GOVERNING EQUATIONS

I. REVIEW OF FLUID MECHANICS

- Shell balances can be performed on any property of interest (e.g. mass, momentum and energy)
- Consider a differential volume in cartesian coordinates and perform shell balances.

1) Shell balance on mass:

\[ [\text{Rate of flow}] \quad - \quad [\text{Rate of flow}] \quad + \quad [\text{Rate of mass in}] \quad \text{of mass out}] \quad \text{generation] } \quad = \quad [\text{Rate of mass}] \quad \text{accumulation}] \]

Assuming that there is no accumulation (steady state) or generation:

\[ \Delta y\Delta z(\rho.V_x)_{x} - \Delta y\Delta z(\rho.V_x)_{x+\Delta x} + \Delta x\Delta z(\rho.V_y)_{y} - \Delta x\Delta z(\rho.V_y)_{y+\Delta y} + \Delta x\Delta y(\rho.V_z)_{z} - \Delta x\Delta y(\rho.V_z)_{z+\Delta z} = \Delta x\Delta y\Delta z. \frac{\partial \rho}{\partial t} \]

Dividing by the differential volume and taking the derivative as \( \Delta x, \Delta y, \Delta z \rightarrow 0 \) yields the continuity equation:

\[ \rho \nabla \cdot \nabla + \nabla \cdot \nabla \rho + \frac{\partial \rho}{\partial t} = 0 \]

If the fluid properties are constant, i.e. \( \rho \) is constant, then:

\[ \nabla \cdot \vec{V} = 0 \]

Or in cartesian coordinates:

\[ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0 \] (steady state and constant properties)

2) Shell balance on momentum in \( x \) direction:

\[ [\text{Rate of x-momentum}] \quad - \quad [\text{Rate of x-momentum}] \quad + \quad [\text{Rate of x-momentum}] \quad \text{in by diffusion}] \quad \text{out by diffusion}] \quad \text{in by convection}] \quad \text{out by convection}] \]

\[ = \quad [\text{Sum of x-forces}] \quad + \quad \text{Accumulation}] \]

Assuming no accumulation or generation and changing the signs:

\[ \Delta y\Delta z\{\tau_{xx} l_{x+\Delta x} - \tau_{xx} l_{x}\} + \Delta x\Delta z\{\tau_{yx} l_{y+\Delta y} - \tau_{yx} l_{y}\} + \Delta x\Delta y\{\tau_{yz} l_{z+\Delta z} - \tau_{yz} l_{z}\} + \Delta y\Delta z\{[V_x(\rho.V_x)]_{x+\Delta x} - [V_x(\rho.V_x)]_{x}\} + \Delta x\Delta z\{[V_y(\rho.V_x)]_{y+\Delta y} - [V_y(\rho.V_x)]_{y}\} + \Delta x\Delta y\{[V_z(\rho.V_x)]_{z+\Delta z} - [V_z(\rho.V_x)]_{z}\} = \Delta y\Delta z\{P_{x+\Delta x} - P_{x}\} + \Delta x\Delta y\Delta z(p.g) \]

\[ + \Delta x\Delta y\Delta z(\rho \frac{\partial V_x}{\partial t}) \]
Dividing by the volume, taking the limit as \( \Delta x, \Delta y, \Delta z \to 0 \) and substituting Newton's law of viscosity yields the **Navier-Stokes equation** in the x-direction:

\[
\rho \frac{DV_x}{Dt} = \rho \left[ \frac{\partial V_x}{\partial t} + \vec{V} \cdot \nabla V_x \right] = -\frac{\partial P}{\partial x} + \mu \nabla^2 V_x + \rho g_x
\]

At steady-state, constant properties (e.g. constant \( \rho \) and \( \mu \)) the equation can be written:

\[
\rho \left( V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) + \rho g_x
\]

Similar expressions can be obtained in the y and z directions.

The equations of continuity and momentum can be used to obtain velocity distributions or calculate \( \tau \) values which in turn can be used to determine friction coefficients.

**II. SHELL BALANCE ON ENERGY IN CARTESIAN COORDINATES**

**Notations:**

- velocity: \( \vec{V} = V_x \vec{x} + V_y \vec{y} + V_z \vec{z} \) (written as \( V_x \hat{x} + V_y \hat{y} + V_z \hat{z} \) in IDW)
- magnitude of velocity: \( \sqrt{V^2} = V_x^2 + V_y^2 + V_z^2 \)
- internal energy per unit mass: \( U \)
- kinetic energy per unit mass: \( K = \frac{V^2}{2} \)
- Potential energy per unit mass:
- total energy per unit mass:

In an differential control volume, every point is at equilibrium and we can use thermodynamics relationships. For an open system at steady-state, we can assume that the energy is conserved and perform a shell balance:

\[
\begin{align*}
\text{[Rate of}
\text{accumulation] = [Rate of energy in by advection] - [Rate of energy out by advection] + [Rate of energy in by conduction] - [Rate of energy out by conduction] - [Net rate of work done by system on sur.]} & \quad (1) + (2) - (3) - (4) - (5) = (6)
\end{align*}
\]

**Term (1):**

\[
\Delta x \Delta y \Delta z \frac{\partial (\rho \hat{E})}{\partial t} = \Delta x \Delta y \Delta z \frac{\partial}{\partial t} \left[ \hat{U} + \frac{V_x^2}{2} \hat{t} + \hat{\phi} \right]
\]

**Term (2):**

\[
\Delta y \Delta z \left( \rho V_x (\hat{U} + \frac{V_x^2}{2} + \hat{\phi}) \right)_x - \left( \rho V_x (\hat{U} + \frac{V_x^2}{2} + \hat{\phi}) \right)_{x+\Delta x} + \text{corresponding terms in } y \text{ and } z
\]
Term (3): \( \Delta y \Delta z \{ q_{x,x} \mid x \Delta x - q_{x,x} \mid x + \Delta x \} \) + corresponding terms in \( y \) and \( z \)

Recall that work = (force).(distance in direction of the force)
Thus rate of work = (force).(velocity in the direction of the force)

Term (4): \( \Delta y \Delta z \{ PV \mid x + \Delta x - PV \mid x \} \)
+ \( y \) and \( z \) terms
rate of work against static pressure in \( x \) direction

Term (5): \( \Delta y \Delta z \{ [\tau_{xx} V_x + \tau_{xy} V_y + \tau_{xz} V_z] \mid x + \Delta x - [\tau_{xx} V_x + \tau_{xy} V_y + \tau_{xz} V_z] \mid x \} \)
+ \( y \) and \( z \) terms
rate of work against viscous forces in \( x \) direction

Note: changes in potential energy due to gravity are already accounted for in \( \phi \).

Term (6): \( \Delta x \Delta y \Delta z \cdot \dot{q}_g \)

Now divide by \( \Delta x \Delta y \Delta z \), take the limit as \( \Delta x, \Delta y, \Delta z \to 0 \), and rearrange using the equations of continuity and momentum. This leads to the equation of energy:

\[
\rho \frac{D}{Dt} (\hat{U} + \frac{V^2}{2} + \hat{\phi}) = (- \nabla \cdot \dot{q}) - (\nabla \cdot \rho \vec{V}) - \nabla \cdot (\tau \cdot \vec{V})
\]

Rate of gain of total energy per unit volume Rate of energy input by conduction per unit volume Rate of reversible change by viscous dissipation per unit volume

Now substitute:
- Thermodynamics relationships: \( h = U + PV \) and \( C_p = \left( \frac{\partial h}{\partial T} \right)_p = C_V \) for incompressible fluid
- Transport properties: \( \dot{q} = -k \nabla T \) and \( \tau = -\mu \nabla \vec{V} \) for constant properties

And in cartesian coordinates:

\[
\rho \cdot C_p \left( \frac{\partial T}{\partial t} + V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial y} + V_z \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \mu \cdot \phi \nu
\]

A similar derivation can be performed in cylindrical and spherical coordinates.
TABLE 10.2-3

THE EQUATION OF ENERGY IN TERMS OF THE TRANSPORT PROPERTIES
(for Newtonian fluids of constant \( \rho \) and \( k \))
(Eq. 10.1-25 with viscous dissipation terms included)

Rectangular coordinates:

\[
\rho C_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] \\
+ 2\mu \left( \left( \frac{\partial v_x}{\partial x} \right)^2 + \left( \frac{\partial v_y}{\partial y} \right)^2 + \left( \frac{\partial v_z}{\partial z} \right)^2 \right) + \mu \left( \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)^2 \right) \\
+ \left( \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)^2 + \left( \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right)^2 \right) \]

\( (A) \)

Cylindrical coordinates:

\[
\rho C_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] \\
+ 2\mu \left( \left( \frac{\partial v_r}{\partial r} \right)^2 + \left[ \frac{1}{r} \left( \frac{\partial v_\theta}{\partial \theta} + v_r \right) \right]^2 + \left( \frac{\partial v_z}{\partial z} \right)^2 \right) + \mu \left( \frac{\partial v_r}{\partial \theta} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} \right)^2 \\
+ \left( \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right)^2 + \left[ \frac{1}{r} \frac{\partial v_r}{\partial \theta} + r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) \right]^2 \right) \]

\( (B) \)

Spherical coordinates:

\[
\rho C_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right) = k \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) \right] \\
+ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \right] + 2\mu \left( \frac{\partial v_r}{\partial r} \right)^2 \\
+ \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)^2 + \left( \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + v_r + \frac{v_\theta \cot \theta}{r} \right)^2 \right) \\
+ \mu \left[ \frac{r}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]^2 + \left[ \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + r \frac{\partial \left( \frac{v_\theta}{r} \right)}{\partial r} \right]^2 \right) \\
+ \left[ \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right]^2 \right) \]

\( (C) \)

Note: The terms contained in braces \{ \} are associated with viscous dissipation and may usually be neglected, except for systems with large velocity gradients.