Market Risk Modeling in S-PLUS

Eric Zivot

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Abstract

This document summarizes the steps for calculating Value-at-Risk (VaR) for a portfolio of equity assets using S-PLUS 7.0 and S+FinMetrics 2.0. Unconditional VaR is computed using empirical quantiles, the normal distribution, and an extreme value theory (EVT) Pareto tail distribution. Conditional VaR is computed using GARCH models, GARCH+EVT. The VaR models are evaluated by examining VaR violations in a backtesting environment.

1 Basic Concepts

This section reviews some basic concepts of asset returns and portfolios, and defines the market risk concepts value-at-risk (VaR) and expected tail loss (ETL) (which is also called expected shortfall (ES)).

1.1 Asset Returns

The portfolio consists of $i = 1, \ldots, N$ equity assets. Let $P_{it}$ denote the price of asset $i$ at time $t$. The one-period simple return on asset $i$ between times $t-1$ and $t$ is

$$R_t = \frac{P_{it} - P_{it-1}}{P_{it-1}}$$

The one-period continuously compounded (log) return is

$$r_{it} = \ln(1 + R_t) = \ln \left( \frac{P_{it}}{P_{it-1}} \right)$$

Note that

$$R_{it} = e^{r_{it}} - 1$$

Log returns are often preferred to simple returns for statistical modeling purposes.
1.2 Portfolios

Let \( R_t = (R_{1t}, \ldots, R_{Nt})' \) and \( r_t = (r_{1t}, \ldots, r_{Nt})' \). A portfolio of \( N \) assets is characterized by a \( N \times 1 \) vector of portfolio weights \( w = (w_1, \ldots, w_N)' \) where \( w_i \) denote the share of wealth invested in asset \( i \) such that \( \sum_{i=1}^N w_i = 1 \). The one-period simple and log returns on the portfolio \( w \) are

\[
R_{w,t} = x'R_t = \sum_{i=1}^N w_i R_{it} \\
r_{w,t} = \ln(1 + R_{w,t}) \\
= \ln(1 + w'R_t) \approx w'r_t = \sum_{i=1}^N w_i r_{it}
\]

Note that the log portfolio return is not exactly equal to the weighted average of the individual asset log returns.

1.3 Value-at-Risk Defined

Consider a one period investment in an asset with simple return \( R \). Let \( W_0 \) denote the initial dollar amount invested. The value of the investment after one period in terms of the simple return is

\[
W_1 = W_0 (1 + R)
\]

and the value of the investment in terms of the log return is

\[
W_1 = W_0 e^r
\]

1.3.1 VaR Based on Simple Returns

For \( \alpha \in (0,1) \), let \( q^R_\alpha \) denote the \( \alpha \times 100\% \) quantile of the probability distribution of the simple return \( R \). Usually, \( q^R_\alpha \) is a low quantile such that \( \alpha = 0.01 \) or \( \alpha = 0.05 \). As a result, \( q^R_\alpha \) is typically a negative number. The \( \alpha \times 100\% \) dollar Value-at-Risk (\( VaR_\alpha \)) is

\[
VaR_\alpha = -W_0 \cdot q^R_\alpha
\]

In words, \( VaR_\alpha \) represents the dollar loss that could occur with probability \( \alpha \). By convention, it is reported as a positive number (hence the minus sign). The VaR as a percentage of the initial portfolio value is simply the (negative) low quantile of the simple return distribution:

\[
VaR_\alpha = \frac{VaR_\alpha}{W_0} = -q^R_\alpha
\]
1.3.2 VaR Based on Log Returns

Let $q^*_r$ denote the $\alpha \times 100\%$ quantile of distribution of the log return $r = \ln(1 + R)$. The simple return quantile $q^R_\alpha$ is related to the log return quantile using the relationship

$$q^R_\alpha = e^{q^*_r} - 1$$

Therefore, $\$VaR_\alpha$ based on log-returns may be computed using

$$\$VaR_\alpha = -\$W_0 \cdot (e^{q^*_r} - 1)$$

and

$$VaR_\alpha = \frac{\$VaR_\alpha}{\$W_0} = -(e^{q^*_r} - 1)$$

1.4 Expected Tail Loss Defined

The $\alpha \times 100\%$ expected tail loss (ETL), in terms of the log return, is defined as

$$ETL_\alpha = -E[r | r < -VaR_\alpha]$$

In words, the ETL is the expected (negative) return conditional on the return being less than the $\alpha \cdot 100\%$ percentage VaR. If the initial investment is $\$W_0$, then the dollar ETL is

$$\$ETL = \$W_0 \times ETL_\alpha$$

2 Example Data

VaR and ETL calculations are illustrated using the daily log-returns on the 30 Dow Jones Industrial Average stocks over the period January 2, 1991 through January 2, 2001. The adjusted closing prices are in the S+FinMetrics “timeSeries” DowJones30. Log returns may be calculated using

> DowJones30.ret = getReturns(DowJones30)

These returns are illustrated in Figure 1.

Summary statistics on these returns are listed below

> summaryStats(DowJones30.ret)

Sample Moments:

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>std</th>
<th>skewness</th>
<th>kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA</td>
<td>0.00066613</td>
<td>0.02027</td>
<td>0.47417262</td>
<td>6.256</td>
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<td>-0.31983600</td>
<td>10.532</td>
</tr>
</tbody>
</table>
Figure 1: Daily log returns on Dow Jones 30 stocks.

<table>
<thead>
<tr>
<th>Company</th>
<th>Mean Log Return</th>
<th>Median Log Return</th>
<th>Min Log Return</th>
<th>Max Log Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAT</td>
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<tr>
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<td>0.00130133</td>
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<td>------</td>
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<td>----------</td>
<td>----------</td>
</tr>
<tr>
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<tr>
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</table>

The skewness and kurtosis values indicate non-normal distributions for the log-returns. Testing the null hypothesis that each return series is normally distributed using the Jarque-Bera statistic gives

```r
> normalTest(DowJones30.ret, method="jb")
```

**Test for Normality: Jarque-Bera**

**Null Hypothesis:** data is normally distributed

**Test Statistics:**

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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
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</tr>
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<td></td>
<td></td>
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</tr>
</tbody>
</table>

5
An equally weighted portfolio of the Dow Jones 30 stocks may be computed using

```r
> port.ret.ts = rowSums(DowJones30.ret, weights=rep(1,30)/30)
```

The daily log-returns are illustrated in Figure 2.

The summary statistics for the portfolio are

```r
> summaryStats(port.ret.ts)
```

**Sample Quantiles:**

```
min  1Q median  3Q    max
-0.0738 -0.00441  0.0007142  0.005973  0.05001
```

**Sample Moments:**
### Mean, Standard Deviation, Skewness, Kurtosis

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>0.0006889</td>
<td>std</td>
<td>0.009707</td>
<td>skewness</td>
</tr>
<tr>
<td>kurtosis</td>
<td>7.663</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Number of Observations: 2527

and the Jarque-Bera test for normality is

```r
> normalTest(port.ret.ts, method="jb")
```

Test for Normality: Jarque-Bera

**Null Hypothesis:** data is normally distributed

**Test Statistics:**
- **Portfolio**
  - Test Stat: 2330.622
  - p.value: 0.000

**Dist. under Null:** chi-square with 2 degrees of freedom
- **Total Observ.:** 2527

## 3 Unconditional Models for VaR and ETL

This section reviews common models for computing unconditional estimates of VaR and ETL. In unconditional models, the multivariate return distribution is assumed to be covariance stationary and ergodic and, hence, have time invariant moments. Usually, the moments of the return distribution are estimated from historical data and are assumed to be fixed over the period in which estimates of VaR and ETL are required. The models considered are historical simulation, normal approximation and extreme value theory.

### 3.1 Historical Simulation

Historical simulation (HS) simply refers to the empirical distribution of the observed returns. As a result, the \( \alpha \times 100\% \) VaR based on HS is just the \( \alpha \times 100\% \) empirical quantile of the return distribution. Similarly, the HS estimate of ETL\( \alpha \) is simply the average of the returns below the HS VaR\( \alpha \) estimate.

**Example 1 \( VaR_{0.01} \) for Dow Jones 30 stocks based on historical simulation**

In S-PLUS, empirical quantiles may be computed using the `quantile` or `colQuantiles` function. To compute \( VaR_{0.01} \) for the equally weighted portfolio of Dow Jones 30 stocks based on HS use...
VaR.hs.01 = quantile(port.ret.ts, probs=0.01)
VaR.hs.01
1%
-0.0247706

With 1% probability the loss is about 2.5% or higher. If there is $1M initially invested in the portfolio, the 1% dollar VaR is

1000000*(exp(VaR.hs.01) - 1)
1%
-24466.33

To compute the VaR.01 for all of the Dow Jones 30 stocks based on HS use

unlist(colQuantiles(DowJones30.ret, probs=0.01))

AA.1% AXP.1% T.1% BA.1% CAT.1% C.1%
-0.04692524 -0.05137398 -0.05050954 -0.04699595 -0.04966563 -0.05252321
KO.1% DD.1% EK.1% XOM.1% GE.1% GM.1%
-0.04102243 -0.04533577 -0.0449885 -0.03381674 -0.0406354 -0.04914696
HWP.1% HD.1% HON.1% INTC.1% IBM.1% IP.1%
-0.06547625 -0.05123644 -0.0545059 -0.07176775 -0.05232081 -0.04953661
JPM.1% JNJ.1% MCD.1% MRK.1% MSFT.1% MMM.1%
-0.05121868 -0.03776024 -0.03966271 -0.04359857 -0.05891276 -0.04029453
MO.1% PG.1% SBC.1% UTX.1% WMT.1% DIS.1%
-0.05290309 -0.03873091 -0.04528876 -0.04237849 -0.05424686 -0.043480624

The bootstrap may be used assess the sampling uncertainty associated with the empirical quantile. The S-PLUS function bootstrap may be used for this purpose. Alternatively, the bootstrap function from the library S+Resample may be used\(^1\). For example, to compute a bootstrap standard error for the HS estimate of VaR.05 for the first Dow Jones 30 stock use

VaR.05.boot = bootstrap(port.ret.ts, statistic=quantile, +
arg.stat=list(probs=0.05))

summary(VaR.05.boot)

Call:
bootstrap(data = port.ret.ts, statistic = quantile, arg.stat = list(probs = 0.05))

\(^1\)The library S+Resample may be downloaded from the Insightful research page. This library greatly extends the boostrapping and resampling functionality in S-PLUS.
Number of Replications: 1000

Summary Statistics:

<table>
<thead>
<tr>
<th>Observed Bias Mean SE</th>
</tr>
</thead>
<tbody>
<tr>
<td>5% -0.01466 -0.00009937 -0.01476 0.0005054</td>
</tr>
</tbody>
</table>

Empirical Percentiles:

<table>
<thead>
<tr>
<th>2.5% 5% 95% 97.5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>5% -0.01588 -0.01561 -0.01403 -0.01388</td>
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</tbody>
</table>

BCa Confidence Limits:

<table>
<thead>
<tr>
<th>2.5% 5% 95% 97.5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>5% -0.01588 -0.01561 -0.01403 -0.01388</td>
</tr>
</tbody>
</table>

The bootstrap standard error estimate is 0.0005 and a 95% bootstrap confidence interval is [-0.0159, -0.0139]. The bootstrap distribution may be visualized using

> plot(VaR.05.boot)

Example 2 ETL₀₁ for Dow Jones 30 stocks based on historical simulation

The HS estimate of ETL₀₁ is simply the average of the returns below the HS VaR₀₁ estimate. For the equally weighted portfolio, the HS estimate of ETL₀₁ is

> mean(port.ret.ts[port.ret.ts < VaR.hs.01])
[1] -0.03496077

The average loss in portfolio value when the return is less than the 1% VaR is about 3.5%.

To compute the HS estimate of ETL₀₁ for all of the Dow Jones 30 stocks use

> ETL.hs.01 = function(x) { mean(x[x < quantile(x, probs=0.01)]) }
> apply(DowJones30.ret, 2, ETL.hs.01)

<table>
<thead>
<tr>
<th></th>
<th>AA</th>
<th>AXP</th>
<th>T</th>
<th>BA</th>
<th>CAT</th>
<th>C</th>
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<tbody>
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<table>
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<th>IP</th>
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<table>
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<th>MCD</th>
<th>MRK</th>
<th>MSFT</th>
<th>MMM</th>
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</table>
3.2 Normal Distribution

Assume the $N \times 1$ vector of log-returns $r$ has a multivariate normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$

$$r \sim N(\mu, \Sigma)$$

where $\mu$ has elements $\mu_i$ ($i = 1, \ldots, N$) and $\Sigma$ has elements $\sigma_{ij}$ ($i, j = 1, \ldots, N$). For an individual asset,

$$r_i \sim N(\mu_i, \sigma_{ii})$$

The $\alpha \times 100\%$ quantile of the normal distribution for $r_i$ is

$$q_i^\alpha = \mu_i + \sigma_i q_\alpha^z$$

where $q_\alpha^z$ is the $\alpha \times 100\%$ quantile of the standard normal distribution. The distribution of $r_i$ given that $r_i \leq q_i^\alpha$ is truncated normal. The mean of this distribution is the normal ETL $\alpha$. Greene (2004) shows that

$$\mathbb{E}[r_i | r_i \leq q_i^\alpha] = \mu_i + \sigma_i \times \frac{\phi(z_i^\alpha)}{\Phi(z_i^\alpha)}$$

where $z_i^\alpha = (\mu_i - \text{VaR}_\alpha) / \sigma_i$, $\phi(z)$ is the standard normal pdf and $\Phi(z)$ is the standard normal CDF.

Given a random sample of size $T$ of observed returns on $N$ assets from the multivariate normal distribution, the mean vector $\mu$ and covariance matrix $\Sigma$ may be estimated using the sample statistics

$$\hat{\mu} = T^{-1} \sum_{t=1}^T r_t, \quad \hat{\Sigma} = T^{-1} \sum_{t=1}^T (r_t - \hat{\mu})(r_t - \mu)'$$

The normal quantile may then be estimated using the plug-in method

$$\hat{q}_i^\alpha = \hat{\mu}_i + \hat{\sigma}_i \hat{q}_\alpha^z$$

where $\hat{\mu}_i$ is the $i$th element of $\hat{\mu}$, and $\hat{\sigma}_i$ is the square root of the $i$th diagonal element of $\hat{\Sigma}$. Similarly, the estimate of normal ETL $\alpha$ is

$$\hat{E}[r_i | r_i \leq q_i^\alpha] = \hat{\mu}_i + \hat{\sigma}_i \times \frac{\phi(z_i^\alpha)}{\Phi(z_i^\alpha)}$$

where $z_i^\alpha = (\hat{\mu}_i - \text{VaR}_\alpha) / \hat{\sigma}_i$, and $\text{VaR}_\alpha = \hat{\mu}_i + \hat{\sigma}_i \hat{q}_\alpha^z$. Standard errors for these estimates may be conveniently computed using the bootstrap.

---

3.2.1 Active versus Passive Portfolio Risk Measurement

In a passive portfolio, the portfolio weight vector $w$ stays fixed over time. The portfolio return is normally distributed

$$r_w = w' r \sim N(\mu_w, \sigma^2_w)$$

with time invariant mean and variance

$$\mu_w = w' \mu, \sigma^2_w = w' \Sigma w$$

The $\alpha \times 100\%$ quantile of the normal distribution for $r_w$ is

$$q^w_\alpha = \mu_w + \sigma_w q^z_\alpha$$

and the $\alpha \times 100\%$ ETL is

$$E[r_w | r_w \le q^w_\alpha] = \mu_w + \sigma_w \times \frac{\phi(z^w_\alpha)}{\Phi(z^w_\alpha)}$$

In a passive portfolio, the portfolio mean and variance may be estimated from observed returns by first computing the portfolio return $r_{w,t} = w' r_t$, and then computing the sample mean and variance from the portfolio returns

$$\hat{\mu}_w = T^{-1} \sum_{t=1}^{T} w' r_t, \quad \hat{\sigma}^2_w = T^{-1} \sum_{t=1}^{T} (r_{w,t} - \hat{\mu}_w)^2$$

In an active portfolio, the weights on the individual assets may change over time as the portfolio manager rebalances the portfolio. Let $w_t$ denote the $N \times 1$ vector of portfolio weights at time $t$. Then the portfolio return is normally distributed

$$r_{w,t} = w'_t r \sim N(\mu_{w,t}, \sigma^2_{w,t})$$

with period specific mean and variance

$$\mu_{w,t} = w'_t \mu, \sigma^2_{w,t} = w'_t \Sigma w_t$$

In an active portfolio, the portfolio mean and variance at time $t$ are usually estimated with

$$\hat{\mu}_{w,t} = w'_t \hat{\mu}, \quad \hat{\sigma}^2_{w,t} = w'_t \hat{\Sigma} w_t$$

3.2.2 S-PLUS Functions

Simple S-PLUS functions for computing VaR and ETL based on the normal distribution are
norm.quantile = function(x, p=0.01, mu=NULL) {
    ## required arguments:
    ## x    timeSeries of returns (simple or continuous)
    ## optional arguments:
    ## p    scalar probability level
    ## mu   vector of mean values.
    ## value:
    ## numeric value giving quantile estimate from estimated normal
    ## distribution
    if(is.null(mu))
        q = colMeans(x) + colStdevs(x)*qnorm(p)
    else
        q = mu + colStdevs(x)*qnorm(p)
    q
}

Example 3  \( \text{VaR}_{0.01} \) for Dow Jones 30 stocks based on normal distribution

The normal estimate of \( \text{VaR}_{0.01} \) for the (passive) equally weighted portfolio is

\[
> \text{norm.quantile(port.ret.ts)}
\]
\[
\text{Portfolio} \quad -0.02189391
\]

A 95% confidence interval for \( \text{VaR}_{0.01} \) based on the bootstrap may be computed using

\[
> \text{VaR.01.boot = bootstrap(port.ret.ts, statistic=norm.quantile)}
\]

\[
> \text{limits.emp(VaR.01.boot)}
\]
\[
\begin{array}{cccc}
    2.5% & 5% & 95% & 97.5% \\
    \text{Portfolio} & -0.02320047 & -0.02299066 & -0.02088363 & -0.0207183
\end{array}
\]

For an active portfolio, the 1% quantile may be estimated using

\[
> \text{mu.hat = colMeans(DowJones30.ret)}
\]
\[
> \text{Sigma.hat = var(DowJones30.ret)}
\]
\[
> w = \text{rep}(1,30)/30
\]
\[
> t(w) \times \text{mu.hat} + \sqrt{(t(w) \times \text{Sigma.hat} \times w)} \times \text{qnorm}(0.01)
\]
\[
[1,] \quad -0.02189391
\]

The normal estimates of \( \text{VaR}_{0.01} \) for all of the Dow Jones 30 stocks are

\[
> \text{norm.quantile(DowJones30.ret)}
\]
\[
\begin{array}{lcccccc}
    \text{AA} & \text{AXP} & \text{T} & \text{BA} & \text{CAT} & \text{C} \\
    -0.04649377 & -0.04803599 & -0.04681135 & -0.04423398 & -0.04728698 & -0.05053085
\end{array}
\]
3.3 Extreme Value Theory

Extreme value theory (EVT) is concerned with modeling the tails of a probability distribution $F$. To illustrate, let $X_1, X_2, \ldots$ be a sequence of iid random variables representing risks or losses with an unknown CDF $F$. A natural measure of extreme events are values of the $X_i$ that exceed a high threshold $u$. Define the excess distribution above the threshold $u$ as the conditional probability:

$$F_u(y) = \Pr\{X - u \leq y | X > u\} = \frac{F(y + u) - F(u)}{1 - F(u)}, \quad y > 0$$

(1)

it can be shown that for large enough $u$ there exists a positive function $\beta(u)$ such that the excess distribution (1) is well approximated by the generalized Pareto distribution (GPD)

$$G_{\xi,\beta(u)}(y) = \begin{cases} 
1 - \left(1 + (\xi y/\beta(u))^{-1/\xi}\right) & \text{for } \xi \neq 0 \\
1 - \exp(-y/\beta(u)) & \text{for } \xi = 0
\end{cases}, \quad \beta(u) > 0$$

(2)

defined for $y \geq 0$ when $\xi \geq 0$ and $0 \leq y \leq -\beta(u)/\xi$ when $\xi < 0$. The parameter $\xi$ determines the tail shape of the distribution of exceedences over the threshold $u$. If $\xi < 0$ then $F$ is in the thin-tailed Weibull family and $G_{\xi,\beta(u)}$ is a Pareto type II distribution; if $\xi = 0$ then $F$ is in the Gumbell family and $G_{\xi,\beta(u)}$ is an exponential distribution; and if $\xi > 0$ then $F$ is in the fat-tailed Fréchet family and $G_{\xi,\beta(u)}$ is a Pareto distribution. For $\xi > 0$, the most relevant case for risk management purposes, it can be shown that $E[X^k] = \infty$ for $k \geq \alpha = 1/\xi$. For example, if $\xi = 0.5$ then $E[X^2] = \infty$ and the distribution of losses, $X$, does not have finite variance. If $\xi = 1$ then $E[X] = \infty$. 

Example 4 $ETL_{01}$ for Dow Jones 30 stocks based on the normal distribution
The threshold parameter $u = u_0$ may be determined by computing the empirical mean excess function

$$e_n(u) = \frac{1}{n_u} \sum_{i=1}^{n_u} (x(i) - u)$$

(3)

where $x(i) (i = 1, \ldots, n_u)$ are the values of $x_i$ such that $x_i > u$, and then plotting $e_n(u)$ against $u$. The plot should be linear in $u$ for $u > u_0$. An upward sloping plot indicates heavy-tailed behavior. In particular, a straight line with positive slope above $u_0$ is a sign of Pareto behavior in tail. A downward trend shows thin-tailed behavior, whereas a line with zero slope shows an exponential tail.

Once the threshold parameter $u = u_0$ is chosen, the remaining parameters $\xi$ and $\beta(u)$ may be estimated by maximum likelihood. Given the ML estimates $\hat{\xi}$ and $\hat{\beta}(u)$, the tails of the loss distribution $F$ may be estimated using

$$\hat{F}(x) = 1 - \frac{k}{n} \left( 1 + \frac{x - u}{\hat{\beta}(u)} \right)^{-\hat{\xi}}$$

(4)

where $k$ denotes the number of exceedences over the threshold $u$. Then, an estimate of VaR sub $\alpha$ based on inverting $\hat{F}(x)$ is

$$\hat{VaR}_\alpha = u + \frac{\hat{\beta}(u)}{\hat{\xi}} \left( \frac{n_k (1 - q)}{k} \right)^{-\hat{\xi}} - 1$$

(5)

An estimate of ETL sub $\alpha$ is

$$\hat{ETL}_\alpha = \frac{\hat{VaR}_\alpha}{1 - \hat{\xi}} + \frac{\hat{\beta}(u) - \hat{\xi} u}{1 - \hat{\xi}}$$

(6)

Remarks:

1. EVT methods are easily applied to passive portfolios.
2. EVT methods are difficult to apply to active portfolios. This usually requires building an EVT model for the full multivariate distribution of returns. The most promising methodology for doing this is based on copulas.

**Example 5** VaR sub $0.01$ and ETL sub $0.01$ for equally weighted portfolio based on EVT

Consider estimating VaR sub $0.01$ and ETL sub $0.01$ based on EVT for the equally weighted portfolio of Dow Jones 30 stocks. The first step is to determine the threshold parameter $u$ using empirical mean excess function. This may be computed using the S+FinMetrics function `meplot`

> tmp = meplot(-port.ret.ts)
which produces the graph in the top panel of Figure 3. The negative returns are used as an input to \texttt{meplot} to investigate the low tail of the distribution. The plot becomes linear with positive slope near $u = 0.015$ suggesting fat tailed behavior in the lower tail. To better pin down the threshold, the S+FinMetrics \texttt{shape.plot} may be used to see how the MLEs of the GPD shape parameter varies with the selected threshold $u$. For example

\begin{verbatim}
> shape.plot(-port.ret.ts, from=0.9, to=0.98)
\end{verbatim}

computes the MLEs of $\xi$ using threshold values from the 90\textit{th} quantile through the 98\textit{th} quantile. These values are illustrated in the lower panel of Figure 3. The two plots suggest a threshold value around $u = 0.015$.

Given the threshold $u = 0.015$, the MLEs of the GPD parameters $\xi$ and $\beta(0.015)$ may be computed using the S+FinMetrics function \texttt{gpd}

\begin{verbatim}
> gpd.fit = gpd(-port.ret.ts, threshold=0.015)
> gpd.fit
Generalized Pareto Distribution Fit --
\end{verbatim}
Total of 2527 observations

Upper Tail Estimated with ml --
Upper Threshold at 0.015 or 4.788 % of the data
ML estimation converged.
Log-likelihood value: 480.9

Parameter Estimates, Standard Errors and t-ratios:

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
<th>Std.Error</th>
<th>t value</th>
</tr>
</thead>
<tbody>
<tr>
<td>xi</td>
<td>0.2109</td>
<td>0.1066</td>
<td>1.9792</td>
</tr>
<tr>
<td>beta</td>
<td>0.0056</td>
<td>0.0008</td>
<td>7.2066</td>
</tr>
</tbody>
</table>

The MLE for $\xi$ is 0.211, with an estimated standard error of 0.107, indicating fat tailed behavior. Since $\hat{\alpha} = 1/\hat{\xi} = 4.74$ the fitted GPD suggests that the equally portfolio return distribution has about four moments finite. The fit of the GPD to the tails of negative returns may be visualized using the S+FinMetrics function tailplot

```r
> tailplot(gpd.fit)
```

which is illustrated in Figure 4. The GPD appears to be a good fit.

From the fitted GPD, estimates of VaR_{0.01} and ETL_{0.01} may be computed using the S+FinMetrics function riskmeasures

```r
> riskmeasures(gpd.fit, p=0.99)

      p quantile  sfall
beta 0.99 0.02539038 0.03526328
```

The estimate of VaR_{0.01} is $-0.0254$ and the estimate of ETL_{0.01} is $-0.0353$. These values are quite close to the HS estimates. 95% confidence intervals for these estimates may be computed using the S+FinMetrics functions gpd.q and gpd.sfall

```r
> gpd.q(pp=0.99, ci.p = 0.95, plot=F)
Lower CI  Estimate  Upper CI
0.023545692 0.02539038 0.02792725
> gpd.sfall(pp=0.99, ci.p = 0.95, plot=F)
Lower CI  Estimate  Upper CI
0.03084265 0.03526328 0.04554279
```

4 Conditional Models for VaR and ETL

In conditional models, it is recognized that the moments of the multivariate return distribution may change over time
Figure 4: Fit of GDP to the lower tail of the equally weighted portfolio return distribution.
4.1 EWMA Estimates

JP Morgan’s RiskMetrics system for market risk management utilizes the following exponentially weighted moving average (EWMA) model for time-varying return variances

\[
\sigma^2_{t+1} = (1 - \lambda) \sum_{\tau=1}^{\infty} \lambda^{\tau-1} r_{t+1-\tau}, \quad 0 < \lambda < 1
\]

which may be re-written as

\[
\sigma^2_{t+1} = \lambda \sigma^2_t + (1 - \lambda) r^2_t
\]

In the RiskMetrics system \( \lambda \) is set at 0.94 for forecasting daily variances.

4.2 GARCH Models

to be completed

4.3 GARCH+EVT

to be completed

5 Backtesting VaR

to be completed

References


