Econ 424

Introduction to Portfolio Theory

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Introduction to Portfolio Theory

Investment in Two Risky Assets

\[ R_A = \text{simple return on asset A} \]
\[ R_B = \text{simple return on asset B} \]
\[ W_0 = \text{initial wealth} \]

Assumptions

- \( R_A \) and \( R_B \) are described by the CER model

\[ R_i \sim \text{iid } N(\mu_i, \sigma_i^2), \ i = A, B \]
\[ \text{cov}(R_A, R_B) = \sigma_{AB}, \ \text{cor}(R_A, R_B) = \rho_{AB} \]
• Investors like high $E[R_i] = \mu_i$

• Investors dislike high $\text{var}(R_i) = \sigma_i^2$

• Investment horizon is one period (e.g., one month or one year)

Note: Traditionally in portfolio theory, returns are simple and not continuously compounded
Portfolios

\[ x_A = \text{share of wealth in asset } A = \frac{\text{$ in A}}{W_0} \]
\[ x_B = \text{share of wealth in asset } B = \frac{\text{$ in B}}{W_0} \]

Long position

\[ x_A, x_B > 0 \]

Short position

\[ x_A < 0 \text{ or } x_B < 0 \]

Assumption: Allocate all wealth between assets A and B

\[ x_A + x_B = 1 \]

Portfolio Return

\[ R_p = x_A R_A + x_B R_B \]
Portfolio Distribution

\[ \mu_p = E[R_p] = x_A \mu_A + x_B \mu_B \]
\[ \sigma_p^2 = \text{var}(R_p) = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB} \]
\[ = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \rho_{AB} \sigma_A \sigma_B \]
\[ R_p \sim \text{iid } N(\mu_p, \sigma_p^2) \]

End of Period Wealth

\[ W_1 = W_0(1 + R_p) = W_0(1 + x_A R_A + x_B R_B) \]
\[ W_1 \sim N(W_0(1 + \mu_p), \sigma_p^2 W_0^2) \]
**Result:** Portfolio SD is not a weighted average of asset SD unless $\rho_{AB} = 1$:

$$
\sigma_p = \left( x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2x_Ax_B\rho_{AB}\sigma_A\sigma_B \right)^{1/2} \\
\neq x_A\sigma_A + x_B\sigma_B \text{ for } \rho_{AB} \neq 1
$$

If $\rho_{AB} = 1$ then

$$
\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B = \sigma_A\sigma_B
$$

and

$$
\sigma_p^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_A\sigma_B \\
= (x_A\sigma_A + x_B\sigma_B)^2 \\
\Rightarrow \sigma_p = x_A\sigma_A + x_B\sigma_B
$$
Example Data

$$\mu_A = 0.175, \mu_B = 0.055$$

$$\sigma_A^2 = 0.067, \sigma_B^2 = 0.013$$

$$\sigma_A = 0.258, \sigma_B = 0.115$$

$$\sigma_{AB} = -0.004875,$$

$$\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B} = -0.164$$

Note: Asset A has higher expected return and risk than asset B.
**Example:** Long only two asset portfolio

Consider an equally weighted portfolio with $x_A = x_B = 0.5$. The expected return, variance and volatility are

\[
\mu_p = (0.5) \cdot (0.175) + (0.5) \cdot (0.055) = 0.115
\]
\[
\sigma_p^2 = (0.5)^2 \cdot (0.067) + (0.5)^2 \cdot (0.013)
+ 2 \cdot (0.5)(0.5)(-0.004875) = 0.01751
\]
\[
\sigma_p = \sqrt{0.01751} = 0.1323
\]

This portfolio has expected return half-way between the expected returns on assets A and B, but the portfolio standard deviation is less than half-way between the asset standard deviations. This reflects risk reduction via diversification.
Example: Long-Short two asset portfolio

Next, consider a long-short portfolio with \( x_A = 1.5 \) and \( x_B = -0.5 \). In this portfolio, asset B is sold short and the proceeds of the short sale are used to leverage the investment in asset A. The portfolio characteristics are

\[
\begin{align*}
\mu_p &= (1.5) \cdot (0.175) + (-0.5) \cdot (0.055) = 0.235 \\
\sigma_p^2 &= (1.5)^2 \cdot (0.067) + (-0.5)^2 \cdot (0.013) \\
&\quad + 2 \cdot (1.5)(-0.5)(-0.004875) = 0.1604 \\
\sigma_p &= \sqrt{0.01751} = 0.4005
\end{align*}
\]

This portfolio has both a higher expected return and standard deviation than asset A.
Portfolio Value-at-Risk

- Assume an initial investment of $W_0$ in the portfolio of assets A and B.

- Given that the simple return $R_p \sim N(\mu_p, \sigma_p^2)$, For $\alpha \in (0, 1)$, the $\alpha \times 100\%$ portfolio value-at-risk is

$$\text{VaR}_{p,\alpha} = q_{p,\alpha}^R W_0 = (\mu_p + \sigma_p q_{\alpha}^z) W_0$$

where $q_{p,\alpha}^R$ is the $\alpha$ quantile of the distribution of $R_p$ and $q_{\alpha}^z = \alpha$ quantile of $Z \sim N(0, 1)$. 
Relationship between Portfolio VaR and Individual Asset VaR

**Result:** Portfolio VaR is not a weighted average of asset VaR

\[ \text{VaR}_{p,\alpha} \neq x_A \text{VaR}_{A,\alpha} + x_B \text{VaR}_{B,\alpha} \]

unless \( \rho_{AB} = 1 \).

Asset VaRs for A and B are

\[ \text{VaR}_{A,\alpha} = q_{0.05}^{R_A} W_0 = (\mu_A + \sigma_A q_{\alpha}) W_0 \]
\[ \text{VaR}_{B,\alpha} = q_{0.05}^{R_B} W_0 = (\mu_B + \sigma_B q_{\alpha}) W_0 \]

Portfolio VaR is

\[ \text{VaR}_{p,\alpha} = (\mu_p + \sigma_p q_{\alpha}) W_0 \]
\[ = \left[ (x_A \mu_A + x_B \mu_B) + \left( x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2x_A x_B \sigma_{AB} \right)^{1/2} q_{\alpha} \right] W_0 \]
Portfolio weighted asset VaR is

\[ x_A \text{VaR}_{A,\alpha} + x_B \text{VaR}_{B,\alpha} = x_A (\mu_A + \sigma_A q^\alpha) W_0 + x_B (\mu_B + \sigma_B q^\alpha) W_0 \]
\[ = [(x_A \mu_A + x_B \mu_B) + (x_A \sigma_A + x_B \sigma_B) q^\alpha] W_0 \]
\[ \neq (\mu_p + \sigma_p q^\alpha) W_0 = \text{VaR}_{p,\alpha} \]

provided \( \rho_{AB} \neq 1. \)

If \( \rho_{AB} = 1 \) then \( \sigma_{AB} = \rho_{AB} \sigma_A \sigma_B = \sigma_A \sigma_B \) and

\[ \sigma_p^2 = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_A \sigma_B = (x_A \sigma_A + x_B \sigma_B)^2 \]
\[ \Rightarrow \sigma_p = x_A \sigma_A + x_B \sigma_B \]

and so

\[ x_A \text{VaR}_{A,\alpha} + x_B \text{VaR}_{B,\alpha} = \text{VaR}_{p,\alpha} \]
**Example:** Portfolio VaR and Individual Asset VaR

Consider an initial investment of $W_0 = $100,000. The 5% VaRs on assets A and B are

\[
\text{VaR}_{A,0.05} = q_{0.05}^R W_0 = (0.175 + 0.258(-1.645)) \cdot 100,000 = -24,937,
\]

\[
\text{VaR}_{B,0.05} = q_{0.05}^R W_0 = (0.055 + 0.115(-1.645)) \cdot 100,000 = -13,416.
\]

The 5% VaR on the equal weighted portfolio with $x_A = x_B = 0.5$ is

\[
\text{VaR}_{p,0.05} = q_{0.05}^R W_0 = (0.115 + 0.1323(-1.645)) \cdot 100,000 = -10,268,
\]

and the weighted average of the individual asset VaRs is

\[
x_A \text{VaR}_{A,0.05} + x_B \text{VaR}_{B,0.05} = 0.5(-24,937) + 0.5(-13,416) = -19,177.
\]
Portfolio Frontier

Vary investment shares $x_A$ and $x_B$ and compute resulting values of $\mu_p$ and $\sigma_p^2$. Plot $\mu_p$ against $\sigma_p$ as functions of $x_A$ and $x_B$

- Shape of portfolio frontier depends on correlation between assets A and B

- If $\rho_{AB} = -1$ then there exists portfolio shares $x_A$ and $x_B$ such that $\sigma_p^2 = 0$

- If $\rho_{AB} = 1$ then there is no benefit from diversification

- Diversification is beneficial even if $0 < \rho_{AB} < 1$
Efficient Portfolios

**Definition**: Portfolios with the highest expected return for a given level of risk, as measured by portfolio standard deviation, are efficient portfolios.

- If investors like portfolios with high expected returns and dislike portfolios with high return standard deviations then they will want to hold efficient portfolios.
• Which efficient portfolio an investor will hold depends on their risk preferences

  – Very risk averse investors dislike volatility and will hold portfolios near the global minimum variance portfolio. They sacrifice expected return for the safety of low volatility

  – Risk tolerant investors don’t mind volatility and will hold portfolios that have high expected returns. They gain expected return by taking on more volatility.
Global Minimum Variance Portfolio

- The portfolio with the smallest possible variance is called the global minimum variance portfolio.

- This portfolio is chosen by the most risk averse individuals.

- To find this portfolio, one has to solve the following constrained minimization problem

\[
\min_{x_A, x_B} \sigma_p^2 = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2x_A x_B \sigma_{AB} \\
\text{s.t. } x_A + x_B = 1
\]
Review of Optimization Techniques: Constrained Optimization

Example: Finding the minimum of a bivariate function subject to a linear constraint

\[ y = f(x, z) = x^2 + z^2 \]
\[ \min_{x,z} y = f(x, z) \]
\[ s.t. \; x + z = 1 \]

Solution methods:

- Substitution
- Lagrange multipliers
Method of Substitution

Substitute $z = x - 1$ in $f(x, z)$ and solve univariate minimization

$y = f(x, x - 1) = x^2 + (1 - x)^2$

$$\min_x f(x, x - 1)$$

First order conditions

$$0 = \frac{d}{dx}(x^2 + (1 - x)) = 2x + 2(1 - x)(-1)$$
$$= 4x - 2$$
$$\Rightarrow x = 0.5$$

Solving for $z$

$$z = 1 - 0.5 = 0.5$$
Method of Lagrange Multipliers

Idea: Augment function to be minimized with extra terms to impose constraints

1. Put constraints in homogeneous form

\[ x + z = 1 \Rightarrow x + z - 1 = 0 \]

2. Form Lagrangian function

\[ L(x, z, \lambda) = x^2 + z^2 + \lambda(x + z - 1) \]

\[ \lambda = \text{Lagrange multiplier} \]
3. Minimize Lagrangian function

\[ \min_{x,z,\lambda} L(x, z, \lambda) \]

First order conditions

\[ 0 = \frac{\partial L(x, z, \lambda)}{\partial x} = 2 \cdot x + \lambda \]
\[ 0 = \frac{\partial L(x, z, \lambda)}{\partial z} = 2 \cdot z + \lambda \]
\[ 0 = \frac{\partial L(x, z, \lambda)}{\partial \lambda} = x + z - 1 \]

We have three linear equations in three unknowns. Solving gives

\[ 2x = 2z = -\lambda \Rightarrow x = z \]
\[ 2z - 1 = 0 \Rightarrow z = 0.5, \ x = 0.5 \]
Example: Finding the Global Minimum Variance Portfolio

Two methods for solution

- Analytic solution using Calculus

- Numerical solution
  - use the Solver in Excel
  - use R function `solve.QP()` in package `quadprog` for quadratic optimization problems with equality and inequality constraints
Calculus Solution

Minimization problem

\[
\min_{x_A, x_B} \quad \sigma_p^2 = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2x_A x_B \sigma_{AB}
\]

\[\text{s.t. } x_A + x_B = 1\]

Use substitution method with

\[x_B = 1 - x_A\]

to give the univariate minimization

\[
\min_{x_A} \quad \sigma_p^2 = x_A^2 \sigma_A^2 + (1 - x_A)^2 \sigma_B^2 + 2x_A (1 - x_A) \sigma_{AB}
\]
First order conditions

\[ 0 = \frac{d}{dx_A} \sigma_p^2 = \frac{d}{dx_A} \left( x_A^2 \sigma_A^2 + (1 - x_A)^2 \sigma_B^2 + 2 x_A (1 - x_A) \sigma_{AB} \right) \]

\[ = 2 x_A \sigma_A^2 - 2 (1 - x_A) \sigma_B^2 + 2 \sigma_{AB} (1 - 2 x_A) \]

\[ \Rightarrow x_A^{\min} = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2 \sigma_{AB}}, \quad x_B^{\min} = 1 - x_A^{\min} \]
Excel Solver Solution

The Solver is an Excel add-in, that can be used to numerically solve general linear and nonlinear optimization problems subject to equality or inequality constraints

- The solver is made by FrontLine Systems and is provided with Excel

- The solver add-in may not be installed in a “default installation” of Excel
  - Tools/Add-Ins and check the Solver Add-In box
  - If Solver Add-In box is not available, the Solver Add-In must be installed from original Excel installation CD
Portfolios with a Risk Free Asset

Risk Free Asset

- Asset with fixed and known rate of return over investment horizon

- Usually use U.S. government T-Bill rate (horizons < 1 year) or T-Note rate (horizon > 1 yr)

- T-Bill or T-Note rate is only nominally risk free
Properties of Risk-Free Asset

\[ R_f = \text{return on risk-free asset} \]
\[ E[R_f] = r_f = \text{constant} \]
\[ \text{var}(R_f) = 0 \]
\[ \text{cov}(R_f, R_i) = 0, \ R_i = \text{return on any asset} \]

Portfolios of Risky Asset and Risk Free Asset

\[ x_f = \text{share of wealth in T-Bills} \]
\[ x_B = \text{share of wealth in asset B} \]
\[ x_f + x_B = 1 \]
\[ x_f = 1 - x_B \]
Portfolio return

\[ R_p = x_f r_f + x_B R_B \]
\[ = (1 - x_B) r_f + x_B R_B \]
\[ = r_f + x_B (R_B - r_f) \]

Portfolio excess return

\[ R_p - r_f = x_B (R_B - r_f) \]

Portfolio Distribution

\[ \mu_p = E[R_p] = r_f + x_B (\mu_B - r_f) \]
\[ \sigma_p^2 = \text{var}(R_p) = x_B^2 \sigma_B^2 \]
\[ \sigma_p = x_B \sigma_B \]
\[ R_p \sim N(\mu_p, \sigma_p^2) \]
**Risk Premium**

\[ \mu_B - r_f = \text{excess expected return on asset B} \]
\[ = \text{expected return on risky asset over return on safe asset} \]

For the portfolio of T-Bills and asset B

\[ \mu_p - r_f = x_B(\mu_B - r_f) \]
\[ = \text{expected portfolio return over T-Bill} \]

The risk premia is an increasing function of the amount invested in asset B.
Leveraged Investment

$$x_f < 0, \ x_B > 1$$

Borrow at T-Bill rate to buy more of asset B

Result: Leverage increases portfolio expected return and risk

$$\mu_p = r_f + x_B(\mu_B - r_f)$$

$$\sigma_p = x_B\sigma_B$$

$$x_B \uparrow \Rightarrow \mu_p \& \sigma_p \uparrow$$
Determining Portfolio Frontier

Goal: Plot $\mu_p$ vs. $\sigma_p$

$$\sigma_p = x_B \sigma_B \Rightarrow x_B = \frac{\sigma_p}{\sigma_B}$$

$$\mu_p = r_f + x_B (\mu_B - r_f)$$

$$= r_f + \frac{\sigma_p}{\sigma_B} (\mu_B - r_f)$$

$$= r_f + \left( \frac{\mu_B - r_f}{\sigma_B} \right) \sigma_p$$

where

$$\left( \frac{\mu_B - r_f}{\sigma_B} \right) = SR_B = \text{Asset B Sharpe Ratio}$$

$$= \text{excess expected return per unit risk}$$
Remarks

- The Sharpe Ratio (SR) is commonly used to rank assets.

- Assets with high Sharpe Ratios are preferred to assets with low Sharpe Ratios.
Efficient Portfolios with 2 Risky Assets and a Risk Free Asset

Investment in 2 Risky Assets and T-Bill

\[ R_A = \text{simple return on asset A} \]
\[ R_B = \text{simple return on asset B} \]
\[ R_f = r_f = \text{return on T-Bill} \]

Assumptions

- \( R_A \) and \( R_B \) are described by the CER model

\[ R_i \sim iid \ N(\mu_i, \sigma_i^2), \ i = A, B \]
\[ \text{cov}(R_A, R_B) = \sigma_{AB}, \ \text{corr}(R_A, R_B) = \rho_{AB} \]
Results:

- The best portfolio of two risky assets and T-Bills is the one with the highest Sharpe Ratio.

- Graphically, this portfolio occurs at the tangency point of a line drawn from $R_f$ to the risky asset only frontier.

- The maximum Sharpe Ratio portfolio is called the “tangency portfolio.”
Mutual Fund Separation Theorem

Efficient portfolios are combinations of two portfolios (mutual funds)

- T-Bill portfolio

- Tangency portfolio - portfolio of assets A and B that has the maximum Shape ratio

Implication: All investors hold assets A and B according to their proportions in the tangency portfolio regardless of their risk preferences.
Finding the tangency portfolio

\[
\max_{x_A, x_B} \text{SR}_p = \frac{\mu_p - rf}{\sigma_p} \text{ subject to }
\]

\[
\begin{align*}
\mu_p &= x_A \mu_A + x_B \mu_B \\
\sigma^2_p &= x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2x_A x_B \sigma_{AB} \\
1 &= x_A + x_B
\end{align*}
\]

Solution can be found analytically or numerically (e.g., using solver in Excel)
Using the substitution method it can be shown that

\[ x_A^{\tan} = \frac{(\mu_A - r_f)\sigma_B^2 - (\mu_B - r_f)\sigma_{AB}}{(\mu_A - r_f)\sigma_B^2 + (\mu_B - r_f)\sigma_A^2 - (\mu_A - r_f + \mu_B - r_f)\sigma_{AB}} \]

\[ x_B^{\tan} = 1 - x_A^{\tan} \]

Portfolio characteristics

\[ \mu_p^{\tan} = x_A^{\tan} \mu_A + x_B^{\tan} \mu_B \]

\[ (\sigma_p^{\tan})^2 = (x_A^{\tan})^2 \sigma_A^2 + (x_B^{\tan})^2 \sigma_B^2 + 2x_A^{\tan} x_B^{\tan} \sigma_{AB} \]
Efficient Portfolios: tangency portfolio plus T-Bills

\[ x_{\text{tan}} = \text{share of wealth in tangency portfolio} \]
\[ x_f = \text{share of wealth in T-bills} \]
\[ x_{\text{tan}} + x_f = 1 \]
\[ \mu_p^e = r_f + x_{\text{tan}}(\mu_p^{\text{tan}} - r_f) \]
\[ \sigma_p^e = x_{\text{tan}}\sigma_p^{\text{tan}} \]

Result: The weights \( x_{\text{tan}} \) and \( x_f \) are determined by an investor’s risk preferences

- Risk averse investors hold mostly T-Bills
- Risk tolerant investors hold mostly tangency portfolio
Example

For the two asset example, the tangency portfolio is

\[
\begin{align*}
x_A^{\text{tan}} &= .46, \quad x_B^{\text{tan}} = 0.54 \\
\mu_p^{\text{tan}} &= (.46)(.175) + (.54)(.055) = 0.11 \\
\left(\sigma_p^{\text{tan}}\right)^2 &= (.46)^2(.067) + (.54)^2(.013) \\
&\quad + 2(.46)(.54)(-.005) \\
&= 0.015 \\
\sigma_p^{\text{tan}} &= \sqrt{0.015} = 0.124
\end{align*}
\]

Efficient portfolios have the following characteristics

\[
\begin{align*}
\mu_p^e &= r_f + x^{\text{tan}}(\mu_p^{\text{tan}} - r_f) \\
&= 0.03 + x^{\text{tan}}(0.11 - 0.03) \\
\sigma_p^e &= x^{\text{tan}}\sigma_p^{\text{tan}} \\
&= x^{\text{tan}}(0.124)
\end{align*}
\]
Problem: Find the efficient portfolio that has the same risk (SD) as asset B? That is, determine $x_{\text{tan}}$ and $x_f$ such that

$$\sigma_p^e = \sigma_B = 0.114 = \text{target risk}$$

Note: The efficient portfolio will have a higher expected return than asset B.
Solution:

\[ .114 = \sigma_p^e = x_{tan}\sigma_p^{tan} \]

\[ = x_{tan}(0.12) \]

\[ \Rightarrow x_{tan} = \frac{0.114}{0.12} = 0.92 \]

\[ x_f = 1 - x_{tan} = 0.08 \]

Efficient portfolio with same risk as asset B has

\[ (.92)(.46) = .42 \text{ in asset A} \]
\[ (.92)(.54) = .50 \text{ in asset B} \]
\[ .08 \text{ in T-Bills} \]

If \( r_f = 0.03 \), then expected Return on efficient portfolio is

\[ \mu_p^e = 0.03 + (.92)(.11 - 0.03) = .104 \]
Problem: Assume that \( r_f = 0.03 \). Find the efficient portfolio that has the same expected return as asset B. That is, determine \( x_{\text{tan}} \) and \( x_f \) such that

\[
\mu_P^e = \mu_B = 0.055 = \text{target expected return}
\]

Note: The efficient portfolio will have a lower SD than asset B
Solution:

\[ 0.055 = \mu_p^e = 0.03 + x_{\tan}(0.11 - 0.03) \]

\[ x_{\tan} = \frac{0.055 - 0.03}{0.11 - 0.03} = 0.31 \]

\[ x_f = 1 - x_{\tan} = 0.69 \]

Efficient portfolio with same expected return as asset B has

\[ (.31)(.46) = 0.14 \text{ in asset A} \]
\[ (.31)(.54) = 0.17 \text{ in asset B} \]
\[ .69 \text{ in T-Bills} \]

The SD of the efficient portfolio is

\[ \sigma_p^e = 0.31(.124) = 0.038 \]