Econ 424/CFRM 462
Statistical Analysis of Efficient Portfolios

Eric Zivot

August 14, 2014
The CER Model and Efficient Portfolios

Let $R_{it}$ denote the return on asset $i$ in month $t$ and assume that $R_{it}$ follows CER model:

$$R_{it} \sim iid \ N(\mu_i, \sigma_i^2),$$

$i = 1, \ldots, N$ (assets)

$t = 1, \ldots, T$ (months)

$$\text{cov}(R_{it}, R_{jt}) = \sigma_{ij}$$

We estimate the CER model parameters using sample statistics giving

$$\hat{\mu}_i, \hat{\sigma}_i^2, \hat{\sigma}_{ij}$$

Remember, the estimates $\hat{\mu}_i, \hat{\sigma}_i^2$ are $\hat{\sigma}_{ij}$ are random variables and are subject to error

Key result: Sharpe ratios and efficient portfolios are functions of $\hat{\mu}_i, \hat{\sigma}_i^2, \hat{\sigma}_{ij}$; they are random variables and are subject to error
Statistical Properties of Efficient portfolios

• Inputs to portfolio theory are estimates from CER model $\hat{\mu}$ and $\hat{\Sigma}$

• Sharpe ratios and efficient portfolios are functions of $\hat{\mu}$ and $\hat{\Sigma}$.

• The estimated Sharpe ratio is

$$ \widehat{SR}_i = \frac{\hat{\mu}_i - rf}{\hat{\sigma}_i} $$

• No easy formula for $SE(\widehat{SR}_i)$
• The estimated global minimum variance portfolio is

\[
\hat{\mathbf{m}} = \frac{\hat{\Sigma}^{-1}\mathbf{1}}{1'\hat{\Sigma}^{-1}\mathbf{1}}
\]

\(\hat{\mathbf{m}}\) is estimated with error because we estimate \(\Sigma\) using \(\hat{\Sigma}\).

• No easy analytic formulas for the standard errors of the elements of \(\hat{\mathbf{m}} = (\hat{m}_1, \ldots, \hat{m}_n)'\); i.e. no easy formula for \(SE(\hat{m}_i)\)

• In addition, the expected return and standard deviation of \(R_{p,\hat{m}} = \hat{\mathbf{m}}'\mathbf{R}\) have additional sources of error due to the error in \(\hat{\mathbf{m}}\). That is,

\[
\hat{\mu}_{p,\hat{m}} = \hat{\mathbf{m}}'\hat{\mu}
\]

\[
\hat{\sigma}_{p,\hat{m}} = (\hat{\mathbf{m}}'\hat{\Sigma}\hat{\mathbf{m}})^{1/2}
\]

No easy analytic formulas for \(SE(\hat{\mu}_{p,\hat{m}})\) and \(SE(\hat{\sigma}_{p,\hat{m}})\)
Optimizers are Error Maximizers

• From our analysis of the CER model, \( \mu_i \) is estimated less precisely than \( \sigma_i \). That is, there is more estimation error in \( \hat{\mu}_i \) than \( \hat{\sigma}_i \).

• Large estimation error in \( \hat{\mu}_i \) greatly impacts efficient portfolios
  
  – Large positive errors (\( \hat{\mu}_i \) much greater than \( \mu_i \)) leads to efficient portfolios being concentrated in asset \( i \)
  
  – Large negative errors (\( \hat{\mu}_i \) much less than \( \mu_i \)) leads to efficient portfolios that avoid asset \( i \) or shorts asset \( i \)

\[ 0 \leq \kappa_i \leq 0.1 \]

• Constraints on portfolio weights can offset the impact of estimation error in \( \hat{\mu}_i \)

  Portfolio of assets have smaller estimation error in \( \hat{\mu} \) than in individual assets
Bootstrapping Efficient Portfolios

The bootstrap can be used to evaluate the sampling uncertainty of Sharpe ratios and efficient portfolios.

Portfolio statistics to bootstrap:

- Portfolio weights
- Portfolio expected returns and standard deviations
Are Efficient Portfolios Constant Over Time?

Result: We have seen evidence that the parameters of the CER model for various assets are not constant over time:

- Rolling estimates of \( \mu, \sigma, \) and \( \sigma_{ij} \) show variation over time

Implication: Since estimates of \( \mu, \sigma, \) and \( \sigma_{ij} \) are inputs to efficient portfolio calculations, then time variation in \( \hat{\mu}, \hat{\sigma}, \) and \( \hat{\sigma}_{ij} \) imply time variation in efficient portfolios
Rolling Efficient Portfolios

Idea: Using rolling estimates of $\mu$ and $\Sigma$ compute rolling efficient portfolios

- global minimum variance portfolio
- efficient portfolio for target return
- tangency portfolio
- efficient frontier

Look at time variation in resulting portfolio weights
Rolling Global Minimum Variance Portfolio

Idea: compute estimates of portfolio weights $m$ over rolling windows of length $n < T$:

$$\min_{m(n)} m_t(n)'\hat{\Sigma}_t(n)m_t(n) \quad \text{s.t.} \quad m_t(n)'1 = 1$$

$$t = n, \ldots, T$$

$\hat{\Sigma}_t(n)$ = rolling estimate of $\Sigma$ in month $t$

If

$$\hat{\Sigma}_n(n) \approx \hat{\Sigma}_{n+1}(n) \approx \cdots \approx \hat{\Sigma}_T(n)$$

then

$$m_n(n) \approx m_{n+1}(n) \approx \cdots \approx m_T(n)$$
Rolling Efficient Portfolios

Idea: compute estimates of portfolio weights $x$ over rolling windows of length $n < T$ for $t = n, \ldots, T$:

$$
\min_{x(n)} \ x_t(n)'\hat{\Sigma}_t(n)x_t(n)
$$

s.t. $x_t(n)'1 = 1$, $x_t(n)'\hat{\mu}_t(n) = \mu_p^{\text{target}}$

$\hat{\mu}_t(n) = \text{rolling estimate of } \mu \text{ in month } t$

$\hat{\Sigma}_t(n) = \text{rolling estimate of } \Sigma \text{ in month } t$

If

$$
\hat{\mu}_n(n) \approx \hat{\mu}_{n+1}(n) \approx \cdots \approx \hat{\mu}_T(n)
$$

$$
\hat{\Sigma}_n(n) \approx \hat{\Sigma}_{n+1}(n) \approx \cdots \approx \hat{\Sigma}_T(n)
$$

then

$$
x_n(n) \approx x_{n+1}(n) \approx \cdots \approx x_T(n)
$$