Econ 424/CFRM 462
Portfolio Theory with Matrix Algebra

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Portfolio Math with Matrix Algebra

Three Risky Asset Example

Let $R_i \ (i = A, B, C)$ denote the return on asset $i$ and assume that $R_i$ follows CER model:

$$R_i \sim \text{iid } N(\mu_i, \sigma_i^2)$$

$$\text{cov}(R_i, R_j) = \sigma_{ij}$$

Portfolio “x”

$x_i$ = share of wealth in asset $i$

$x_A + x_B + x_C = 1$

Portfolio return

$$R_{p,x} = x_A R_A + x_B R_B + x_C R_C.$$
<table>
<thead>
<tr>
<th>Stock $i$</th>
<th>$\mu_i$</th>
<th>$\sigma_i$</th>
<th>Pair (i,j)</th>
<th>$\sigma_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (Microsoft)</td>
<td>0.0427</td>
<td>0.1000</td>
<td>(A,B)</td>
<td>0.0018</td>
</tr>
<tr>
<td>B (Nordstrom)</td>
<td>0.0015</td>
<td>0.1044</td>
<td>(A,C)</td>
<td>0.0011</td>
</tr>
<tr>
<td>C (Starbucks)</td>
<td>0.0285</td>
<td>0.1411</td>
<td>(B,C)</td>
<td>0.0026</td>
</tr>
</tbody>
</table>

Table 1: Three asset example data.

In matrix algebra, we have

$$
\begin{align*}
\mu &= \begin{pmatrix}
\mu_A \\
\mu_B \\
\mu_C
\end{pmatrix} = 
\begin{pmatrix}
0.0427 \\
0.0015 \\
0.0285
\end{pmatrix} \\
\Sigma &= 
\begin{pmatrix}
\sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\
\sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\
\sigma_{AC} & \sigma_{BC} & \sigma_C^2
\end{pmatrix} = 
\begin{pmatrix}
(0.1000)^2 & 0.0018 & 0.0011 \\
0.0018 & (0.1044)^2 & 0.0026 \\
0.0011 & 0.0026 & (0.1411)^2
\end{pmatrix}
\end{align*}
$$
Matrix Algebra Representation

\[
\mathbf{R} = \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix}, \quad \mathbf{\mu} = \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

\[
\mathbf{x} = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix}
\]

Portfolio weights sum to 1

\[
\mathbf{x}' \mathbf{1} = \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = x_1 + x_2 + x_3 = 1
\]
Portfolio return

\[ R_{p,x} = \mathbf{x}' \mathbf{R} = (x_A \ x_B \ x_C) \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix} \]

\[ = x_A R_A + x_B R_B + x_C R_C \]

Portfolio expected return

\[ \mu_{p,x} = \mathbf{x}' \mathbf{\mu} = (x_A \ x_B \ x_X) \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix} \]

\[ = x_A \mu_A + x_B \mu_B + x_C \mu_C \]
R formula

\[ t(x.\text{vec}) \times \mu.\text{vec} \]
\[ \text{crossprod}(x.\text{vec}, \mu.\text{vec}) \]

Excel formula

\[ \text{MMULT} \left( \text{transpose}(x.\text{vec}), \mu.\text{vec} \right) \]
\[ <\text{ctrl}>-<\text{shift}>-<\text{enter}> \]
Portfolio variance

\[ \sigma_{p,x}^2 = x' \Sigma x \]

\[ = \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} \]

\[ = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + x_C^2 \sigma_C^2 \]

\[ + 2x_A x_B \sigma_{AB} + 2x_A x_C \sigma_{AC} + 2x_B x_C \sigma_{BC} \]

Portfolio distribution

\[ R_{p,x} \sim N(\mu_{p,x}, \sigma_{p,x}^2) \]
R formulas

\[ t(x.\text{vec}) \times \sigma.\text{mat} \times x.\text{vec} \]

Excel formulas

\begin{align*}
\text{MMULT} & (\text{TRANSPOSE}(x\text{vec}), \text{MMULT}(\sigma, x\text{vec})) \\
\text{MMULT} & (\text{MMULT}(\text{TRANSPOSE}(x\text{vec}), \sigma), x\text{vec}) \\
<\text{ctrl}> & -<\text{shift}> - <\text{enter}> 
\end{align*}
Covariance Between 2 Portfolio Returns

2 portfolios

\[ x = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \quad y = \begin{pmatrix} y_A \\ y_B \\ y_C \end{pmatrix} \]

\[ x'1 = 1, \quad y'1 = 1 \]

Portfolio returns

\[ R_{p,x} = x'R \]
\[ R_{p,y} = y'R \]

Covariance

\[ \text{cov}(R_{p,x}, R_{p,y}) = x'\Sigma y \]
\[ = y'\Sigma x \]
R formula

\[ t(x.vect) \times \sigma.mat \times y.vec \]

Excel formula

\[
\text{MMULT} (\text{TRANSPOSE}(xvec), \text{MMULT}(\sigma, yvec)) \\
\text{MMULT} (\text{TRANSPOSE}(yvec), \text{MMULT}(\sigma, xvec))
\]

\(<\text{ctrl}>-<\text{shift}>-<\text{enter}>\)
Derivatives of Simple Matrix Functions

Let $A$ be an $n \times n$ symmetric matrix, and let $x$ and $y$ be an $n \times 1$ vectors. Then

$$\frac{\partial}{\partial x} x'y = \begin{pmatrix} \frac{\partial}{\partial x_1} x'y \\ \vdots \\ \frac{\partial}{\partial x_n} x'y \end{pmatrix} = y,$$ (1)

$$\frac{\partial}{\partial x} x'Ax = \begin{pmatrix} \frac{\partial}{\partial x_1} x'Ax \\ \vdots \\ \frac{\partial}{\partial x_n} x'Ax \end{pmatrix} = 2Ax.$$ (2)
Computing Global Minimum Variance Portfolio

Problem: Find the portfolio \( \mathbf{m} = (m_A, m_B, m_C)' \) that solves

\[
\min_{m_A, m_B, m_C} \sigma_{p,m}^2 = \mathbf{m}'\Sigma\mathbf{m} \text{ s.t. } \mathbf{m}'\mathbf{1} = 1
\]

1. Analytic solution using matrix algebra

2. Numerical Solution in Excel Using the Solver (see 3firmExample.xls)
Analytic solution using matrix algebra

The Lagrangian is

$$L(m, \lambda) = m' \Sigma m + \lambda (m'1 - 1)$$

First order conditions (use matrix derivative results)

$$
\begin{align*}
0_{(3 \times 1)} &= \frac{\partial L(m, \lambda)}{\partial m} = \frac{\partial m' \Sigma m}{\partial m} + \frac{\partial}{\partial m} \lambda (m'1 - 1) = 2 \cdot \Sigma m + \lambda \mathbf{1} \\
0_{(1 \times 1)} &= \frac{\partial L(m, \lambda)}{\partial \lambda} = \frac{\partial m' \Sigma m}{\partial \lambda} + \frac{\partial}{\partial \lambda} \lambda (m'1 - 1) = m'1 - 1
\end{align*}
$$

Write FOCs in matrix form

$$
\left( \begin{array}{cc}
2 \Sigma & 1' \\
1' & 0
\end{array} \right) \left( \begin{array}{c}
m \\
\lambda
\end{array} \right) = \left( \begin{array}{c}
0_{3 \times 1} \\
1_{1 \times 1}
\end{array} \right)
$$
The FOCs are the linear system

\[ A_mz_m = b \]

where

\[ A_m = \begin{pmatrix} 2\Sigma & 1 \\ 1' & 0 \end{pmatrix}, \quad z_m = \begin{pmatrix} m \\ \lambda \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

The solution for \( z_m \) is

\[ z_m = A_m^{-1}b. \]

The first three elements of \( z_m \) are the portfolio weights \( m = (m_A, m_B, m_C)' \) for the global minimum variance portfolio with expected return \( \mu_{p,m} = m'\mu \) and variance \( \sigma_{p,m}^2 = m'\Sigma m \).
Alternative Derivation of Global Minimum Variance Portfolio

The first order conditions from the optimization problem can be expressed in matrix notation as

\[
\begin{align*}
0 &= \frac{\partial L(m, \lambda)}{\partial m} = 2 \cdot \Sigma m + \lambda \cdot 1, \\
0 &= \frac{\partial L(m, \lambda)}{\partial \lambda} = m'1 - 1. \quad \Rightarrow \quad m'1 = 1
\end{align*}
\]

Using first equation, solve for \( m \):

\[
m = -\frac{1}{2} \cdot \lambda \Sigma^{-1}1.
\]

\[
\begin{align*}
2 \cdot \Sigma m + \lambda \cdot 1 &= 0 \\
\Rightarrow \quad 2 \cdot \Sigma m &= -\lambda \cdot 1 \\
\Rightarrow \quad \Sigma m &= -\frac{\lambda}{2} \cdot 1 \\
\Rightarrow \quad m &= -\frac{\lambda}{2} \cdot \Sigma^{-1}1
\end{align*}
\]
Next, multiply both sides by $1'$ and use second equation to solve for $\lambda$:

$$1 = 1'm = -\frac{1}{2} \cdot \lambda 1'\Sigma^{-1}1$$

$$\Rightarrow \lambda = -2 \cdot \frac{1}{1'\Sigma^{-1}1}.$$ 

Finally, substitute the value for $\lambda$ in the equation for $m$:

$$m = -\frac{1}{2} (-2) \frac{1}{1'\Sigma^{-1}1} \Sigma^{-1}1$$

$$= \frac{\Sigma^{-1}1}{1'\Sigma^{-1}1}.$$
Efficient Portfolios of Risky Assets: Markowitz Algorithm

Problem 1: find portfolio \( x \) that has the highest expected return for a given level of risk as measured by portfolio variance

\[
\max_{x_A, x_B, x_C} \mu_{p,x} = x'\mu \quad \text{s.t.} \quad 
\sigma^2_{p,x} = x'\Sigma x = \sigma^0 = \text{target risk} \\
x'1 = 1
\]

Problem 2: find portfolio \( x \) that has the smallest risk, measured by portfolio variance, that achieves a target expected return.

\[
\min_{x_A, x_B, x_C} \sigma^2_{p,x} = x'\Sigma x \quad \text{s.t.} \quad 
\mu_{p,x} = x'\mu = \mu^0 = \text{target return} \\
x'1 = 1
\]

Remark: Problem 2 is usually solved in practice by varying the target return between a given range.
Solving for Efficient Portfolios:

1. Analytic solution using matrix algebra

2. Numerical solution in Excel using the solver
Analytic solution using matrix algebra

The Lagrangian function associated with Problem 2 is

\[ L(x, \lambda_1, \lambda_2) = x'\Sigma x + \lambda_1(x'\mu - \mu_p, 0) + \lambda_2(x'1 - 1) \]

The FOCs are

\[
\begin{align*}
0 \quad & = \frac{\partial L(x, \lambda_1, \lambda_2)}{\partial x} = 2\Sigma x + \lambda_1\mu + \lambda_21, \\
0 \quad & = \frac{\partial L(x, \lambda_1, \lambda_2)}{\partial \lambda_1} = x'\mu - \mu_p,0, \\
0 \quad & = \frac{\partial L(x, \lambda_1, \lambda_2)}{\partial \lambda_2} = x'1 - 1.
\end{align*}
\]

These FOCs consist of five linear equations in five unknowns

\((x_A, x_B, x_C, \lambda_1, \lambda_2)\).
We can represent the FOCs in matrix notation as

\[
\begin{pmatrix}
2 \Sigma & \mu & 1 \\
\mu' & 0 & 0 \\
1' & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
\lambda_1 \\
\lambda_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
\mu_{p,0} \\
1
\end{pmatrix}
\]

or

\[A_x z_x = b_0\]

where

\[A_x = \begin{pmatrix}
2 \Sigma & \mu & 1 \\
\mu' & 0 & 0 \\
1' & 0 & 0
\end{pmatrix}, \quad z_x = \begin{pmatrix}
x \\
\lambda_1 \\
\lambda_2
\end{pmatrix} \text{ and } b_0 = \begin{pmatrix}
0 \\
\mu_{p,0} \\
1
\end{pmatrix}\]
The solution for $z_x$ is then

$$z_x = A_x^{-1}b_0.$$ 

The first three elements of $z_x$ are the portfolio weights $x = (x_A, x_B, x_C)'$ for the efficient portfolio with expected return $\mu_{p,x} = \mu_{p,0}$. 
**Example:** Find efficient portfolios with the same expected return as MSFT and SBUX

For MSFT, we solve

$$\min_{x_A, x_B, x_C} \sigma_{p,x}^2 = x' \Sigma x \quad \text{s.t.}$$

$$\mu_{p,x} = x' \mu = \mu_{MSFT} = 0.0427$$

$$x'1 = 1$$

For SBUX, we solve

$$\min_{y_A, y_B, y_C} \sigma_{p,y}^2 = y' \Sigma y \quad \text{s.t.}$$

$$\mu_{p,y} = y' \mu = \mu_{SBUX} = 0.0285$$

$$y'1 = 1$$
Example continued

Using the matrix algebra formulas (see R code in Powerpoint slides) we get

\[ x = \begin{pmatrix} x_{msft} \\ x_{nord} \\ x_{sbux} \end{pmatrix} = \begin{pmatrix} 0.8275 \\ -0.0908 \\ 0.2633 \end{pmatrix}, \quad y = \begin{pmatrix} y_{msft} \\ y_{nord} \\ y_{sbux} \end{pmatrix} = \begin{pmatrix} 0.5194 \\ 0.2732 \\ 0.2075 \end{pmatrix} \]

Also,

\[
\mu_{p,x} = x'\mu = 0.0427, \quad \mu_{p,y} = y'\mu = 0.0285
\]

\[
\sigma_{p,x} = (x'\Sigma x)^{1/2} = 0.09166, \quad \sigma_{p,y} = (y'\Sigma y)^{1/2} = 0.07355
\]

\[
\sigma_{xy} = x'\Sigma y = 0.005914, \quad \rho_{xy} = \sigma_{xy}/(\sigma_{p,x}\sigma_{p,y}) = 0.8772
\]
Computing the Portfolio Frontier

Result: The portfolio frontier can be represented as convex combinations of any two frontier portfolios. Let $x$ be a frontier portfolio that solves

$$\min_x \sigma^2_{p,x} = x' \Sigma x \quad \text{s.t.}$$

$$\mu_{p,x} = x' \mu = \mu_p^0$$
$$x'1 = 1$$

Let $y \neq x$ be another frontier portfolio that solves

$$\min_y \sigma^2_{p,y} = y' \Sigma y \quad \text{s.t.}$$

$$\mu_{p,y} = y' \mu = \mu_p^1 \neq \mu_p^0$$
$$y'1 = 1$$
Let $\alpha$ be any constant. Then the portfolio

$$z = \alpha \cdot x + (1 - \alpha) \cdot y$$

is a frontier portfolio. Furthermore

$$\mu_{p,z} = z' \mu = \alpha \cdot \mu_{p,x} + (1 - \alpha) \mu_{p,y}$$
$$\sigma^2_{p,z} = z' \Sigma z$$
$$= \alpha^2 \sigma^2_{p,x} + (1 - \alpha)^2 \sigma^2_{p,y} + 2\alpha(1 - \alpha)\sigma_{x,y}$$
$$\sigma_{x,y} = \text{cov}(R_{p,x}, R_{p,y}) = x' \Sigma y$$
Example: 3 asset case

\[ z = \alpha \cdot x + (1 - \alpha) \cdot y \]

\[ = \alpha \cdot \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} + (1 - \alpha) \begin{pmatrix} y_A \\ y_B \\ y_C \end{pmatrix} \]

\[ = \begin{pmatrix} \alpha x_A + (1 - \alpha)y_A \\ \alpha x_B + (1 - \alpha)y_B \\ \alpha x_C + (1 - \alpha)y_C \end{pmatrix} = \begin{pmatrix} z_A \\ z_B \\ z_C \end{pmatrix} \]
**Example:** Compute efficient portfolio as convex combination of efficient portfolio with same mean as MSFT and efficient portfolio with same mean as SBUX.

Let \( x \) denote the efficient portfolio with the same mean as MSFT, \( y \) denote the efficient portfolio with the same mean as SBUX, and let \( \alpha = 0.5 \). Then

\[
z = \alpha \cdot x + (1 - \alpha) \cdot y
\]

\[
= 0.5 \cdot \begin{pmatrix} 0.82745 \\ -0.09075 \\ 0.26329 \end{pmatrix} + 0.5 \cdot \begin{pmatrix} 0.5194 \\ 0.2732 \\ 0.2075 \end{pmatrix}
\]

\[
= \begin{pmatrix} (0.5)(0.82745) \\ (0.5)(-0.09075) \\ (0.5)(0.26329) \end{pmatrix} + \begin{pmatrix} (0.5)(0.5194) \\ (0.5)(0.2732) \\ (0.5)(0.2075) \end{pmatrix} = \begin{pmatrix} 0.6734 \\ 0.0912 \\ 0.2354 \end{pmatrix} = \begin{pmatrix} z_A \\ z_B \\ z_C \end{pmatrix}.
\]
Example continued

The mean of this portfolio can be computed using:

\[
\mu_{p,z} = z'\mu = (0.6734, 0.0912, 0.2354)'
\begin{pmatrix}
  0.0427 \\
  0.0015 \\
  0.0285
\end{pmatrix} = 0.0356
\]

\[
\mu_{p,z} = \alpha \cdot \mu_{p,x} + (1 - \alpha)\mu_{p,y} = 0.5(0.0427) + (0.5)(0.0285) = 0.0356
\]

The variance can be computed using

\[
\sigma_{p,z}^2 = z'\Sigma z = 0.00641
\]
\[
\sigma_{p,z}^2 = \alpha^2\sigma_{p,x}^2 + (1 - \alpha)^2\sigma_{p,y}^2 + 2\alpha(1 - \alpha)\sigma_{xy}
\]
\[
= (0.5)^2(0.09166)^2 + (0.5)^2(0.07355)^2 + 2(0.5)(0.5)(0.005914) = 0.00641
\]
**Example:** Find efficient portfolio with expected return 0.05 from two efficient portfolios

Use

$$0.05 = \mu_{p,z} = \alpha \cdot \mu_{p,x} + (1 - \alpha)\mu_{p,y}$$

to solve for $\alpha$:

$$\alpha = \frac{0.05 - \mu_{p,y}}{\mu_{p,x} - \mu_{p,y}} = \frac{0.05 - 0.0285}{0.0427 - 0.0285} = 1.514$$

Then, solve for portfolio weights using

$$z = \alpha \cdot x + (1 - \alpha) \cdot y$$

$$= 1.514 \begin{pmatrix} 0.8275 \\ -0.0908 \\ 0.2633 \end{pmatrix} - 0.514 \begin{pmatrix} 0.5194 \\ 0.2732 \\ 0.2075 \end{pmatrix} = \begin{pmatrix} 0.9858 \\ -0.2778 \\ 0.2920 \end{pmatrix}$$
Strategy for Plotting Portfolio Frontier

1. Set global minimum variance portfolio = first frontier portfolio

\[
\min_{m} \sigma_{p,m}^2 = m' \Sigma m \quad \text{s.t.} \quad m'1 = 1
\]
and compute \( \mu_{p,m} = m' \mu \)

2. Find asset \( i \) that has highest expected return. Set target return to \( \mu^0 = \max(\mu) \) and solve

\[
\min_{x} \sigma_{p,x}^2 = x' \Sigma x \quad \text{s.t.}
\]
\[
\mu_{p,x} = x' \mu = \mu^0 = \max(\mu)
\]
\[
x'1 = 1
\]
3. Create grid of $\alpha$ values, initially between 1 and $-1$, and compute

$$z = \alpha \cdot m + (1 - \alpha) \cdot x$$

$$\mu_{p,z} = \alpha \cdot \mu_{p,m} + (1 - \alpha) \mu_{p,x}$$

$$\sigma^2_{p,z} = \alpha^2 \sigma^2_{p,m} + (1 - \alpha)^2 \sigma^2_{p,x} + 2\alpha(1 - \alpha)\sigma_{m,x}$$

$$\sigma_{m,x} = m'\Sigma x$$

4. Plot $\mu_{p,z}$ against $\sigma_{p,z}$. Expand or contract the grid of $\alpha$ values if necessary to improve the plot.
Finding the Tangency Portfolio

The tangency portfolio \( t \) is the portfolio of risky assets that maximizes Sharpe’s slope:

\[
\max_t \text{ Sharpe’s ratio} = \frac{\mu_{p,t} - r_f}{\sigma_{p,t}}
\]

subject to

\[ t'1 = 1 \]

In matrix notation,

\[
\text{Sharpe’s ratio} = \frac{t'\mu - r_f}{(t'\Sigma t)^{1/2}}
\]
Solving for Efficient Portfolios:

1. Analytic solution using matrix algebra

2. Numerical solution in Excel using the solver
Analytic solution using matrix algebra

The Lagrangian for this problem is

\[ L(t, \lambda) = \left( t' \mu - r_f \right) (t' \Sigma t)^{-\frac{1}{2}} + \lambda (t' 1 - 1) \]

Using the chain rule, the first order conditions are

\[
\begin{align*}
0 = \frac{\partial L(t, \lambda)}{\partial t} &= \mu (t' \Sigma t)^{-\frac{1}{2}} - \left( t' \mu - r_f \right) (t' \Sigma t)^{-3/2} \Sigma t + \lambda 1 \\
0 = \frac{\partial L(t, \lambda)}{\partial \lambda} &= t' 1 - 1 = 0
\end{align*}
\]

After much tedious algebra, it can be shown that the solution for \( t \) is

\[
t = \frac{\Sigma^{-1}(\mu - r_f \cdot 1)}{1' \Sigma^{-1}(\mu - r_f \cdot 1)}
\]
Remarks:

- If the risk free rate, \( r_f \), is less than the expected return on the global minimum variance portfolio, \( \mu_{g_{\text{min}}} \), then the tangency portfolio has a positive Sharpe slope.

- If the risk free rate, \( r_f \), is equal to the expected return on the global minimum variance portfolio, \( \mu_{g_{\text{min}}} \), then the tangency portfolio is not defined.

- If the risk free rate, \( r_f \), is greater than the expected return on the global minimum variance portfolio, \( \mu_{g_{\text{min}}} \), then the tangency portfolio has a negative Sharpe slope.
Mutual Fund Separation Theorem Again

Efficient Portfolios of T-bills and Risky assets are combinations of two portfolios (mutual funds)

• T-bills

• Tangency portfolio
Efficient Portfolios

\[ x_t = \text{share of wealth in tangency portfolio } t \]

\[ x_f = \text{share of wealth in T-bills} \]

\[ x_t + x_f = 1 \Rightarrow x_f = 1 - x_t \]

\[ \mu_p^e = r_f + x_t(\mu_{p,t} - r_f), \quad \mu_{p,t} = t'\mu \]

\[ \sigma_p^e = x_t\sigma_{p,t}, \quad \sigma_{p,t} = (t'\Sigma t)^{1/2} \]
**Remark:** The weights $x_t$ and $x_f$ are determined by an investor’s risk preferences

- Risk averse investors hold mostly T-Bills ($x_t \approx 0$)

- Risk tolerant investors hold mostly tangency portfolio ($x_t \approx 1$)

- If Sharpe’s slope for the tangency portfolio is negative then the efficient portfolio involve shorting the tangency portfolio
**Example:** Find efficient portfolio with target risk (SD) equal to 0.02

Solve

\[ 0.02 = \sigma_p^e = x_t \sigma_{p,t} = x_t(0.1116) \]

\[ \Rightarrow x_t = \frac{0.02}{0.1116} = 0.1792 \]

\[ x_f = 1 - x_t = 0.8208 \]

Also,

\[ \mu_p^e = r_f + x_t(\mu_{p,t} - r_f) = 0.005 + (0.1116)(0.05189 - 0.005) = 0.0134 \]

\[ \sigma_p^e = x_t \sigma_{p,t} = (0.1792)(0.1116) = 0.02 \]
Example: Find efficient portfolio with target ER equal to 0.07

Solve

\[ 0.07 = \mu_p^e = r_f + x_t (\mu_{p,t} - r_f) \]

\[ \Rightarrow x_t = \frac{0.07 - r_f}{\mu_{p,t} - r_f} = \frac{0.07 - 0.005}{0.05189 - 0.005} = 1.386 \]

Also,

\[ \sigma_p^e = x_t \sigma_{p,t} = (1.386)(0.1116) = 0.1547 \]
**Portfolio Value-at-Risk**

Let $x = (x_1, \ldots, x_n)'$ denote a vector of asset share for a portfolio. Portfolio risk is measured by $\text{var}(R_{p, x}) = x'\Sigma x$. Alternatively, portfolio risk can be measured using Value-at-Risk:

$$\text{VaR}_\alpha = W_0 q^R_\alpha$$

$W_0$ = initial investment
$q^R_\alpha = 100 \cdot \alpha\%$ Simple return quantile
$\alpha$ = loss probability

If returns are normally distributed then

$$q_\alpha = \mu_{p, x} + \sigma_{p, x} q^Z_\alpha$$

$\mu_{p, x} = x'\mu$

$$\sigma_{p, x} = (x'\Sigma x)^{1/2}$$

$q^Z_\alpha = 100 \cdot \alpha\%$ quantile from $N(0, 1)$
Example: Using VaR to evaluate an efficient portfolio

Invest in 3 risky assets (Microsoft, Starbucks, Nordstrom) and T-bills. Assume $r_f = 0.005$

1. Determine efficient portfolio that has same expected return as Starbucks

2. Compare VaR_{0.05} for Starbucks and efficient portfolio based on $100,000 investment
Solution for 1.

\[ \mu_{SBUX} = 0.0285 \]
\[ \mu^e_p = r_f + x_t(\mu_{p,t} - r_f) \]
\[ r_f = 0.005 \]
\[ \mu_{p,t} = t^\prime \mu = 0.05186, \sigma_{p,t} = 0.111 \]

Solve

\[ 0.0285 = 0.005 + x_t(0.05186 - 0.005) \]
\[ x_t = \frac{0.0285 - 0.005}{0.05186 - 0.005} = 0.501 \]
\[ x_f = 1 - 0.501 = 0.499 \]

Note:

\[ \mu^e_p = 0.005 + 0.501 \cdot (0.05186 - 0.005) = 0.0285 \]
\[ \sigma^e_p = x_t \sigma_{p,t} = (0.501)(0.111) = 0.057 \]
Solution for 2.

\[ q_{0.05}^{SBUX} = \mu_{SBUX} + \sigma_{SBUX} \cdot (-1.645) \]
\[ = 0.0285 + (0.141) \cdot (-1.645) \]
\[ = -0.203 \]

\[ q_{0.05}^e = \mu_p^e + \sigma_p^e \cdot (-1.645) \]
\[ = .0285 + (.057) \cdot (-1.645) \]
\[ = -0.063 \]

Then

\[ \text{VaR}_{0.05}^{SBUX} = 100,000 \cdot q_{0.05}^{SBUX} \]
\[ = 100,000 \cdot (-0.203) = -20,300 \]

\[ \text{VaR}_{0.05}^e = 100,000 \cdot q_{0.05}^e \]
\[ = 100,000 \cdot (-0.063) = -6,300 \]