Appendix

**Proof of Lemma 1:**

Define:

\[
V_i(Q, \Delta, t) = \sum_{d_1=0}^{U_1} \sum_{d_2=0}^{U_2} \ldots \sum_{d_{N-1}=0}^{U_{N-1}} dp(d_i; \lambda t) \prod_{j \neq i} p(d_j; \lambda t). \tag{A.1}
\]

Using \(\frac{dp(d_i; \lambda t)}{dt} = \lambda (p(d_i - 1; \lambda t) - p(d_i; \lambda t))\) and (A.1), with some effort, it can be shown that:

\[
V_i(Q, \Delta, t) = V_N(Q, \Delta, t) = -\lambda \sum_{d_1=0}^{U_1} \sum_{d_2=0}^{U_2} \ldots \sum_{d_{N-1}=0}^{U_{N-1}} (p(U_N; \lambda t)) \prod_{j \neq N} p(d_j; \lambda t). \tag{A.2}
\]

Next, the probability density function of the cycle time, \(\tau\), can be written as:

\[
f(t) = \frac{dF(t)}{dt} = -\sum_{i=1}^{N} V_i(Q, \Delta, t).
\]

Using (A.2), \(f(t)\) reduces to:

\[
f(t) = -NV_N(Q, \Delta, t)
= NV_N(Q, \Delta, t) = NV_N(Q, \Delta, t) \prod_{j \neq N} p(d_j; \lambda t). \tag{A.3}
\]

The proof is now complete.

**Proof of Lemma 2:**

Without loss of generality, assume that the retailer of interest is Retailer 1. The event “\(D_i(\tau) \geq n\)” is equivalent to the event that the \(n^{th}\) demand arrives at Retailer 1 before the replenishment has been triggered. Considering all the possible combinations of demand from other retailers (as long as the total is less than \(Q-I\)), we have:

\[
\Pr(D_i(\tau) \geq n) = \int \sum_{i=0}^{U_1} \sum_{d_2=0}^{U_2} \ldots \sum_{d_{N-2}=0}^{U_{N-2}} \prod_{i=0}^{N} p(d_i; \lambda t) \frac{e^{-\lambda t} \lambda t^{n-1}}{(n-1)!} dt \tag{A.4}
\]
where, for \( i=2,..,N \): \( U_i = \min(\Delta - 1, Q - 1 - d_i - .. - d_{i-1}) \) and \( d_1 = n \).

Similarly, for the outbound quantity, the expression will be the same as in (A.4) except that the total can be equal to \( Q \) now (instead of \( Q-1 \)). Thus, we have the following:

\[
\Pr(Z_1(\tau) \geq n) = \int_0^\infty \sum_{i=0}^{U_1} \sum_{d_2=0}^{U_2} \ldots \sum_{d_N=0}^{U_N} \prod_{i=2}^{N} p(d_i; \lambda t) \frac{e^{-\lambda t} \lambda t^{n-1}}{(n-1)!} dt. \tag{A.5}
\]

where, for \( i=2,..,N \): \( U_i = \min(\Delta - 1, Q - 1 - d_i - .. - d_{i-1}) \) and \( d_1 = n - 1 \).

Therefore when \( Q>\Delta+(N-1)*\Delta \), we have:

\[
\Pr(D_1(\tau) \geq n) = \int_0^\infty \sum_{i=0}^{\Delta-1} \sum_{d_2=0}^{\Delta-1} \prod_{i=2}^{N} p(d_i; \lambda t) \frac{e^{-\lambda t} \lambda t^{n-1}}{(n-1)!} dt = \int_0^\infty \lambda p(n-1;\lambda t)[1 - P(\Delta; \lambda t)]^{(N-1)} dt.
\]

The same expression holds for the outbound quantity.

\[\square.\]

**Proof of Lemma 4:**

\[E[Z_0] = E_\tau[E[Z_0 | \tau]] = E_\tau[N\lambda \tau] = N\lambda E[\tau]. \] Therefore, \( E[\tau] = \frac{E[Z_0]}{N\lambda} \). \tag{A.6}

\[\square.\]

**Proof of Lemma 5:**

We know that the expected cycle time for given \( Q, \Delta, \) and \( N \) can be written as:

\[
E[\tau] = \int_0^\infty \Pr(\tau \geq t) dt.
\]

\[
= \int \sum_{i=0}^{U_1} \sum_{d_2=0}^{U_2} \ldots \sum_{d_N=0}^{U_N} p(d_i; \lambda t) dt
\]

\[
= \sum_{d_1=0}^{U_1} \ldots \sum_{d_N=0}^{U_N} \frac{1}{d_1! \ldots d_N!} \int_0^\infty e^{-N\lambda t} (\lambda t)^{d_1} dt
\]

\[
= \sum_{d_1=0}^{U_1} \ldots \sum_{d_N=0}^{U_N} D_0 \frac{D_0!}{N\lambda (N)^{D_0}} \frac{h(d_1, d_2, ..., d_N)}{N\lambda}
\]

where, \( D_0 = \sum_{i=1}^{N} d_i, \ h(d_1, d_2, ..., d_N) = \frac{D_0!}{N^{D_0} \prod_{i=1}^{N} d_i!} \)

\[\square.\]
Using (A.7), we can write the following recursive relation of the expected cycle time with respect to $Q$ for $N-1$ retailers:

$$E[\tau \mid Q + 1 - n, \Delta, N - 1] - E[\tau \mid Q - n, \Delta, N - 1] = \frac{1}{(N-1)!} \sum_{D_n = Q-n} D_0! \frac{D_1! \ldots D_{N-1}!}{(N-1)^{Q+n}}$$

(A.8)

Note that here $D_0 = d_1 + \ldots + d_{N-1}$.

Equation (A.8) yields:

$$\sum_{D_n = Q-n} \frac{1}{d_1! \ldots d_{N-1}!} = \frac{\lambda (N-1)^{(Q+n)}}{(Q-n)!} (E[\tau \mid Q + 1 - n, \Delta, N - 1] - E[\tau \mid Q - n, \Delta, N - 1])$$

(A.9)

And we know from (A.5) that:

$$\Pr(Z_i(\tau) \geq n) = \int \ldots \int p(d_1; \lambda t) e^{-\lambda t} \lambda (\lambda t)^{(n-1)} \frac{1}{(n-1)!} \left( \frac{1}{N} \right)^{D_0} \frac{1}{D_1! \ldots D_{N-1}!}$$

Therefore, we can write the following recursive relation of $\Pr(Z \geq n)$ with respect to $Q$ as follows when $Q > \Delta$:

$$P(Z \geq n \mid Q, \Delta, N) - P(Z \geq n \mid Q-1, \Delta, N) = \frac{1}{N} \sum_{D_n = Q-n} (D_0 + n-1)! \frac{1}{(n-1)!} \frac{1}{D_1! \ldots D_{N-1}!}$$

(A.10)

Finally, substituting (A.9) in (A.10) yields:

$$P(Z \geq n \mid Q, \Delta, N) - P(Z \geq n \mid Q-1, \Delta, N)$$

$$= \left( \frac{1}{N} \right)^{(Q-n)} \frac{1}{(n-1)!} \sum_{D_n = Q-n} (D_0 + n-1)! \frac{1}{(n-1)!} \frac{1}{D_1! \ldots D_{N-1}!}$$

$$= \left( \frac{1}{N} \right)^{(Q-n)} \frac{\lambda (N-1)^{(Q+n)}}{(n-1)!} (E[\tau \mid Q + 1 - n, \Delta, N - 1] - E[\tau \mid Q - n, \Delta, N - 1])$$

$$= \lambda \left( \frac{1}{N} \right)^{(Q-n)} \left( 1 - \frac{1}{N} \right)^{(Q+n)} (E[\tau \mid Q + 1 - n, \Delta, N - 1] - E[\tau \mid Q - n, \Delta, N - 1])$$
And when $Q \leq \Delta$, the hybrid policy reduces to Policy 1, and the probability of the outbound quantity being greater than $n = \min(\Delta, Q)$, is just equal to the outbound quantity being equal to $Q$. And it is given by:

$$\Pr(Z \geq \min(\Delta, Q)) = P(Z = \min(\Delta, Q)) = P(Z = Q) = \left(\frac{Q}{Q} \right)^{\frac{Q}{N}} \left(1 - \frac{1}{N}\right)^{Q - Q} = \left(\frac{1}{N}\right)^{Q}$$

**Proof of Proposition 1:**

The first term in (11) is the warehouse ordering cost per unit time. A fixed cost of $K_0$ independent of the order size is incurred at the warehouse each time an order is placed, so the expected ordering cost per unit time at the warehouse is (Ross 1993, page 318):

$$E[\text{Ordering Cost}] = \frac{K_0}{E[\tau]}.$$  

The second term represents the fixed ordering cost at the retailers, and is similar to the first one. Without loss of generality, let Retailer 1 be the triggering retailer, so there is definitely a shipment to Retailer 1. As to the other retailers, there will be a shipment if and only if there is at least one demand. Hence, the total is

$$K + \sum_{i=2}^{N} K \Pr(Z_i \geq 1).$$

Since the demand during leadtime is Poisson with rate $\lambda*LT$, we obtain the following by conditioning on $IP$:

$$E[IL^*] = \sum_{j=(\max(5-Q+1,5-\Delta+1))}^{S} \sum_{l=0}^{j} (j-l) p(l, \lambda*LT) \Pr\{IP = j\} = E[(IP - D(LT))^+] \quad (A.11)$$

$$E[IL] = \sum_{j=\max(5-Q+1,5-\Delta+1)}^{\infty} \sum_{l=0}^{j} (l-j) p(l, \lambda*LT) \Pr\{IP = j\} + \sum_{j=\max(5-Q+1,5-\Delta+1)}^{\infty} \sum_{l=j+1}^{\infty} (l-j) p(l, \lambda*LT) \Pr\{IP = j\}$$

$$= E[(D(LT) - IP)^+] \quad (A.12)$$
Using (A.11) and (A.12), one can express the expected backorders in terms of the expected on-hand inventory as follows:

\[ E[IL^-] = E[IL^+] + E[D(LT)] - E[IP] = E[IL^+] + E[D(LT)] - \sum_{j=\max(0,\Delta)}^{S} j \Pr(IP = j). \quad (A.13) \]

Consequently, using (A.13), the sum of the expected holding and shortage costs at any retailer will be:

\[ E[\text{Holding + Shortage Cost at a retailer}] = hE[IL^+] + \pi E[IL^-] \]
\[ = (h + \pi)E[IL^+] + \pi(E[D(LT)] - E[IP]). \]

The last two terms of (11) are the expected cost of transportation penalty costs due to inbound and outbound quantities exceeding the capacity limits, respectively. \( \Box \)

**Proof of Proposition 2:**

The second order difference with respect to \( S \) is always greater than zero as follows:

\[ \Delta^2 CR = N(h + \pi) \sum_{j=\max(0,\Delta)}^{S} \Pr(IP = j) p(j; \lambda LT) > 0. \]

Therefore, the following first order difference can be used to find the optimal \( S \) given \( \Delta \) and \( Q \):

\[ \Delta CR = N \left( h + \pi \left( \sum_{j=\max(0,\Delta)}^{S} \Pr(IP = j)(1 - P(j; \lambda * LT)) \right) - \pi \right). \quad \Box \]

**Proof of Lemma 7:**

**Proof:** Recall that (8) states

\[ P(IP = j \mid D(\tau) = n) = \frac{1}{n+1} \quad \text{for } j = S - n, S - n + 1, \ldots, S. \]

Since the quantity demanded from any retailer is the random disaggregation of the total demand, it is binomially distributed:

\[ \Pr(D(\tau) = n) = \binom{Q}{n} \left( \frac{1}{N} \right)^n \left( 1 - \frac{1}{N} \right)^{Q-1-n} \quad , 0 \leq n \leq Q - 1. \quad (A.14) \]

Using (A.14) to remove the condition on “\( n \)”, we obtain:
Pr(IP = j) = \sum_{n=S-j}^{Q-1} Pr(IP = j \mid D(n) = n) Pr(D(n) = n) = \sum_{n=S-j}^{Q-1} \left( \frac{1}{n+1} \right) \left( \frac{Q-1}{n} \right) \left( \frac{1}{N} \right) \left( \frac{1}{N} \right)^{Q-1-n} \quad (A.15)

Simplifying (A.15), we obtain the expression in (15).

**B(1,S-1,T) and the first order difference w.r.t. S under Policy 2:**

These expressions can also be found in Hadley & Whitin (1963):

\[ B(1,S-1,T) = \frac{1}{T} \left( \frac{\lambda}{2} \left[ (LT + T)^2 P(S - 1; \lambda(LT + T)) - LT^2 P(S - 1; \lambda * LT) \right] + \frac{S(S + 1)}{2\lambda} \left[ P(S + 1; \lambda(LT + T)) - P(S + 1; \lambda * LT) \right] - S \left[ (LT + T)P(S; \lambda(LT + T)) - LT P(S; \lambda * LT) \right] \right) \]

and

\[ \Delta \text{CR} = \text{CR} \big|_{S} - \text{CR} \big|_{S-1} = h + \frac{S}{\lambda T} \left( h + \pi \left[ P(S; \lambda(LT + T)) - P(S; \lambda * LT) \right] \right) \]

\[ - \left( \frac{h + \pi}{T} \right) \left[ (LT + T)P(S; \lambda(LT + T)) - LT P(S; \lambda * LT) \right] - \left( \frac{(h + \pi)S}{\lambda T} \right) \left[ P(S; \lambda(LT + T)) - P(S; \lambda * LT) \right] \quad (A.16) \]

For a given \( T \), the optimal value of \( S \) will be the largest integer where the expression in (A.16) is non-positive. To find the optimal \( T \), we search over possible values of \( T \).

**Variance of demand rate at the supplier:**

Denote by \( X \) the random variable for the supplier’s demand rate, \( Y \) the number of orders received by the supplier per time unit, and \( Z_0 \) the quantity demanded in each order. It is easy to see that \( E[X] = N\lambda \) under all the policies. Next we derive the variance of the demand rate.

For the Non-coordinated Model, we use equation (4) in Svoronos and Zipkin (1988) for the variance of the number of orders received by the supplier during time \( L_w \):

\[ \text{Var}(D_w) = \frac{\lambda L_n N}{Q} + \frac{N}{Q^2} \sum_{k=1}^{Q-1} \left[ \frac{1 - \exp(-\alpha_k \lambda L_w) \cos(\beta_k \lambda L_w)}{\alpha_k} \right] \]
where, \( \alpha_k = 1 - \cos(2\pi k / Q) \) and \( \beta_k = \sin(2\pi k / Q) \).

Then the demand rate variance can be derived as follows:

\[
Var(X) = \lim_{L_w \to \infty} \frac{Q^2 Var(D_w)}{L_w} = N\lambda.
\]

For Policy 2, cycle time is a constant \( T \). So the demand received by the supplier every cycle is Poisson with rate \( N\lambda T \), which is also its variance. Therefore, the demand rate variance is \( N\lambda \).

For Policy 1, each order at the supplier has a constant size \( Q \), and the cycle time has Erlang distribution, with a mean of \( Q/(N\lambda) \) and a variance of \( Q/(N\lambda)^2 \). According to Deurmeyer and Schwarz (1981), the demand rate variance at the supplier is again \( N\lambda \).

Finally, we use the following relation to derive the supplier’s demand rate variance. Because this relation assumes independence between \( Y \) and \( Z \), our derivation below is an approximation.

\[
Var(X) = Var(Y)E^2(Z_0) + E(Y)Var(Z_0)
\]  \((A.17)\)

From Ross (1993) page 317, the asymptotic mean and variance for \( Y \) is \( 1/\mu \) and \( \sigma^2/\mu^3 \), respectively, in the steady state where \( \mu \) and \( \sigma^2 \) are the mean and variance of the cycle time in our problem. To compute \( \mu \) and \( \sigma^2 \), we use the following relations:

\[
\mu = E[\tau] = \frac{E[Z_0]}{N\lambda},
\]  \((A.18)\)

and

\[
E[\tau^2] = \int_{t=0}^{\infty} t^2 f(t) dt = \sum_{d_1=0}^{U_1} \ldots \sum_{d_k=0}^{U_k} \sum_{d_{k+1}=0}^{U_{k+1}} \frac{1}{N\lambda^{d_1} \ldots d_k N^{d_{k+1}}} e^{-N\lambda t} Nz \sum_{d_{k+2}=0}^{U_{k+2}} dt...
\]  \((A.19)\)

\[
= \sum_{d_1=0}^{U_1} \ldots \sum_{d_{k+1}=0}^{U_{k+1}} \sum_{d_{k+2}=0}^{U_{k+2}} \frac{(D_0 + 2)!}{(N\lambda)^2 N^{d_1} \ldots d_k N^{d_{k+1}}} \sum_{d_{k+3}=0}^{U_{k+3}} \ldots \sum_{d_{k+2}=0}^{U_{k+2}} \frac{(D_0 + 2)(D_0 + 1)h(d_k, dz, \ldots, d_n)}{(N\lambda)^2}
\]

\[
= \frac{1}{(N\lambda)^2} \sum_{k=0}^{O-1} (k+2)(k+1)P(D_0 = k) = \frac{1}{(N\lambda)^2} \sum_{n=0}^{O} n(n+1)P(Z_0 = n)
\]

\[
= \frac{1}{(N\lambda)^2} \left( E[Z_0^2] + E[Z_0] \right).
\]
Using (A.18) and (A.19), we get the variance of the cycle time for the hybrid policy and Policy 0 as follows:

\[
\sigma^2 = \text{Var}[\tau] = E[\tau^2] - E^2[\tau] = \left(\frac{1}{N \lambda}\right)^2 \left( E[Z_0^2] + E[Z_0] - E^2[Z_0] \right)
\]

\[
= \left(\frac{1}{N \lambda}\right)^2 \left( \text{Var}[Z_0] + E[Z_0] \right).
\]

(A.20)

Equations (A.17), (A.18), and (A.20) yield the following for the demand rate variance at the supplier for the hybrid policy and Policy 0 as follows:

\[
\text{Var}(X) = N \lambda \left(1 + 2 \frac{\text{Var}(Z_0)}{E(Z_0)} \right)
\]

The following table summarizes the variance and mean of \(X\) for all policies:

<table>
<thead>
<tr>
<th></th>
<th>Hybrid policy</th>
<th>Policy 0</th>
<th>Policy 1</th>
<th>Policy 2</th>
<th>Non-coordinated Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(X)</td>
<td>(N \lambda)</td>
<td>(N \lambda)</td>
<td>(N \lambda)</td>
<td>(N \lambda)</td>
<td>(N \lambda)</td>
</tr>
<tr>
<td>Var(X)</td>
<td>(N \lambda \left(1 + 2 \frac{\text{Var}(Z_0)}{E(Z_0)} \right))</td>
<td>(N \lambda \left(1 + 2 \frac{\text{Var}(Z_0)}{E(Z_0)} \right))</td>
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