Econ 424/CFRM 462
Portfolio Risk Budgeting

Eric Zivot

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Portfolio Risk Budgeting

Idea: Additively decompose a measure of portfolio risk into contributions from the individual assets in the portfolio.

- Show which assets are most responsible for portfolio risk
- Help make decisions about rebalancing the portfolio to alter the risk
- Construct “risk parity” portfolios where assets have equal risk contributions
Example: 2 risky asset portfolio

\[
R_p = x_1R_1 + x_2R_2
\]
\[
\sigma_p^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12}
\]
\[
\sigma_p = \left( x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12} \right)^{1/2}
\]

Case 1: \( \sigma_{12} = 0 \)

\[
\sigma_p^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 = \text{additive decomposition}
\]
\[
x_1^2\sigma_1^2 = \text{portfolio variance contribution of asset 1}
\]
\[
x_2^2\sigma_2^2 = \text{portfolio variance contribution of asset 2}
\]
\[
\frac{x_1^2\sigma_1^2}{\sigma_p^2} = \text{percent variance contribution of asset 1}
\]
\[
\frac{x_2^2\sigma_2^2}{\sigma_p^2} = \text{percent variance contribution of asset 2}
\]

In a risky two portfolio the 90% confidence interval

\( \text{could be 50%} \)
Note

\[ \sigma_p = \sqrt{x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2} \neq x_1 \sigma_1 + x_2 \sigma_2. \]

To get an additive decomposition we use

\[ \frac{x_1^2 \sigma_1^2}{\sigma_p} + \frac{x_2^2 \sigma_2^2}{\sigma_p} = \frac{\sigma_p^2}{\sigma_p} = \sigma_p \]

\[ \frac{x_1^2 \sigma_1^2}{\sigma_p} = \text{portfolio sd contribution of asset 1} \]

\[ \frac{x_2^2 \sigma_2^2}{\sigma_p} = \text{portfolio sd contribution of asset 2} \]

Notice that percent sd contributions are the same as percent variance contributions.
Case 2: $\sigma_{12} \neq 0$

\[
\sigma_p^2 = x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2 + 2x_1x_2\sigma_{12} \\
= \left(x_1^2 \sigma_1^2 + x_1 x_2 \sigma_{12}\right) + \left(x_2^2 \sigma_2^2 + x_1 x_2 \sigma_{12}\right).
\]

Here we can split the covariance contribution $2x_1x_2\sigma_{12}$ to portfolio variance evenly between the two assets and define

\[
x_1^2 \sigma_1^2 + x_1 x_2 \sigma_{12} = \text{variance contribution of asset 1}
\]
\[
x_2^2 \sigma_2^2 + x_1 x_2 \sigma_{12} = \text{variance contribution of asset 2}
\]
We can also define an additive decomposition for $\sigma_p$

$$
\sigma_p = \frac{x_1^2 \sigma_1^2 + x_1 x_2 \sigma_{12}}{\sigma_p} + \frac{x_2^2 \sigma_2^2 + x_1 x_2 \sigma_{12}}{\sigma_p}
$$

$$
\frac{x_1^2 \sigma_1^2 + x_1 x_2 \sigma_{12}}{\sigma_p} = \text{sd contribution of asset 1}
$$

$$
\frac{x_2^2 \sigma_2^2 + x_1 x_2 \sigma_{12}}{\sigma_p} = \text{sd contribution of asset 2}
$$
Euler’s Theorem and Risk Decompositions

• When we used $\sigma_p^2$ or $\sigma_p$ to measure portfolio risk, we were able to easily derive sensible risk decompositions.

• If we measure portfolio risk by value-at-risk or some other risk measure it is not so obvious how to define individual asset risk contributions.

• For portfolio risk measures that are homogenous functions of degree one in the portfolio weights, Euler’s theorem provides a general method for additively decomposing risk into asset specific contributions.
Homogenous functions and Euler’s theorem

First we define a homogenous function of degree one.

Definition 1  homogenous function of degree one

Let $f(x_1, \ldots, x_n)$ be a continuous and differentiable function of the variables $x_1, \ldots, x_n$. $f$ is homogeneous of degree one if for any constant $c$, $f(c \cdot x_1, \ldots, c \cdot x_n) = c \cdot f(x_1, \ldots, x_n)$.

Note: In matrix notation we have $f(x_1, \ldots, x_n) = f(x)$ where $x = (x_1, \ldots, x_n)'$. Then $f$ is homogeneous of degree one if $f(c \cdot x) = c \cdot f(x)$.
Examples

Let \( f(x_1, x_2) = x_1 + x_2 \). Then
\[
f(c \cdot x_1, c \cdot x_2) = c \cdot x_1 + c \cdot x_2 = c \cdot (x_1 + x_2) = c \cdot f(x_1, x_2)
\]

Let \( f(x_1, x_2) = x_1^2 + x_2^2 \). Then
\[
f(c \cdot x_1, c \cdot x_2) = c^2 x_1^2 + c^2 x_2^2 = c^2(x_1^2 + x_2^2) \neq c \cdot f(x_1, x_2)
\]

Let \( f(x_1, x_2) = \sqrt{x_1^2 + x_2^2} \) Then
\[
f(c \cdot x_1, c \cdot x_2) = \sqrt{c^2 x_1^2 + c^2 x_2^2} = c \sqrt{(x_1^2 + x_2^2)} = c \cdot f(x_1, x_2)
\]
Repeat examples using matrix notation

Define \( x = (x_1, x_2)' \) and \( 1 = (1, 1)' \).

Let \( f(x_1, x_2) = x_1 + x_2 = x'1 = f(x) \). Then
\[
  f(c \cdot x) = (c \cdot x)'1 = c \cdot (x'1) = c \cdot f(x).
\]

Let \( f(x_1, x_2) = x_1^2 + x_2^2 = x'x = f(x) \). Then
\[
  f(c \cdot x) = (c \cdot x)'(c \cdot x) = c^2 \cdot x'x \neq c \cdot f(x).
\]

Let \( f(x_1, x_2) = \sqrt{x_1^2 + x_2^2} = (x'x)^{1/2} = f(x) \). Then
\[
  f(c \cdot x) = \left((c \cdot x)'(c \cdot x)\right)^{1/2} = c \cdot (x'x)^{1/2} = c \cdot f(x).
\]
Consider a portfolio of $n$ assets $\mathbf{x} = (x_1, \ldots, x_n)'$

$$\mathbf{R} = (R_1, \ldots, R_n)'$$

$$\mathbf{x} = (x_1, \ldots, x_n)'$$

$$E[\mathbf{R}] = \mu, \quad \text{cov}(\mathbf{R}) = \Sigma$$

Define

$$R_p = R_p(\mathbf{x}) = \mathbf{x}'\mathbf{R},$$

$$\mu_p = \mu_p(\mathbf{x}) = \mathbf{x}'\mu$$

$$\sigma_p^2 = \sigma_p^2(\mathbf{x}) = \mathbf{x}'\Sigma\mathbf{x}, \quad \sigma_p = \sigma_p(\mathbf{x}) = (\mathbf{x}'\Sigma\mathbf{x})^{1/2}$$

**Result**: Portfolio return $R_p(\mathbf{x})$, expected return $\mu_p(\mathbf{x})$ and standard deviation $\sigma_p(\mathbf{x})$ are homogenous functions of degree one in the portfolio weight vector $\mathbf{x}$. 
The key result is for volatility $\sigma_p(x) = (x'\Sigma x)^{1/2}$:

$$\sigma_p(c \cdot x) = ((c \cdot x)'\Sigma(c \cdot x))^{1/2}$$

$$= c \cdot (x'\Sigma x)^{1/2}$$

$$= c \cdot \sigma_p(x)$$
**Theorem 2** *Euler’s theorem*

Let \( f(x_1, \ldots, x_n) = f(x) \) be a continuous, differentiable and homogenous of degree one function of the variables \( x = (x_1, \ldots, x_n)' \). Then

\[
    f(x) = x_1 \cdot \frac{\partial f(x)}{\partial x_1} + x_2 \cdot \frac{\partial f(x)}{\partial x_2} + \cdots + x_n \cdot \frac{\partial f(x)}{\partial x_n} \\
    = x' \frac{\partial f(x)}{\partial x},
\]

where

\[
    \frac{\partial f(x)}{\partial x} = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix}
\]
Verifying Euler’s theorem

The function \( f(x_1, x_2) = x_1 + x_2 = f(x) = x'1 \) is homogenous of degree one, and

\[
\frac{\partial f(x)}{\partial x_1} = \frac{\partial f(x)}{\partial x_2} = 1
\]

\[
\frac{\partial f(x)}{\partial x} = \left( \begin{array}{c} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \end{array} \right) = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = 1
\]

By Euler’s theorem,

\[
f(x) = x_1 \cdot 1 + x_2 \cdot 1 = x_1 + x_2
\]

\[
= x'1
\]
The function \( f(x_1, x_2) = (x_1^2 + x_2^2)^{1/2} = f(x) = (x'x)^{1/2} \) is homogenous of degree one, and

\[
\frac{\partial f(x)}{\partial x_1} = \frac{1}{2} (x_1^2 + x_2^2)^{-1/2} 2x_1 = x_1 (x_1^2 + x_2^2)^{-1/2},
\]

\[
\frac{\partial f(x)}{\partial x_2} = \frac{1}{2} (x_1^2 + x_2^2)^{-1/2} 2x_2 = x_2 (x_1^2 + x_2^2)^{-1/2}.
\]

By Euler’s theorem

\[
f(x) = x_1 \cdot x_1 (x_1^2 + x_1^2)^{-1/2} + x_2 \cdot x_2 (x_2^2 + x_2^2)^{-1/2}
\]

\[
= (x_1^2 + x_2^2) (x_1^2 + x_2^2)^{-1/2}
\]

\[
= (x_1^2 + x_2^2)^{1/2}.
\]
Using matrix algebra we have

\[
\frac{\partial f(x)}{\partial x} = \frac{\partial (x'x)^{1/2}}{\partial x} = \frac{1}{2} (x'x)^{-1/2} \frac{\partial x'x}{\partial x} = \frac{1}{2} (x'x)^{-1/2} 2x = (x'x)^{-1/2} \cdot x
\]

so by Euler’s theorem

\[
f(x) = x' \frac{\partial f(x)}{\partial x} = x'(x'x)^{-1/2} \cdot x = (x'x)^{-1/2} x'x = (x'x)^{1/2}
\]
Risk decomposition using Euler’s theorem

Let $R_{M_p}(x)$ denote a portfolio risk measure that is a homogenous function of degree one in the portfolio weight vector $x$. For example,

$$R_{M_p}(x) = \sigma_p(x) = (x'\Sigma x)^{1/2}$$

Euler’s theorem gives the additive risk decomposition

$$R_{M_p}(x) = \sum_{i=1}^{n} x_i \frac{\partial R_{M_p}(x)}{\partial x_i} = x' \frac{\partial R_{M_p}(x)}{\partial x}$$
Here, \( \frac{\partial \text{RM}_p(x)}{\partial x_i} \) are called *marginal contributions to risk* (MCRs):

\[
\text{MCR}_i^{RM} = \frac{\partial \text{RM}_p(x)}{\partial x_i} = \text{marginal contribution to risk of asset } i,
\]

The *contributions to risk* (CRs) are defined as the weighted marginal contributions:

\[
\text{CR}_i^{RM} = x_i \cdot \text{MCR}_i^{RM} = \text{contribution to risk of asset } i,
\]

Then

\[
\text{RM}_p(x) = x_1 \cdot \text{MCR}_1^{RM} + x_2 \cdot \text{MCR}_2^{RM} + \cdots + x_n \cdot \text{MCR}_n^{RM}
\]

\[
\text{PC}_1^{RM} + \text{PC}_2^{RM} + \cdots + \text{PC}_n^{RM}
\]
If we divide the contributions to risk by $RM_p(x)$ we get the percent contributions to risk (PCRs)

$$1 = \frac{CR_1^{RM}}{RM_p(x)} + \cdots + \frac{CR_n^{RM}}{RM_p(x)} = PCR_1^{RM} + \cdots + PCR_n^{RM},$$

where

$$PCR_i^{RM} = \frac{CR_i^{RM}}{RM_p(x)} = \text{percent contribution of asset } i$$

The asset with the highest $PCR_i^{RM}$ is the "riskiest" asset in the portfolio.

The asset with the lowest $PCR_i^{RM}$ is the "safest" asset in the portfolio.

It's possible for $CR_i^{RM} < 0$ and $PCR_i < 0$. 

Risk Decomposition for Portfolio SD

\[
\begin{bmatrix}
\frac{1}{\sigma p(x)} \\
\frac{1}{\sigma p(x)} \\
\vdots \\
\frac{1}{\sigma p(x)} \\
\end{bmatrix}
\frac{(\sum x)^i}{\sigma p(x)}
\]

Because \(\sigma p(x)\) is homogenous of degree 1 in \(x\), by Euler’s theorem

\[
\sigma p(x) = x_1 \frac{\partial \sigma p(x)}{\partial x_1} + x_2 \frac{\partial \sigma p(x)}{\partial x_2} + \cdots + x_n \frac{\partial \sigma p(x)}{\partial x_n} = x' \frac{\partial \sigma p(x)}{\partial x}
\]

Now

\[
\frac{\partial \sigma p(x)}{\partial x} = \frac{\partial (x' \sum x)^{1/2}}{\partial x} = \frac{1}{2} (x' \sum x)^{-1/2} 2 \sum x
\]

\[
= \frac{\sum x}{(x' \sum x)^{1/2}} = \frac{\sum x}{\sigma p(x)}
\]

\[
\Rightarrow \frac{\partial \sigma p(x)}{\partial x_i} = \text{MCR}_i^\sigma = \text{ith row of } \frac{\sum x}{\sigma p(x)}
\]

Remark: In R, the **PerformanceAnalytics** function `StdDev()` performs this decomposition
**Example:** 2 asset portfolio

\[
\sigma_p(x) = (x' \Sigma x)^{1/2} = \left( x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2 + 2x_1 x_2 \sigma_{12} \right)^{1/2}
\]

\[
\Sigma x = \begin{pmatrix}
\sigma_1^2 & \sigma_{12} \\
\sigma_{12} & \sigma_2^2 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix}
= \begin{pmatrix}
x_1 \sigma_1^2 + x_2 \sigma_{12} \\
x_2 \sigma_2^2 + x_1 \sigma_{12} \\
\end{pmatrix}
\]

\[
\frac{\Sigma x}{\sigma_p(x)} = \begin{pmatrix}
\frac{x_1 \sigma_1^2 + x_2 \sigma_{12}}{\sigma_p(x)} \\
\frac{x_2 \sigma_2^2 + x_1 \sigma_{12}}{\sigma_p(x)} \\
\end{pmatrix}
\]

so that

\[
MCR_1^\sigma = \frac{x_1 \sigma_1^2 + x_2 \sigma_{12}}{\sigma_p(x)}
\]

\[
MCR_2^\sigma = \frac{x_2 \sigma_2^2 + x_1 \sigma_{12}}{\sigma_p(x)}
\]
Then

\[
\text{MCR}_1^\sigma = \left( x_1 \sigma_1^2 + x_2 \sigma_{12} \right) / \sigma_p(x)
\]

\[
\text{MCR}_2^\sigma = \left( x_2 \sigma_2^2 + x_1 \sigma_{12} \right) / \sigma_p(x)
\]

\[
\text{CR}_1^\sigma = x_1 \times \left( x_1 \sigma_1^2 + x_2 \sigma_{12} \right) / \sigma_p(x) = \left( x_1^2 \sigma_1^2 + x_1 x_2 \sigma_{12} \right) / \sigma_p(x)
\]

\[
\text{CR}_2^\sigma = x_2 \times \left( x_2 \sigma_2^2 + x_1 \sigma_{21} \right) / \sigma_p(x) = \left( x_2^2 \sigma_2^2 + x_1 x_2 \sigma_{12} \right) / \sigma_p(x)
\]

and

\[
\text{PCR}_1^\sigma = \text{CR}_1^\sigma / \sigma_p(x) = \left( x_1^2 \sigma_1^2 + x_1 x_2 \sigma_{12} \right) / \sigma_p^2(x)
\]

\[
\text{PCR}_2^\sigma = \text{CR}_2^\sigma / \sigma_p(x) = \left( x_2^2 \sigma_2^2 + x_1 x_2 \sigma_{12} \right) / \sigma_p^2(x)
\]

Note: This is the decomposition we derived at the beginning of lecture.
How to Interpret and Use $MCR_i^\sigma$

\[
MCR_i^\sigma = \frac{\partial \sigma_p(x)}{\partial x_i} \approx \frac{\Delta \sigma_p}{\Delta x_i}
\]

\[
\Rightarrow \Delta \sigma_p \approx MCR_i^\sigma \cdot \Delta x_i
\]

However, in a portfolio of $n$ assets

\[
x_1 + x_2 + \cdots + x_n = 1
\]

so that increasing or decreasing $x_i$ means that we have to decrease or increase our allocation to one or more other assets. Hence, the formula

\[
\Delta \sigma_p \approx MCR_i^\sigma \cdot \Delta x_i
\]

ignores this re-allocation effect.
If the increase in allocation to asset $i$ is offset by a decrease in allocation to asset $j$, then

$$\Delta x_j = -\Delta x_i$$

and the change in portfolio volatility is approximately

$$\Delta \sigma_p \approx MCR^\sigma_i \cdot \Delta x_i + MCR^\sigma_j \cdot \Delta x_j$$

$$= MCR^\sigma_i \cdot \Delta x_i - MCR^\sigma_j \cdot \Delta x_i$$

$$= \left( MCR^\sigma_i - MCR^\sigma_j \right) \cdot \Delta x_i$$
Consider two portfolios:

- **equal weighted portfolio** $x_1 = x_2 = 0.5$

- **long-short portfolio** $x_1 = 1.5$ and $x_2 = -0.5$.

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\sigma_1^2$</th>
<th>$\sigma_2^2$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\sigma_{12}$</th>
<th>$\rho_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.175</td>
<td>0.055</td>
<td>0.067</td>
<td>0.013</td>
<td>0.258</td>
<td>0.115</td>
<td>-0.004875</td>
<td>-0.164</td>
</tr>
</tbody>
</table>

Table 1: Example data for two asset portfolio.
Table 2: Risk decomposition using portfolio standard deviation.

**Interpretation:** For equally weighted portfolio, increasing $x_1$ from 0.5 to 0.6 decreases $x_2$ from 0.5 to 0.4. Then

$$\Delta \sigma_p \approx (MCR_{1}^\sigma - MCR_{2}^\sigma) \cdot \Delta x_i$$

$$= (0.23310 - 0.03158)(0.1)$$

$$= 0.02015$$

So $\sigma_p$ increases from 13% to 15%
For the long-short portfolio, increasing $x_1$ from 1.5 to 1.6 decreases $x_2$ from -0.5 to -0.6. Then

$$
\Delta \sigma_p \approx (\text{MCR}^\sigma_1 - \text{MCR}^\sigma_2) \cdot \Delta x_i
$$

$$
= [0.25540 - (-0.03474)](0.1)
$$

$$
= 0.02901
$$

So $\sigma_p$ increases from 40% to 43%
Beta as a Measure of Asset Contribution to Portfolio Volatility

For a portfolio of $n$ assets with return

$$R_p(x) = x_1 R_1 + \cdots + x_n R_n = x' R$$

we derived the portfolio volatility decomposition

$$\sigma_p(x) = x_1 \frac{\partial \sigma_p(x)}{\partial x_1} + x_2 \frac{\partial \sigma_p(x)}{\partial x_2} + \cdots + x_n \frac{\partial \sigma_p(x)}{\partial x_n} = x' \frac{\partial \sigma_p(x)}{\partial x}$$

$$\frac{\partial \sigma_p(x)}{\partial x} = \frac{\Sigma x}{\sigma_p(x)} \frac{\partial \sigma_p(x)}{\partial x_i} = \text{ith row of } \frac{\Sigma x}{\sigma_p(x)}$$

With a little bit of algebra we can derive an alternative expression for

$$\text{MCR}^\sigma_i = \frac{\partial \sigma_p(x)}{\partial x_i} = \text{ith row of } \frac{\Sigma x}{\sigma_p(x)}$$
**Definition:** The beta of asset $i$ with respect to the portfolio is defined as

$$\beta_i = \frac{\text{cov}(R_i, R_p(x))}{\text{var}(R_p(x))} = \frac{\text{cov}(R_i, R_p(x))}{\sigma_p^2(x)}$$

**Result:** $\beta_i$ measures asset contribution to $\sigma_p(x)$:

- $\text{MCR}_i^\sigma = \frac{\partial \sigma_p(x)}{\partial x_i} = \beta_i \sigma_p(x)$
- $\text{CR}_i^\sigma = x_i \beta_i \sigma_p(x)$
- $\text{PCR}_i^\sigma = x_i \beta_i$

The slope $\beta_i$ is the change in the portfolio variance due to a change in asset $i$.

\[ \beta_i = \frac{\text{PCR}_i}{x_i} \]

\[ \text{slope} = \beta_i \]
Remarks

• By construction, the beta of the portfolio is 1

\[ \beta_p = \frac{\text{cov}(R_p(x), R_p(x))}{\text{var}(R_p(x))} = \frac{\text{var}(R_p(x))}{\text{var}(R_p(x))} = 1 \]

• When \( \beta_i = 1 \)

\[
\begin{align*}
\text{MCR}_i^\sigma &= \sigma_p(x) \\
\text{CR}_i^\sigma &= x_i \sigma_p(x) \\
\text{PCR}_i^\sigma &= x_i
\end{align*}
\]

If we increase the allocation to asset \( i \) with \( \beta_i = 1 \) and decrease allocation to asset \( j \) with \( \beta = 1 \), the

\[ \Delta \text{EF} = (\text{MCR}_i - \text{MCR}_j) \Delta x_i = (\sigma^p - \sigma^j) \Delta x_i = 0 \]
• When $\beta_i > 1$

$\text{MCR}_i^\sigma > \sigma_p(x)$  
$\text{CR}_i^\sigma > x_i \sigma_p(x)$  
$\text{PCR}_i^\sigma > x_i$

• When $\beta_i < 1$

$\text{MCR}_i^\sigma < \sigma_p(x)$  
$\text{CR}_i^\sigma < x_i \sigma_p(x)$  
$\text{PCR}_i^\sigma < x_i$

If $\beta_i > 1$ then add asset $i$ to portfolio increases portfolio risk.

If $\beta_i < 1$ then add asset $i$ to portfolio decreases risk.
<table>
<thead>
<tr>
<th></th>
<th>$\sigma_i$</th>
<th>$x_i$</th>
<th>MCR$^\sigma_i$</th>
<th>CR$^\sigma_i$</th>
<th>PCR$^\sigma_i$</th>
<th>$\beta_i = \text{PCR}_i^\sigma / x_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 1</td>
<td>0.258</td>
<td>0.5</td>
<td>0.23310</td>
<td>0.11655</td>
<td>0.8807</td>
<td>1.761</td>
</tr>
<tr>
<td>Asset 2</td>
<td>0.115</td>
<td>0.5</td>
<td>0.03158</td>
<td>0.01579</td>
<td>0.1193</td>
<td>0.239</td>
</tr>
</tbody>
</table>

$\sigma_p = 0.1323$

Table 3: Risk decomposition using portfolio standard deviation.

**Example**

- Asset 1 has $\beta_1 = 1.761 \Rightarrow$ Asset 1’s percent contribution to risk (PCR$^\sigma_i$) is much greater than its allocation weight ($x_i$)

- Asset 2 has $\beta_2 = 0.239 \Rightarrow$ Asset 1’s percent contribution to risk (PCR$^\sigma_i$) is much less than its allocation weight ($x_i$)
Derivation of Result:

Recall,

\[
\frac{\partial \sigma_p(x)}{\partial x} = \frac{\Sigma x}{p(x)}
\]

Now,

\[
\Sigma x =
\begin{pmatrix}
\sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\
\sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{1n} & \sigma_{n2} & \cdots & \sigma_n^2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
\]

1st row of \( \Sigma x \):

\[\begin{align*}
&= x_1 \sigma_1^2 + x_2 \cdot \sigma_{12} + x_3 \cdot \sigma_{13} + \cdots + x_n \cdot \sigma_{1n}
\end{align*}\]
The first row of $\Sigma x$ is

$$x_1\sigma_1^2 + x_2\sigma_{12} + \cdots + x_n\sigma_{1n} = \text{cov}(R_1, R_p)$$

Now consider

$$\text{cov}(R_1, R_p) = \text{cov}(R_1, x_1R_1 + \cdots + x_nR_n)$$

$$= \text{cov}(R_1, x_1R_1) + \cdots + \text{cov}(R_1, x_nR_n)$$

$$= x_1\sigma_1^2 + x_2\sigma_{12} + \cdots + x_n\sigma_{1n}$$

Next, note that

$$\beta_1 = \frac{\text{cov}(R_1, R_p)}{\sigma^2_p(x)} \Rightarrow \text{cov}(R_1, R_p) = \beta_1 \sigma^2_p(x)$$
An asset with $\beta > 1$ is very correlated (positively) with the other assets in the portfolio.

Hence, the first row of $\Sigma x$ is

$$x_1\sigma_1^2 + x_2\sigma_{12} + \cdots + x_n\sigma_{1n} = \beta_1\sigma_p^2(x)$$

and so

$$MCR_1^\sigma = \frac{\partial \sigma_p(x)}{\partial x_1} = \text{first row of } \frac{\Sigma x}{\sigma_p(x)}$$

$$= \frac{\beta_1\sigma_p^2(x)}{\sigma_p(x)} = \beta_1\sigma_p(x)$$

In a similar fashion, we have

$$MCR_i^\sigma = \frac{\partial \sigma_p(x)}{\partial x_i} = \text{i’th row of } \frac{\Sigma x}{\sigma_p(x)}$$

$$= \frac{\beta_i\sigma_p^2(x)}{\sigma_p(x)} = \beta_i\sigma_p(x)$$

$$\beta_i = \frac{\text{Cov}(r_i, r_p)}{\sigma_p \cdot (i+1)}$$
Decomposition of Portfolio Volatility

Recall,

\[ \text{MCR}^R_i = \frac{\partial \sigma_p(x)}{\partial x_i} = \text{ith row of } \frac{\Sigma x}{\sigma_p(x)} = \frac{\text{cov}(R_i, R_p(x))}{\sigma_p(x)} \]

Using

\[ \rho_{i,p} = \text{corr}(R_i, R_p(x)) = \frac{\text{cov}(R_i, R_p(x))}{\sigma_i \sigma_p(x)} \]

\[ \Rightarrow \text{cov}(R_i, R_p(x)) = \rho_{i,p} \sigma_i \sigma_p(x) \]

gives

\[ \text{MCR}^R_i = \frac{\rho_{i,p} \sigma_i \sigma_p(x)}{\sigma_p(x)} = \frac{\rho_{i,p} \sigma_i}{\sigma_i} = \rho_{i,p} \]

\[ \rho_{i,p} = \frac{\text{MCR}^R_i}{\sigma_i} \]
Then

\[ \text{CR}_i^\sigma = x_i \times \text{MCR}_i^\sigma = x_i \times \sigma_i \times \rho_{i,p} \]

= allocation \times \text{standalone risk} \times \text{correlation with portfolio}

Remarks:

- \[ x_i \times \sigma_i = \text{standalone contribution to risk} \text{ (ignores correlation effects with other assets)} \]

- \[ \text{CR}_i^\sigma = x_i \times \sigma_i \text{ only when } \rho_{i,p} = 1 \]

- If \[ \rho_{i,p} \neq 1 \] then \[ \text{CR}_i^\sigma < x_i \times \sigma_i \]
### Table 4: Risk decomposition using portfolio standard deviation.

<table>
<thead>
<tr>
<th>Asset</th>
<th>$\sigma_i$</th>
<th>$x_i$</th>
<th>$\rho_i$</th>
<th>$\text{MCR}_i^\sigma$</th>
<th>$\text{CR}_i^\sigma$</th>
<th>$\text{PCR}_i^\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.258</td>
<td>0.5</td>
<td>0.90</td>
<td>0.23310</td>
<td>0.11655</td>
<td>0.8807</td>
</tr>
<tr>
<td>2</td>
<td>0.115</td>
<td>0.5</td>
<td>0.27</td>
<td>0.03158</td>
<td>0.01579</td>
<td>0.1193</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\sigma_p = 0.1323$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.258</td>
<td>1.5</td>
<td>0.99</td>
<td>0.25540</td>
<td>0.38310</td>
<td>0.95663</td>
</tr>
<tr>
<td>2</td>
<td>0.115</td>
<td>-0.5</td>
<td>-0.30</td>
<td>-0.03474</td>
<td>0.01737</td>
<td>0.04337</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\sigma_p = 0.4005$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Remarks:

- For the equally weighted portfolio, both assets are positively correlated with the portfolio.

- For the long-short portfolio, Asset 2 is negatively correlated with the portfolio.
Beta as a Measure of Portfolio Risk

Key points:

- Asset specific risk can be diversified away by forming portfolios. What remains is “portfolio risk”.

- Riskiness of an asset should be judged in a portfolio context - portfolio risk demands a risk premium; asset specific risk does not

- Beta measures the portfolio risk of an asset

- In a large diversified portfolio of all traded assets, portfolio risk is the same as “market risk”
Beta and Risk Return Tradeoff

\[ R_p = \text{return on any portfolio} \]
\[ R_i = \text{return on any asset } i \]
\[ \beta_{i,p} = \frac{\text{cov}(R_i, R_p)}{\text{var}(R_p)} = \frac{\sigma_{i,p}}{\sigma_p^2} \]

Conjecture: If \( \beta_{i,p} \) is the appropriate measure of the risk of an asset, then the asset’s expected return, \( \mu_i \), should depend on \( \beta_{i,p} \). That is

\[ E[R_i] = \mu_i = f(\beta_{i,p}) \]

The Capital Asset Pricing Model (CAPM) formalizes this conjecture.

\[ E[R_i] = \mu + \beta_{i,m} (E[R_m] - \mu) \]