In theory, currency option prices contain information about market sentiment and risk premia over and above that contained in macro-economic variables such as GDP and interest rates, as well as the information in sources such as survey data or prices of forward contracts. Furthermore, over-the-counter currency market conventions are particularly suited for the extraction and intuitive interpretation of the information contained in these options. In fact, policy makers such as central bankers have the risk-neutral probability density function extracted from a cross-section of currency option prices to as a measure of market sentiment. This naturally leads to the question: how useful is this extra information for the purpose of forecasting the dynamics of future spot rate. Using a parsimonious and model-free framework to extract the risk-neutral density, we first show that currency option prices do not forecast the density of future spot rate well. Using daily data from 2007 to 2010 for five currency pairs and their term structures, we further analyze how option prices tell us about the events surrounding the 2008 financial crisis.
I. Introduction and motivations.

Does the cross section of observed over-the-counter (o-t-c) currency option prices contain information about the future dynamics of the spot exchange rate? Using a parsimonious and model-free density forecasting framework, we find that the risk-neutral probability density function extracted from a cross section of 1-month, 3-month, and 12-month o-t-c option prices do not provide accurate density forecasts of the future spot rate. These results provide insights on the size and dynamics of the currency market risk premium, and therefore provide insights on both the uncovered interest parity puzzle (UIP) and exchange rate disconnect puzzle. If the risk-neutral probability density can forecast well, then the time-varying risk premia explanation for the UIP puzzle. If the risk-neutral probability density does not forecast well, then accounting for the time varying risk premium have to be given more attention.

The economics of exchange rates is one of the most important sub-fields of international economics. This is because, as Sarno and Valente(1997) point out, understanding exchange rate determination and the ability to accurately forecast exchange rate movements is important for, inter-alia, assessing the uncertainty in the prices of imports and exports, evaluating the uncertainty in the value of international reserves, evaluating the domestic value of foreign currency denominated debt payments, and for facilitating investment, lending, borrowing and hedging decisions. These issues are of importance to a wide variety of economics agents, including households, policymakers, as well as investment and risk analysts.

Many theoretical models of FX suggest that FX should be determined by macro-economic fundamentals, such as foreign and domestic money supply, price levels, interest rates, GDP and the balance of payments. However, the failure of structural exchange rate models based on
macro-economic fundamentals suggested by economic theory to accurately forecast nominal exchange rate movements has been extensively studied and documented, in what has become known as the “exchange rate disconnect puzzle.” The seminal work for this literature is the paper by Meese and Rogoff (1983). These authors find that traditional empirical and structural exchange rate models based on macroeconomic theory were out-performed by a simple random walk model in out-of-sample point forecasting. Actually, Meese (1990) notes that “the proportion of monthly or quarterly exchange rates that the models can explain is essentially zero. This finding has potentially huge economic implications, because of the reasons mentioned in the first paragraph.

A related empirical regularity is what has become known as the “forward discount puzzle;” the finding that not only are currency forward prices bad predictors of the future spot rate, the predict with the wrong sign.

Three possible reasons for the forward discount puzzle have been given in the literature, see Taylor and Sarno:

- Failure to capture changes in risk premia
- Failure to capture changes in market expectations.
- Rational bubbles
- Leaning about regime shifts
- Peso problems
- Inefficiencies in information processing
This has been circumvented by the use of survey data.

Furthermore, the failure of these structural models can be attributed to the failure of these structural models to the same reasons as above: If a model/empirical framework cannot capture changes in risk premia or in market expectations, then it will not be very successful for forecasting purposes.

Research in this area has therefore focused on developing and implementing empirical frameworks that are more likely to capture changes in risk premia and in market expectations, therefore making these frameworks better forecasting frameworks.

Literature exploiting information in the term structure of interest rate.

- Ang and Chen

An approach that has also been relatively successful is the one exploiting the information contained in the prices of currency derivatives such as forward contracts, options and swaps. Along these lines, using weekly data, Clarida and Taylor fit a linear VECM model to the term structure of currency forward premia, and show that this set up is able to capture the information contained in forward contracts, and that the linear VECM framework leads to forecasting framework that is better than the benchmark random walk model of Meese and Rogoff (1983). That is, they find that, even if the forward rate is a biased estimator of the future
spot rate (forward discount puzzle), it still contains information useful for forecasting nominal
spot exchange rates.

Motivated by the increasing popularity of Markov-Switching models in modeling
macroeconomic data, as well as the findings of Hamilton and Engel() suggesting non-linear
dynamics in nominal exchange rates, Clarida et al() extend the analysis of Clarida and Taylor to
the case of non-linear VECM. They account for possible non-linearities by fitting a markov-
switching VECM model to the term structure of forward premia. These authors find that
accounting for non-linearities improved the point forecasting performance of the term structure
of forward premia. These results highlight the importance of incorporating non-linearity.

Sarno and Valente then extend the analysis of Clarida and Taylor, and Clarida et al, to the case of
density forecasting. Their study is motivated by the increasing importance of risk management
exercises such as calculating Value at Risk (VaR) and Conditional Value at Risk(CVaR),
for both money/portfolio managers and policy makers. They find that the linear VECM
framework also performs better than the benchmark random walk model in density forecasting.

This paper builds on the work of Clarida and Taylor, Clarida et al, and Sarno and Valente. The
paper extends the literature by investigating the density forecasting performance of over-the-
counter currency options. Using a rich data set of daily O-T-C currency options maturities of 1-
month,3-months and 12-months, we investigate the density forecasting performance of a cross-
section of options prices and ask: “What does the volatility smile tell us about FX density
predictability.” The contributions of this paper are as follows: first, I use data observed at a
higher frequency (daily) than those usually used in the literature. Second, by using currency
options, I explicitly disentangle term structure effects and cross-sectional money-ness effects,
which cannot be done using, forward prices or interest rate differentials. The main difference between this paper and the above mentioned literature is that we consider a cross sectional approach: On a given day, we use a cross section of currency options, with the same to maturity but different levels of money-ness.

The link between forward contracts, interest rate differentials and option contracts is as follows: forward rates and the interest rate differential are connected through the covered interest parity condition. On the other hand, the forward contract can be viewed as a special type of European option contract: one with an exercise price of zero, as argued in section 2. On the other hand, the Breeden-Litzenberger result relates the call option prices with the risk-neutral density of the spot rate. The theoretical first moment of this risk neutral density equals the forward rate. Finally, the Galman-Kolhagen formula, used for pricing options, explicitly includes the interest rate differential.

The paper is motivated by the following: first, in theory, as argued in section 2, currency options prices contain information about market expectations and the risk premium, over and above that contained in forward prices, spot exchange rates, survey data and interest rate differentials. Therefore, for example, one would expect currency option prices to at least do as well as forward prices in density forecasting. This paper seeks to explicitly answer this question: “What does the volatility smile tell us about FX density predictability?”

Second, the relative success of survey data in capturing market expectations in shedding light on the FX disconnect puzzle and the forward discount puzzle provides another motivation for this paper. As Malz argue, survey data are based are based on what market participants say, instead
of what they do. On the other hand, derivative prices such as options are deeply rooted in market participant behavior, and will therefore be more likely to capture market participants’ perception.

The third motivation is, as Chen (1997) points out, there tends to be a disconnect between the economics of exchange rates and the currency option pricing literature, “although both ill benefit from the joint study of the two.” This paper adds to the growing number of papers that try to bridge the gap between these two fields.

Finally: An increasing number of financial institutions have started to use currency options as a measure of market sentiment. Is this warranted? Literature studying the forecasting ability of these extracted densities is scant. This paper contributes to this literature. Are option-implied risk-neutral densities worth calculating?

The rest of the paper is organized as follows: section 2 explains the theoretical information content of currency option prices. Section 3 describes the data and o-t-c market conventions, placing particular focus on how these conventions make it easier to extract the information contained in currency option prices. Section 4 describes the methodology for extracting and evaluating option-implied risk-neutral densities from currency option prices. Section 5 contains the empirical results and a discussion of these results. Section 6 concludes.

II. The information contained in currency in o-t-c option prices.

As () point out, for satisfactory forecast of the dynamics of future spot rate, an approach should be able to capture changes in both market expectations and the market risk premium. In this section, I argue that a combination of theory and o-t-c market conventions make currency options particularly attractive for the purpose of capturing market sentiment and the market
risk premium, and therefore for forecasting the dynamics of the nominal exchange rates. This is especially so when contrasted with forward contract prices, spot prices and survey data.

- *Siegel (1997) writes that “information derived from current option prices is appealing because it can react quickly to market changes and does not have the statistical sampling error associated with classical measures based on samples of historical data.”*

First, although options are priced in the risk-neutral world, the theoretical prices of currency options still contain information about both investor beliefs and their preferences, as shown from the following formula for the price of a European currency option:

\[
c(t, K, T) = \int_K^\infty M_{t,T} (S_T - K) \pi(S_T) dS_T
\]

\[
= \int_K^\infty (S_T - K) \pi^*(S_T) dS_T
\]

where \( M_{t,T} \) is the investor’s stochastic discount factor, which captures investor preferences, \( \pi(S_T) \) is the physical probability density function of the spot exchange rate at maturity, which captures the investors’ beliefs. Last, \( \pi^*(S_T) \) is the risk neutral probability density function at maturity, thus with beliefs and risk preferences co-mingled.

Thus, a change in the price of a currency option can be due to either a change in market beliefs or due to a change in the risk premium. Dis-entangling belief and preferences is potentially useful, and is a subject pursued in the second Chapter. However, for the purpose of forecasting exchange rate movements, this dis-entangling is not necessary.
Second, in theory, currency option prices contain more information about perceived moments of the future spot rate than forward prices. In fact, a forward contract can be viewed as a European call option with an exercise price of zero. Using no-arbitrage arguments, the forward price formula is given by the formula:

\[ F_{t,T} = e^{-rt} \int_0^\infty S_T \pi^*(S_T) dS_T. \]

Now, evaluating the equation for the call option at \( K = 0 \), we get:

\[ c(t, 0, T) = e^{-rt} \mathbb{E}^*(\max(S_T - K, 0)) = e^{-rt} \int_0^\infty S_T \pi^*(S_T) dS_T = F_{t,T} \]

As discussed in section I, Clarida and Taylor find that the forward exchange rate and the whole term structure of forward premia contain information valuable for forecasting the future spot rate. Therefore, at worst, currency option prices should do as well as forward prices.

A related argument goes as follows: Both forward contracts and currency options are forward-looking by construction. Furthermore, the risk-neutral probability density implied by the pricing equations for both derivatives is the same. However, for the purpose of making inferences about this density, the option price equation is more useful because, for a given tenor and currency pair, a cross section of option prices with different strike prices are observed, whereas there is only one forward price observed for a given time to maturity. This is illustrated below:

<table>
<thead>
<tr>
<th>Date</th>
<th>spot</th>
<th>fwd</th>
<th>l</th>
<th>r*</th>
<th>l</th>
<th>atm</th>
<th>l</th>
<th>r*</th>
<th>l</th>
<th>str</th>
<th>l</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day1</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
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<tr>
<td>Day2</td>
<td>x</td>
<td>-</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
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<tr>
<td>Day3</td>
<td>x</td>
<td>-</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
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</tbody>
</table>
For a given currency pair, day and time-to-maturity, we observe three option prices as compared to just one forward price. Furthermore, the data that I use are observed on a daily frequency. In their analysis using forward prices, Clarida and Taylor, Clarida et al, and Sarno and Valente use data observed weekly. Most structural models in the FX forecasting literature use data observed at even lower frequencies. In stable macro-economic conditions, the difference between forecasts that use higher and lower frequencies might not be huge. However, in times of economic turbulence, the differences might be huge. Therefore, in addition to currency options theoretically containing more information than forward prices for any given sampling frequency, the daily data used in this paper therefore contain a second dimension in which currency options data can result in improved ability to capture changes in risk premia and market expectation, and therefore, forecasting.

The price difference between two otherwise identical European call options that differ in their strike prices, is closely related to the probability that $S_T$, the spot rate at maturity, is between $K$ and $K'$. Currency option prices are therefore more informative than, say, forward prices, because they contain information about higher order moments of market participants’ expectation of exchange rate movements. Therefore, by looking at a cross section of strike prices and across different maturities, we can disentangle the information content of option prices into cross

<table>
<thead>
<tr>
<th>Day4</th>
<th>x</th>
<th>-</th>
<th>x</th>
<th>x</th>
<th>x</th>
<th>x</th>
<th>x</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day5</td>
<td>x</td>
<td>-</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Day 6</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Day 7</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Day 8</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
</tbody>
</table>
sectional effects and the term structure effects. For a given time to maturity and currency pair, there is no cross-section of forward prices.

When using the term structure of forward premia to make inferences about $\pi^*(S_T)$, as in Sarno and Valente, one has to first fit a time series model that sufficiently captures issues such as regime shifts and unit roots. This fitted model is then used to make density forecasts. However, the aforementioned econometric issues are not straightforward to deal with. The currency-option approach has the advantage that it allows for the extraction and evaluation of the risk-neutral density without having to fit a model for the underlying exchange rate: whatever the process that is followed by the underlying exchange rate, we try to extract the density implied from the observed option prices.

Furthermore, when using the currency option price approach, we do not have to specify how agents form their beliefs: whatever way agents form their expectations, we get to extract the distribution implied by their actions. Thus, in comparison to say, survey data, currency options are more rooted in market participant behavior, and use the investor’s information set, instead of that used by the econometrician. Time series models using macro-fundamentals or the term structure of forward premia are based on the information set specified by the econometrician. This information set may be different from that used by the investor.

An important theoretical result that highlights the information content of currency option prices is the observation by Breeden and Litzenberger (1978) that the risk-neutral probability density function of the underlying asset at maturity is equal to the second derivative of the market call price with respect to the exercise price. Using Leibnitz’s rule, first derivative is given by
\[ \frac{\partial c(t,K,T)}{\partial K} = -e^{-rT}(1 - \Pi^*(S_T)), \] where \( \Pi^*(S_T) \) is the risk-neutral cumulative distribution function of \( S_T \).

\[ \frac{\partial^2 c(t,K,T)}{\partial K^2} = e^{-rT}\pi^*(S_T) \Rightarrow \pi^*(S_T) = \frac{\partial^2 c(t,K,T)}{\partial K^2} e^{rT}. \]

Thus, in principle, the risk-neutral probability density function of \( S_T \) can be estimated from the prices of call options with different strike prices that mature at time \( T \). However, for the estimated probability density to be accurate, we need a continuous range of exercise prices observed in the market. However, there are only a few option combinations traded in the o-t-c currency option market. Furthermore, as discussed in the next section, currency option prices are quoted in terms of implied volatilities instead of currency units, and the degree of moneyness is measured by delta instead of the strike price. These market conventions mean using the Breeden-Litzenberger results for estimating the risk-neutral pdf requires a way to move from the delta-implied volatility space used in the market to the exercise price-call price space used in the Breeden–Litzenberger result.

Finally, as argued by, the risk-neutral probability density function corresponds to Arrow-Debreu prices in the continuum of states case. Arrow-Debreu prices and the volatility smile at a given date for a given maturity basically convey the same information about investors’ preferences.

**Further explain this connection between the risk-neutral density and Arrow-Debreu securities.**

### III. Data description, o-t-c market conventions and stylized facts

An exchange rate is the price of one currency in terms of another. An exchange rate pair is denoted XXXYYY, interpreted as units of YYY per unit of XXX. XXX is the “foreign” or base
currency and YYY is the numairare or “domestic” currency. The table below gives a summary of
the FX marketing quoting conventions for major currency pairs.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Numaraire currency</th>
<th>FX fall means</th>
</tr>
</thead>
<tbody>
<tr>
<td>GBPUSD</td>
<td>US dollars per GBP</td>
<td>USD</td>
<td>USD appreciation</td>
</tr>
<tr>
<td>GBPCHF</td>
<td>Swiss Francs per GBP</td>
<td>CHF</td>
<td>CHF appreciation</td>
</tr>
<tr>
<td>GBPJPY</td>
<td>JPY per GBP</td>
<td>JPY</td>
<td>JPY appreciation</td>
</tr>
<tr>
<td>USDJPY</td>
<td>JPY per USD</td>
<td>JPY</td>
<td>JPY appreciation</td>
</tr>
<tr>
<td>USDSEK</td>
<td>SEK per USD</td>
<td>SEK</td>
<td>Koruna appreciation</td>
</tr>
<tr>
<td>AUDUSD</td>
<td>USD per AUD</td>
<td>USD</td>
<td>USD appreciation</td>
</tr>
<tr>
<td>EURUSD</td>
<td>USD per EUR</td>
<td>USD</td>
<td>USD appreciation</td>
</tr>
<tr>
<td>USDCAD</td>
<td>CAD per USD</td>
<td>CAD</td>
<td>CAD appreciation</td>
</tr>
</tbody>
</table>

Currency options are traded over-the-counter or on organized exchange markets. As Malz points out, for the purpose of extracting the risk-neutral density of future spot rate, there are several advantages that come with using o-t-c currency option price data. First, exchange –traded options are not exactly at the money, therefore variations in the level of implied volatility is co-mingled with variations in the curvature of the volatility smile. Using at-the-money o-t-c options circumvents this distortion.
Second, fresh 1-month, 3-month, and 12-month to maturity o-t-c options are issued daily, so that it is possible to have a time series of constant-maturity implied volatility. This therefore avoids the maturity mis-match that come with the co-mingling of variations in the term structure of implied volatility with variations in the level of volatility.

Third, most of the currency options are traded in the o-t-c market are European–style. Therefore, by using European options data, we avoid the complications that come with having to adjust for possible early exercises. Lastly, most of the liquidity in the currency option prices is concentrated in the o-t-c market.

Using information on the standard o-t-c option combinations, we have the following equations:

\[ atm_t = \sigma(0.5) \]

\[ rr_t = \sigma(0.25) - \sigma(0.75) \]

\[ str_t = \frac{\sigma(0.25) + \sigma(0.75)}{2} - \sigma(0.5) \]

Rearranging the above three equations, we have the following formulae for the three implied volatilities of vanilla options, at deltas 0.25, 0.5 and 0.75:

\[ \sigma(0.25) = str_t + atm_t + \frac{rr_t}{2} \]

\[ \sigma(0.5) = atm_t \]

\[ \sigma(0.75) = str_t + atm_t - \frac{rr_t}{2} \]
The o-t-c options data for this paper consists of daily quotes of $atm_t, str_t$, and $rr_t$ European style option combinations for the currency pairs shown in table 2. The data are from January 2001 to Dec 2008. These quotes are from the British Bankers Association.

The risk-free interest rate comes from the BBA’s 11am LIBOR fixings. As argued by Hull, there are several advantages of using LIBOR rate instead of the treasury rate as a proxy for the risk-free rate. He notes that the treasury rate is too low to be used as the risk-free rate because, first, treasury bills and bonds must be purchased by financial institutions to meet a variety of regulatory requirements. This increases the demand for this treasury instruments, thus driving down their yields. Second, the amount of capital a bank is required to support an investment in treasury bills is lower than that required for similar investments in other low-risk investments. Third, at least in the United States, treasury instruments are given favorable tax treatment compared with other fixed income investments because they are not taxed at the state level.

For the purpose of density forecast evaluation, we need to adopt a day-count convention so that we can match today’s data, with the spot rate one month from now. For this, I follow the modified following rule day-count convention, as laid out in Benhamon and Castagna. For 1 month options, $T$ falls on same date in month 2. If the day is a weekend or holiday, then move to the next trading day. However, if the next trading day falls in month 3, go back and use the last trading day of month 2 as the $T$.

The empirical procedure for this paper follows the following procedure:

1. For each trading day, I match the option price data for that day with the spot rate 1-month from now (or 1 year or 3 month), found using the “modified following rule.”
ii. For the given trading day, I extract the risk-neutral probability density function, using the Malz (1997) “volatility smile function” methodology.

iii. I then evaluate the forecasting ability of the extracted densities.

IV. Methodology

4.1 The Malz method

Castagna(2010) points out three criteria for the representation of the volatility surface: parsimony, consistency and intuitiveness.

- **Parsimony**: The representation contains the smallest amount of information needed to retrieve the entire volatility smile.

- **Consistency**: The information contained in the representation is along expiries and strikes so as to make the integration of missing points, either by interpolation or extrapolation easily possible.

- **Intuitiveness**: The information provides the user with a clear picture of the volatility surface. Volatilities do not move independently from one another. From principal component analysis exercises, one may reasonably assume that the degrees of freedom are only three: level, slope, and curvature.

To extract the risk-neutral probability density of future spot rate for each day, I use the “volatility smile function” method of Malz. Malz posits a continuous volatility smile function constructed by interpolating a quadratic functional form through the observed prices of option combinations. He assumes the specification:

\[ \hat{\sigma}_\delta(\delta) = b_0 atm_t + b_1 rr_t(\delta - 0.50) + b_2 str_t(\delta - 0.50)^2 \]
He then chooses parameters $b_0$, $b_1$, $b_2$ such that $atm_t$, $rr_t$, and $str_t$ are on the fitted volatility smile on an given day. This requires that $b_0 = 1, 16, b_2=-2^1$. Thus, the specification provides a perfect fit three volatility smile points that make up our data for a given day.

Equation (1) warrants some discussion regarding the aforementioned criteria. The specification is certainly parsimonious since it uses only three option combinations that are actively traded in the o-t-c market. With regard to intuitiveness, this functional form captures the basic information about the smile expressed by each of the option combinations. The straddle volatility captures the general level of implied volatility, the 25-delta strangle captures the curvature of the volatility smile, and the 25-delta risk-reversal captures the slope of the smile. Note that the atm, straddle corresponds to the forward price. Therefore, the currency option approach tries to capture the information about the level, slope, and curvature of the volatility smile, whereas methodologies using just forward prices captures information about the level of the smile.

As for consistency, the method has, however, been shown to underestimate the implied volatility at the details of the density. This might not be surprising, given that the most commonly used option combinations are 25-delta, and therefore might not extrapolate well to the tails. Use of 10-delta options might help solving this problem.

Castagna (2010) points out that another shortcoming of the Malz methodology is that there is no real financial justification for the methodology, other than that it fits the data perfectly and is simple. A better methodology in this regard is the vanna-volga method of Castagna (), which is based on a vega-hedging argument.

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1 The derivation is carried out in the appendix.
Last, the methodology assumes that the relationship between the delta of a call and the delta of a put is given by the following:

\[ \delta_{\text{put}} = 1 - \delta_{\text{call}}. \]

However, the put call parity condition means that this relationship between \( \delta_{\text{put}} \) and \( \delta_{\text{call}} \) is only an estimate, and is only exact when the foreign interest rate is zero.

### 4.2 Density Forecast Evaluation

The framework for density forecast evaluation employed in this paper is introduced in Diebold, Gunther and Tay().

Let \( S_T \) be the realization of the spot exchange rate at maturity. Then, \( U_t \), the probability integral transform of \( S_T \) is given by

\[ u_t(S_T) = \int_{-\infty}^{S_T} \hat{f}_t(u)du = \hat{F}_T(S_T) = \text{PIT}(S_T) \]

That is, \( U_t(S_T) \) is the probability, using the estimated risk-neutral pdf, of observing a spot rate at maturity that is less than \( S_T \).

Diebold, Gunther and Tay show that if the estimated density has the same predictive ability as the true density, then it should not be able to predict the probability that, at maturity, the spot rate will be less than \( S_T \). Therefore, under the null hypothesis that the extracted risk-neutral density is the true density, then we should not be able to predict the probability of getting a spot rate less than \( S_T \). Under the same hypothesis, the time series of \( U_t(S_T) \) should therefore be independent and identically distributed observations from the uniform(0,1) distribution. Furthermore, using the inverse transform theorem, the series

\[ z_t = \Phi^{-1}(U_t) = \Phi^{-1}\left(\int_{-\infty}^{S_T} \hat{f}_t(u)du\right) \]
, where $\Phi$ is the standard normal cumulative distribution function, would be i.i.d $N(0,1)$. Berkowitz (2001) recommends working with the $z_t$ series instead of the $u_t$ is because it is easier to work with the normal distribution.

4.2.1 Separate tests of uniformity and independence

I follow the work of Clements and Smith, Siliverstovs and Dijik, Rapach and Wohar in using the Kolmogorov-Smirnov test for uniformity on the $U_t(S_T)$ series. To test normality of the $z_t$ series, I use the Jarque-Bera test as well as the Anderson-Darling test. I also plot normal Q-Q plots of the $z_t$ for each currency pair.

In testing for normality or uniformity, the above methodologies assume independence. To test for this independence, I follow the approach Rapach and Wohar (2005) and look for first order autocorrelation patterns of powers of the normal transform of the time series of probability integral transforms:

$$\left\{ \left( z_{t+1|t} - \bar{z}_{t+1|t} \right)^k \right\}_{t=1}^{20} \text{ for } k = 1, \ldots, 4$$

I then use the Ljung-Box test statistic to test for the significance of autocorrelations of the above series.

4.2.2 Joint tests of normality and independence

A major weakness of the above density forecast evaluation tests for normality and independence and normality considered above is that they are separate tests. A joint test of independence and normality of $z_t$ would increase the power of the test. I follow the method of Berkowitz, who suggested testing the joint hypothesis of normality and independence using a likelihood-ratio
like test. Berkowitz (2001) posits an AR (1) model for the normal transform of the probability integral transform:

\[ z_t - \mu = \rho(z_{t-1} - \mu) + \varepsilon_t \quad , \quad \varepsilon_t \sim \mathcal{N}(0,1) \]

The joint hypothesis that the \( z_t \) is standard normal and independent is equivalent to imposing the following restrictions on the AR (1) model for \( z_t : \mu = 0, \rho = 0, \) and \( \text{var}(\varepsilon_t) = 1. \) The likelihood ratio statistic for this joint test is given by:

\[ LR = -2(L(0,1,0) - L(\hat{\mu}, \hat{\sigma}^2, \hat{\rho})) \]

and has a \( \chi^2(3) \) under the joint null hypothesis. The log-likelihood function for this AR(1) model, whose derivation is straightforward but tedious (see, for example John and Cryer()), is given by the following expression:

\[
l(\mu, \sigma^2, \rho) = -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log\left(\frac{\sigma^2}{(1-\rho)}\right) - \frac{(z_1 - \frac{\mu}{(1-\rho)})^2}{2\sigma^2(1-\rho)} - \frac{n-1}{2} \log(2\pi) - \frac{n-1}{2} \log(2\sigma^2) - \sum_{t=2}^{T} \left[\frac{(z_t - \mu(1-\rho) - \rho(z_{t-1}))}{2\sigma^2}\right]
\]

V. EMPIRICAL RESULTS AND DISCUSSIONS

I analyze data for three currency pairs: GBPUSD and USDCAD currency pairs.
Figure 1. The panel of diagrams above shows the time series evolution of the time series evolutions of the spot exchange rate, at-the-money implied volatility, 25-delta risk reversals and 25-delta strangles. Note that both risk-reversal and strangle are mostly zero before starting to move significantly around mid-2007. All implied volatilities are in percentages.
Figure 2: The panel of diagrams above shows the volatility smile for a given day in delta-sigma space (top-left) through to the extracted estimate of the risk-neutral probability density function (bottom right). The question is: how good is the extracted risk-neutral density in capturing changes in market risk-premia and market sentiment?
Figure 3: The above diagram shows extracted densities for a few days. Note that there is no term structure element here.

DENSITY FORECAST EVALUATION
Figure 4: Show time series of probability integral transform. Under the null hypothesis that the estimated (extracted) risk-neutral density has the same predictive ability as the true density, this series should be i.i.d(0,1). It does not look like that in the above diagram.

HISTOGRAMS OF PROBABILITY INTEGRAL TRANSFORMS
Figure 5: Some histograms to indicate the failure of i.i.d uniform for the probability integral transform. Under the null, the frequencies should be roughly same height.

The results of the normality tests for the currencies under consideration are summarized in the following tables:

Figure 6: Berkowitz recommends working with the normal distribution instead of the uniform distribution. Saying \( u_t \) series is iid \( U(0,1) \) is equivalent to saying \( z_t = \Phi^{-1}(u_t) \) series is iid \( N(0,1) \). The above q-q plot suggests this might not be the case.

One step-ahead density forecast evaluation: Separate tests of uniformity/normality and independence
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**CONCLUSIONS and POSSIBLE EXTENSIONS**

Using the GBPUSD currency pair it seems like risk-neutral probability density functions extracted from currency options do not forecast the future exchange rate well.
Using currency options data for the GBPUSD pair, there is strong evidence to suggest that currency options are not any useful for the purpose of predicting the dynamics of the future spot rate, at least for the one month tenor. That is, the superior theoretical information content of option prices does not really translate to an empirical reality. It seems that the risk-neutral probability density functions extracted from a cross section of currency options do not really capture market sentiments.

Note: The code still needs some additional fixing, but I do not expect it to dramatically change the results.

RELATED QUESTIONS BEING/TO BE ADDRESSED/: MOTIVATIONS

1) Are option-implied risk-neutral densities worth calculating? An increasing number of financial institutions have started to use currency options as a measure of market sentiment. Is this warranted? (Fed, BIS, etc.). But, are option prices a good measure of market sentiment?

Literature studying the forecasting ability of these extracted densities is scant. This paper contributes to that literature.

2) Is it worth it to go over the trouble of calculating the risk neutral pdf: How correlated are the moments from the risk-neutral pdf with the “short cut” method of just using the raw option prices to sub for these moments: a-t-m for standard deviation, risk reversal for skewness and strangle for the kurtosis?

BIBLIOGRAPHY

APPENDIX

- Derivation of the values of $b_0, b_1,$ and $b_2$

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