Present Value Methodology

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Present Value Concept

Wealth in Fisher Model:
\[ W = Y_0 + Y_1/(1+r) \]
The consumer/producer’s wealth is their current endowment plus the future endowment discounted back to the present by the rate of interest (rate at which present and future consumption can be exchanged).

• Why do this?
  – Purpose of comparison—apples to apples (temporal) comparison with multiple agents or apples to apples comparison of investment/consumption opportunities

• Uniform method for valuing present and future streams of consumption in order for appropriate decision making by consumer/producer

• Useful concept for valuing multiple period investments and pricing financial instruments

Calculating Present Value

Present value calculations are the reverse of compound growth calculations:

Suppose \( V_0 \) = a value today (time 0)
\[ r \] = fixed interest rate (annual)
\[ T \] = amount of time (years) to future period

The value in \( T \) years we calculate as:
\[ V_T = V_0 (1+r)^T \] (Future Value)
Example

• A $30,000 Certificate of Deposit with 5% annual interest in 10 years will be worth:
  \[ V_T = V_0 (1 + r)^T = 30,000 \times (1 + 0.05)^{10} = 48,866.84 \]

• Note: Computation is easy to do in Excel
  \[ = 30,000 \times (1 + 0.05)^{10} \]

Present Value

In reverse:

\[ V_0 = \frac{V_T}{(1+r)^T} \] (Present Value)

*The present value amount is the future value discounted (divided) by the compounded rate of interest*

Example: A $48,866.84 Certificate of Deposit received 10 years from now is worth today:

\[ V_0 = \frac{48,866.84}{(1+0.05)^{10}} = 30,000 \]

Exam Review

• Be able to calculate present and future values

• For any three of four variables: \((V_0, r, T, V_T)\) you should be able to determine the value of the fourth variable.

• How do changes to \(r\) and \(T\) impact \(V_0\) and \(V_T\)?
Example: Rule of 70

• Q: How many years, T, will it take for an initial investment of $V_0$ to double if the annual interest rate is $r$?
• A: Solve $V_0 (1 + r)^T = 2V_0$
• => $(1 + r)^T = 2$
• => $T \ln(1 + r) = \ln(2)$
• => $T = \frac{\ln(2)}{\ln(1+r)}$
• = 0.69/ln(1 + r) ≈ 0.70/r for $r$ not too big

Present Value of Future Cash Flows

• A cash flow is a sequence of dated cash amounts received (+) or paid (-): $C_0, C_1, ..., C_T$
• Cash amounts received are positive; whereas, cash amounts paid are negative
• The present value of a cash flow is the sum of the present values for each element of the cash flow

Discount factors: Intertemporal Price of $1$ with constant interest rate $r$

• $1/(1+r) = \text{price of } 1 \text{ to be received 1 year from today}$
• $1/(1+r)^2 = \text{price of } 1 \text{ to be received 2 years from today}$
• $1/(1+r)^T = \text{price of } 1 \text{ to be received } T \text{ years from today}$
Present Value of a Cash Flow

- \{C_0, C_1, C_2, \ldots, C_T\} represents a sequence of cash flows where payment
- \(C_i\) is received at time \(i\). Let \(r\) = the interest or discount rate.

Q: What is the present value of this cash flow?

A: The present value of the sequence of cash flows is the sum of the present values:

\[
P V = C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_T}{(1+r)^T}
\]

Summation Notation

\[
PV = \sum_{t=0}^{T} \frac{C_t}{(1 + r)^t}
\]

= \(C_0 + \sum_{t=1}^{T} \frac{C_t}{(1 + r)^t}\)

Example

You receive the following cash payments:
- time 0: -$10,000 (Your initial investment)
- time 1: $4,000
- time 2: $4,000
- time 3: $4,000

The discount rate = 0.08 (or 8%)

\[
P V = -$10,000 + \frac{4,000}{(1+0.08)} + \frac{4,000}{(1+0.08)^2} + \frac{4,000}{(1+0.08)^3}
\]

= -$10,000 + $3,703.70 + $3,429.36 + $3,175.33

= $308.39

See econ422PresentValueProblems.xls for Excel calculations
PV Calculations in Excel

Excel function **NPV**:

\[ \text{NPV(rate, value1, value2, \ldots, value29)} \]

- **Rate** = per period fixed interest rate
- **value1** = cash flow in period 1
- **value2** = cash flow in period 2
- \( \ldots \)
- **value29** = cash flow in 29th period

*Note: NPV function does not take account of initial period cash flow!*

Present Value Calculation Short-cuts

**PERPETUITY:**

A perpetuity pays an amount \( C \) starting next period and pays this same constant amount \( C \) in each period forever:

\[ C_1 = C, C_2 = C, C_3 = C, C_4 = C, \ldots \]

\[ \text{PV(Perpetuity)} = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^n} + \ldots \]

\[ = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r} \]

PV of Perpetuity

Based on the infinite sum property, we can write PV as:

\[ \text{PV} = \text{Initial Term} \cdot \frac{1 - \text{Common Ratio}}{1 - \text{Common Ratio}} \]

\[ = \frac{C}{(1+r)} \cdot \frac{1 - (1/(1+r))}{1 - (1/(1+r))} \]

\[ = \frac{C}{r} \]

Initial Term = \( C/(1 + r) \)

Common Ratio = \( 1/(1 + r) \)
PV(Perpetuity) = C/(1 + r) + C/(1 + r)^2 + C/(1 + r)^3 + \ldots + C/(1 + r)^t + \ldots

Let \( a = C/(1 + r) \) = initial term
\( x = 1/(1 + r) \) = common ratio
Rewriting:
\[ PV = a (1 + x + x^2 + x^3 + \ldots) \] (1.)
Post multiplying by \( x \):
\[ PVx = a(x + x^2 + x^3 + \ldots) \] (2.)
Subtracting (2.) from (1.):
\[ PV(1 - x) = a \]
\[ \rightarrow PV = a/(1 - x) \]
Multiplying through by \( (1 + r) \):
\[ PV = C/r \]

Example
The *preferred stock* of a secure company will pay the owner of the stock $100/year forever, starting next year.

Q: If the interest rate is 5%, what is the share worth?

A: The share should be worth the value to you as an investor today of the future stream of cash flows.

This share of preferred stock is an example of a perpetuity, such that

\[ \text{PV(preferred stock)} = \frac{100}{0.05} = 2,000 \]

Example Continued

- Q: What if the interest rate is 10%?

\[ \text{PV(preferred stock)} = \frac{100}{0.10} = 1,000 \]

- Notice: That when the interest rate doubled, the present value of the preferred stock decreased by \( \frac{1}{2} \).
The preferred stock of a secure company will pay the owner of the stock $100/year forever, starting this year.

Q: If the interest rate is 5%, what is the share worth?

A: The share should be worth the value to you as an investor today of the future stream of cash flows (perpetuity component) plus the $100 received this year.

\[ PV(\text{preferred stock}) = \$100 + \$100 \times 0.05 = \$100 + \$2,000 = \$2,100 \]

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GROWING PERPETUITY

Suppose the cash flow starts at amount \( C \) at time 1, but grows at a rate of \( g \) thereafter, continuing forever:

\[ C_1 = C, \ C_2 = C(1+g), \ C_3 = C(1+g)^2, \ldots \]

\[ PV(\text{Perpetuity}) = \frac{C}{(1+r)} \frac{(1+g)}{(1+r)} + \frac{(1+g)^2}{(1+r)^2} + \cdots + \frac{(1+g)^t}{(1+r)^t} + \cdots \]

\[ = \sum_{t=1}^{\infty} \frac{(1+g)^t}{(1+r)^t} \]

Based on the infinite sum property, we can write this as:

\[ PV = \frac{\text{Initial Term}}{1 - \text{Common Ratio}} \]

\[ = \frac{C}{(1 + r)} \left[ 1 - ((1 + g)/(1 + r)) \right] \]

\[ = \frac{C}{(r - g)} \]

Note: This formula requires \( r > g \).
Example

• Your next year’s cash flow or parental stipend will be $10,000. Your parents have generously agreed to increase the yearly amount to account for increases in cost of living as indexed by the rate of inflation.

• Your parents have established a trust vehicle such that after their death you will continue to receive this cash flow, so effectively this will continue forever.

• Assume the rate of inflation is 3%.

• Assume the market interest rate is 8%.

• Q: What is the value to you today of this parental support?

Answer

This is a growing perpetuity with

\[ C = 10,000, \quad r = 0.08, \quad g = 0.03 \]

Therefore,

\[ PV = \frac{10,000}{0.08 - 0.03} = 200,000 \]

FINITE ANNUITY

A finite annuity will pay a constant amount \( C \) starting next period through period \( T \), so that there are \( T \) total payments (e.g., financial vehicle that makes finite number of payments based on death of owner or joint death or term certain number of payments, etc.)

\[ C_1 = C, \quad C_2 = C, \quad C_3 = C, \quad \ldots, \quad C_T = C \]

\[
PV(\text{Finite Annuity}) = \sum_{t=1}^{T} \frac{C}{(1+r)^t} = C \sum_{t=1}^{T} \frac{1}{(1+r)^t}
\]
Finite Annuity

Formula Result:

\[ PV \text{ (Finite Annuity)} = C \left( \frac{1}{r} \right) \left[ 1 - \frac{1}{(1+r)^T} \right] \]

where

\[ PVA(r, T) = \left( \frac{1}{r} \right) \left[ 1 - \frac{1}{(1+r)^T} \right] \]

= PV of annuity that pays $1 for T years

Value of Finite Annuity = Difference Between Two Perpetuities

Consider the Finite Annuity cash flow: \( C_1 = C, \ C_2 = C, \ C_3 = C, \ C_4 = C, \ldots \ C_T = C \)
Suppose you want to determine the present value of this future stream of cash.

Recall a perpetuity cash flow (#1):

\( C_1 = C, \ C_2 = C, \ C_3 = C, \ C_4 = C, \ldots C_T = C, C_{T+1} = C, \ldots \)
From our formula, the value today of this perpetuity = \( \frac{C}{r} \)

Consider a second perpetuity (#2) starting at time \( T+1 \):

\( C_{T+1} = C, \ C_{T+2} = C, \ C_{T+3} = C, \ldots \)
The value today of this perpetuity starting at \( T+1 \):

\[ = \frac{C}{r} \left( \frac{1}{1+(1+r)} \right) \] (why?)

Note: The Annuity = Perpetuity #1 – Perpetuity #2

\[ = \frac{C}{r} - \frac{C}{r} \left( \frac{1}{1+(1+r)} \right) \]
\[ = \frac{C}{r} \left( 1 - \frac{1}{1+(1+r)} \right) \]

PV (Finite Annuity) = \( C \left( \frac{1}{1+r} \right) + C \left( \frac{1}{1+r} \right)^2 + C \left( \frac{1}{1+r} \right)^3 + \ldots + C \left( \frac{1}{1+r} \right)^{T-1} \)
Let \( a = \frac{C}{1+r} \)
\( x = \left( \frac{1}{1+r} \right) \)
Rewriting:

\[ PV = a \left( 1 + x + x^2 + x^3 + \ldots + x^{T-1} \right) \] (1.)

Multiplying by \( x \):

\[ PVx = \left( x + x^2 + x^3 + \ldots + x^T \right) \] (2.)

Subtracting (2.) from (1.):

\[ PV(1-x) = a \left( 1-x \right) \]
\[ PV = a \left( \frac{1-x^T}{1-x} \right) \]
\[ \frac{PV}{(1+r)} = a \left( 1-x^T \right) \]
\[ PV = a \left( \frac{1-x^T}{1-x} \right) \]
\[ PV(1+r) = \frac{a}{1-x} \]

Multiplying the \( 1+r \) in the denominator thru:

\[ PV \text{ (Finite Annuity)} = C \left( \frac{1}{1+(1+r)} \right) \]
Example
Find the value of a 5 year car loan with annual payments of $3,600 per year starting next year (i.e., 5 payments of $3,600 in the future). The cost of capital or opportunity cost of capital is 6%.

\[ PV = \frac{3,600}{0.06}[1 - \frac{1}{(1.06)^5}] \]
\[ = 15,164.51 \]

Example Continued
Suppose you had also made a down-payment for the car of $5,000 to lower your monthly loan payments. The total cost/value of the car you purchased is then:

\[ PV(\text{down payment}) + PV(\text{loan annuity}) = 5,000 + 15,164.51 = 20,164.51 \]

Computing Present Value of Finite Annuities in Excel
Excel function PV:
\[ PV(\text{Rate}, \text{Nper}, \text{Pmt}, \text{Fv}, \text{Type}) \]
- Rate = per period interest rate
- Nper = number of annuity payments
- Fv = cash balance after last payment
- Type = 1 if payments start in first period; 0 if payments start in initial period
Example

• Borrow $200,000 to buy a house.
• Annual interest rate = 10%
• Loan is to be paid back in 30 years
• Q: What is the annual payment?
• PV = $200,000 = C*PVA(0.10, 30)
• =>$C = \frac{200,000}{PVA(0.10, 30)}$
• PVA(0.10, 30) = \left(\frac{1}{0.10}\right)[1 - \frac{1}{(1.10)^{30}}] = 9.427
• =>$C = \frac{200,000}{9.427} = 21,215.85$

Computing Payments from Finite Annuities in Excel

Excel function PMT:
PMT(Rate, Nper, Pv, Fv, Type)
Rate = per period interest rate
Nper = number of annuity payments
Pv = initial present value of annuity
Fv = future value after last payment
Type = 1 if payments are due at the beginning of the period; 0 if payments are due at the end of the period

Example

• You win the $5 million lottery!
• 25 annual installments of $200,000 starting next year
• Q: What is the PV of winnings if r = 10%?
• PV = $200,000 * PVA(0.10, 25)
• PVA = \left(\frac{1}{0.10}\right)[1 - \frac{1}{(1.10)^{25}}] = 9.07704
• =>$PV= \frac{200,000 \times 9.07704}{9.07704} = 1,815,408 < $5M!$
Future Value of an Annuity

- Invest $C$ every year, starting next year, for $T$ years at a fixed rate $r$
- How much will investment be worth in year $T$?
- Trick: $FVA(r, T) = PVA(r, T)(1+r)^T$
  - $= \frac{1}{r} \left[ 1 - \frac{1}{(1+r)^T} \right] \times (1+r)^T$
- Therefore
  - $FV = C \times FVA(r, T)$
- where $FVA(r, T)$ = FV of $1$ invested every year for $T$ years at rate $r$

Example

- Save $1,000 per year, starting next year, for 35 years in IRA
- Annual rate = 7%
- Q: How much will you have saved in 35 years?
- $FV = 1,000 \times FVA(0.07, 35)$
- $FVA(0.07, 35) = \frac{1}{0.07} \times \left[ (1.07)^{35} - 1 \right] = 138.23688$
- $\Rightarrow FV = 1,000 \times (138.23688) = 138,236.88$

Computing Future Value of Finite Annuities in Excel

Excel function $FV$:

$FV(\text{Rate, Nper, Pmt, Pv, Type})$

- Rate = per period interest rate
- Nper = number of annuity payments
- Pmt = payment made each period
- Pv = present value of future payments
- Type = 1 if payments start in first period; 0 if payments start in initial period
Finite Growing Annuities

• Similar to how we amended the Perpetuity formula for ‘Growing’ Perpetuities, we can amend the Annuity formula to account for a ‘Growing’ Annuity.

• The cash flow for a finite growing annuity pays an amount C, starting next period, with the cash flow growing thereafter at a rate of g, through period T:

\[PV = \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \ldots + \frac{C(1+g)^{T-1}}{(1+r)^T}\]

\[= \sum_{t=1}^{T} \frac{C(1+g)^{t-1}}{(1+r)^t}\]

\[= \frac{C}{r-g} \left[1 - \frac{(1+g)^T}{(1+r)^T}\right]\]

Class Example

• An asset generates a cash flow that is $1 next year, but is expected to grow at 5% per year indefinitely.
• Suppose the relevant discount rate is 7%.

Q: After receiving the third payment, what can you expect to sell the asset for?

Q: What is the present value of the asset you held?

Compounding Frequency

• Cash flows can occur annually (once per annum), semi-annually (twice per annum), quarterly (four times per annum), monthly (twelve times per annum), daily (365 times per annum), etc.

• Based on the cash flows, the formulas for compounding and discounting can be adjusted accordingly:

**General formula:** For stated annual interest rate \(r\) compounded for \(T\) years \(n\) times per year:

\[FV = V_0 \times [1 + \frac{r}{n}]^{nT}\]
### Compounding Frequency

**Effective Annual Rate** (annual rate that gives the same FV with compounding $n$ times per year):

\[
\left[1 + \frac{r_{\text{EAR}}}{n}\right]^n = \left[1 + \frac{r}{n}\right]^{nT}
\]

\[
=> r_{\text{EAR}} = \left[1 + \frac{r}{n}\right]^n - 1
\]

### Example

- Invest $1,000 for 1 year
- Annual rate (APR) $r = 10\%$
- Semi-annual compounding: semi-annual rate = $0.10/2 = 0.05$
- FV = $1,000 \times (1 + 0.05)^2 = $1,102.50$
- Note: $1,000 \times (1 + 0.05)^2 = 1,000 \times (1 + 2 \times (0.05) + (0.05)^2)$
- $= $1,000 + $100 + $2.5$
- $= \text{principal} + \text{simple interest} + \text{interest on interest}$
- Effective annual rate:
  - $(1 + r_{\text{EAR}}) = (1 + \text{APR}/2)^2$
  - $=> r_{\text{EAR}} = (1.05)^2 - 1 = 0.1025 \ or \ 10.25\%$

### Example: The Difference In Compounding

<table>
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<th>Compounding Frequency</th>
<th>Times per Annum</th>
<th>One plus Effective Rate</th>
</tr>
</thead>
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<tr>
<td>Yearly</td>
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<tr>
<td>By the second</td>
<td>31,536,000</td>
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</tr>
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</table>
Example
• Take out (borrow) $300,000 30 year fixed rate mortgage
• Annual rate = 8%, monthly rate = 0.08/12 = 0.0067
• 30*12 = 360 monthly payments
• Q: What is the monthly payment?

\[ \text{PV} = \frac{300,000}{C \cdot \text{PVA}(0.08/12, 360)} \]
\[ \text{PVA}(0.0067, 360) = 136.283 \]
\[ \Rightarrow C = \frac{300,000}{136.283} = \$2,201.30 \]
\[ \text{Note: total amount paid over 30 years is} \]
\[ 360 \times \$2,201.30 = \$792,468 \]

Example
• Consider previous 30 year mortgage
• Suppose the day after the mortgage is issued, the annual rate on new mortgages shoots up to 15%
• Q: How much is the old mortgage worth?

\[ \text{PV} = \frac{2,201}{C \cdot \text{PVA}(0.15/12, 360)} \]
\[ \text{PVA}(0.15/12, 360) = 79.086 \]
\[ \Rightarrow \text{PV} = \frac{2,201 \times 79.086}{79.086} = \$174,092 < \$300,000! \]

Continuous Compounding
Increasing the frequency of compounding to continuously:
\[ \lim_{n \to \infty} [1 + \frac{r}{n}]^n = (2.718)^r = e^r \]

Effective Annual Rate:
\[ [1 + r_{\text{EAR}}]^r = e^r \]
\[ \Rightarrow r_{\text{EAR}} = e^r - 1 \]
Example

• $r =$ annual (simple) interest rate $= 10\%$, $T =$ 1 year
• FV of 1$ with annual compounding:
  • $FV = 1(1+r) = 1.10$
• FV of 1$ with continuous compounding:
  • $FV = 1* e^r = 2.71818^{0.10} = 1.10517$
• Effective annual rate
  • $1 + \text{r}_{\text{EAR}} = 1.10517 \Rightarrow \text{r}_{\text{EAR}} = 0.10517 = 10.517\%$

Further Insight on Continuous Compounding

Example: Invest $V_0$ for 1 year with annual rate $r$ and continuous compounding

$$V_1 = V_0 e^{rt} \Rightarrow \left(\frac{V_1}{V_0}\right) = e^r$$

$$\Rightarrow \ln \left(\frac{V_1}{V_0}\right) = r$$

$$\Rightarrow \ln V_1 - \ln V_0 = r$$

Test/Practical Tips

• General formula will always work by may be tedious
• Short-cuts exist if you can recognize them
• Use short-cuts!
• Break down complicated problems into simple pieces