stationary.

Note: A non-stationary process violates one or more of the properties of covariance stationarity.

Example: Deterministically trending process

\[ Y_t = \beta_0 + \beta_1 t + \varepsilon_t, \varepsilon_t \sim WN(0, \sigma^2) \]
\[ E[Y_t] = \beta_0 + \beta_1 t \text{ depends on } t \]

Note: A simple detrending transformation yield a stationary process:

\[ X_t = Y_t - \beta_0 - \beta_1 t = \varepsilon_t \]
Example: Random Walk

\[ Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2_{\varepsilon}), \quad Y_0 \text{ is fixed} \]

\[ = Y_0 + \sum_{j=1}^{t} \varepsilon_j \Rightarrow var(Y_t) = \sigma^2_{\varepsilon} \times t \quad \text{depends on } t \]

Note: A simple detrending transformation yield a stationary process:

\[ \Delta Y_t = Y_t - Y_{t-1} = \varepsilon_t \]
**Time Series Models**

Defn: A time series model is a probability model to describe the behavior of a stochastic process \( \{Y_t\} \).

Note: Typically, a time series model is a simple probability model that describes the time dependence in the stochastic process \( \{Y_t\} \).
Moving Average (MA) Processes

Idea: Create a stochastic process that only exhibits one period linear time dependence

MA(1) Model

\[ Y_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}, \quad -\infty < \theta < \infty \]
\[ \varepsilon_t \sim iid \ N(0, \sigma^2_\varepsilon) \] (i.e., \( \varepsilon_t \sim GWN(0, \sigma^2_\varepsilon) \))
\[ \theta \] determines the magnitude of time dependence

Properties

\[ E[Y_t] = \mu + E[\varepsilon_t] + \theta E[\varepsilon_{t-1}] \]
\[ = \mu + 0 + 0 = \mu \]
\[
\text{var}(Y_t) = \sigma^2 = E[(Y_t - \mu)^2] \\
\quad = E[(\varepsilon_t + \theta \varepsilon_{t-1})^2] \\
\quad = E[\varepsilon_t^2] + 2\theta E[\varepsilon_t \varepsilon_{t-1}] + \theta^2 E[\varepsilon_{t-1}^2] \\
\quad = \sigma_\varepsilon^2 + 0 + \theta^2 \sigma_\varepsilon^2 = \sigma_\varepsilon^2 (1 + \theta^2)
\]

\[
\text{cov}(Y_t, Y_{t-1}) = \gamma_1 = E[(\varepsilon_t + \theta \varepsilon_{t-1})(\varepsilon_{t-1} + \theta \varepsilon_{t-2})] \\
\quad = E[\varepsilon_t \varepsilon_{t-1}] + \theta E[\varepsilon_t \varepsilon_{t-2}] \\
\quad + \theta E[\varepsilon_{t-1}^2] + \theta^2 E[\varepsilon_{t-1} \varepsilon_{t-2}] \\
\quad = 0 + 0 + \theta \sigma_\varepsilon^2 + 0 = \theta \sigma_\varepsilon^2
\]
Furthermore,

\[ \text{cov}(Y_t, Y_{t-2}) = \gamma_2 = E[(\varepsilon_t + \theta \varepsilon_{t-1})(\varepsilon_{t-2} + \theta \varepsilon_{t-3})] \]

\[ = E[\varepsilon_t \varepsilon_{t-2}] + \theta E[\varepsilon_t \varepsilon_{t-3}] \]

\[ + \theta E[\varepsilon_{t-1} \varepsilon_{t-2}] + \theta^2 E[\varepsilon_{t-1} \varepsilon_{t-3}] \]

\[ = 0 + 0 + 0 + 0 = 0 \]

Similar calculation show that

\[ \text{cov}(Y_t, Y_{t-j}) = \gamma_j = 0 \text{ for } j > 1 \]
Autocorrelations

$$\rho_1 = \frac{\gamma_1}{\sigma^2} = \frac{\theta \sigma^2}{\sigma^2(1 + \theta^2)} = \frac{\theta}{(1 + \theta^2)}$$

$$\rho_j = \frac{\gamma_j}{\sigma^2} = 0 \text{ for } j > 1$$

Note:

$$\rho_1 = 0 \text{ if } \theta = 0$$

$$\rho_1 > 0 \text{ if } \theta > 0$$

$$\rho_1 < 0 \text{ if } \theta < 0$$

Result: MA(1) is covariance stationary for any value of $\theta$. 
Example: MA(1) model for overlapping returns

Let $r_t$ denote the 1–month cc return and assume that

$$r_t \sim \text{iid } N(\mu_r, \sigma_r^2)$$

Consider creating a monthly time series of 2–month cc returns using

$$r_t(2) = r_t + r_{t-1}$$

These 2–month returns observed monthly overlap by 1 month

$$r_t(2) = r_t + r_{t-1}$$
$$r_{t-1}(2) = r_{t-1} + r_{t-2}$$
$$r_{t-2}(2) = r_{t-2} + r_{t-3}$$
\vdots

Claim: The stochastic process $\{r_t(2)\}$ follows a MA(1) process
Autoregressive (AR) Processes

Idea: Create a stochastic process that exhibits multi-period geometrically decaying linear time dependence

AR(1) Model (mean-adjusted form)

\[ Y_t - \mu = \phi(Y_{t-1} - \mu) + \varepsilon_t, \quad -1 < \phi < 1 \]

\[ \varepsilon_t \sim \text{iid } N(0, \sigma^2_\varepsilon) \]

Result: AR(1) model is covariance stationary provided \(-1 < \phi < 1\)
Properties

\[ E[Y_t] = \mu \]
\[ \text{var}(Y_t) = \sigma^2 = \sigma^2_{\epsilon}/(1 - \phi^2) \]
\[ \text{cov}(Y_t, Y_{t-1}) = \gamma_1 = \sigma^2 \phi \]
\[ \text{corr}(Y_t, Y_{t-1}) = \rho_1 = \gamma_1/\sigma^2 = \phi \]
\[ \text{cov}(Y_t, Y_{t-j}) = \gamma_j = \sigma^2 \phi^j \]
\[ \text{corr}(Y_t, Y_{t-j}) = \rho_j = \gamma_j/\sigma^2 = \phi^j \]

Note: Since \(|\phi| < 1\)

\[ \lim_{j \to \infty} \rho_j = \phi^j = 0 \]
AR(1) Model (regression model form)

\[ Y_t - \mu = \phi (Y_{t-1} - \mu) + \varepsilon_t \Rightarrow \]
\[ Y_t = \mu - \phi \mu + \phi Y_{t-1} + \varepsilon_t \]
\[ = c + \phi Y_{t-1} + \varepsilon_t \]

where

\[ c = (1 - \phi) \mu \Rightarrow \mu = \frac{c}{1 - \phi} \]

Remarks:

- Regression model form is convenient for estimation by linear regression
The AR(1) model and Economic and Financial Time Series

The AR(1) model is a good description for the following time series

- Interest rates on U.S. Treasury securities, dividend yields, unemployment

- Growth rate of macroeconomic variables
  - Real GDP, industrial production, productivity
  - Money, velocity, consumer prices
  - Real and nominal wages