Finding cosmic filament by the directional ridge finding algorithm

Yen-Chi Chen

Department of Statistics
University of Washington
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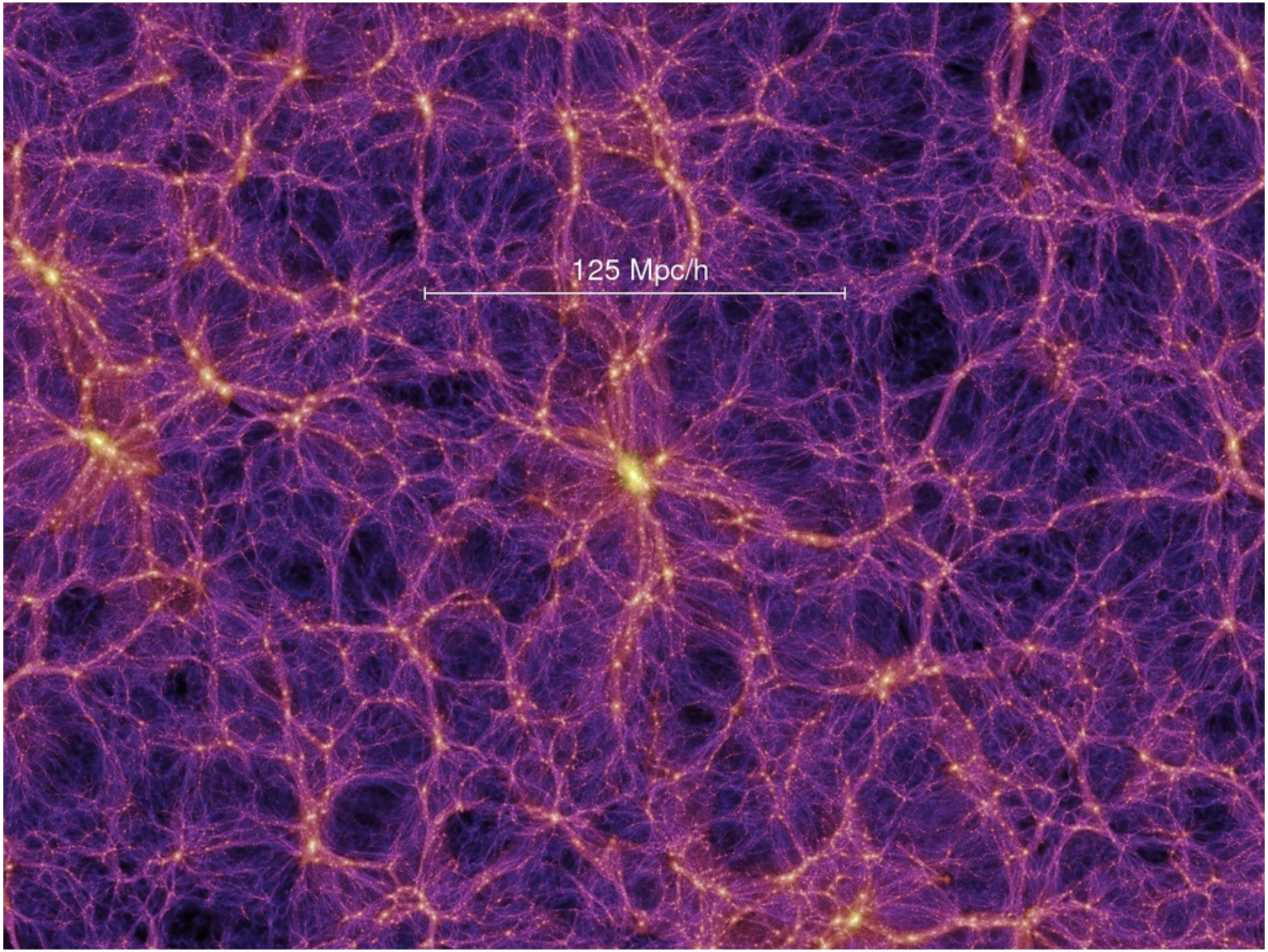
Joint work with Yikun Zhang

Cosmic Web: What Does Our Universe Look Like



Credit: Millennium Simulation

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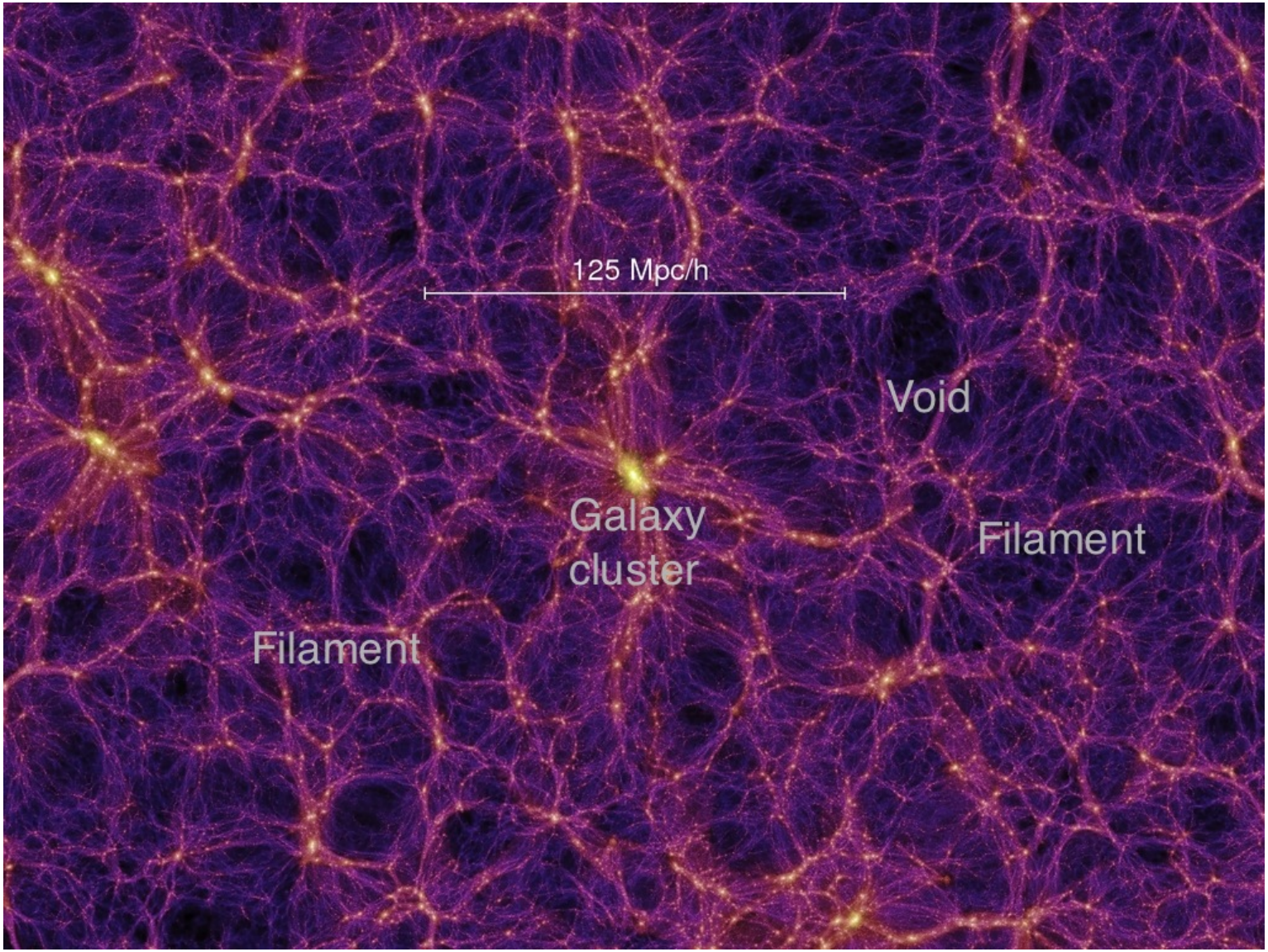

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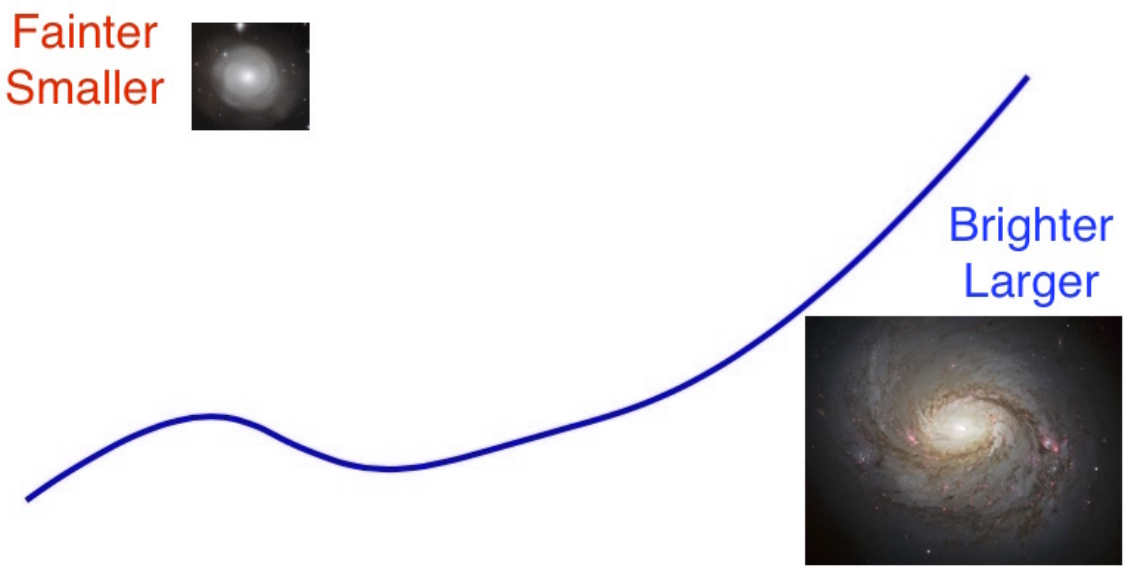
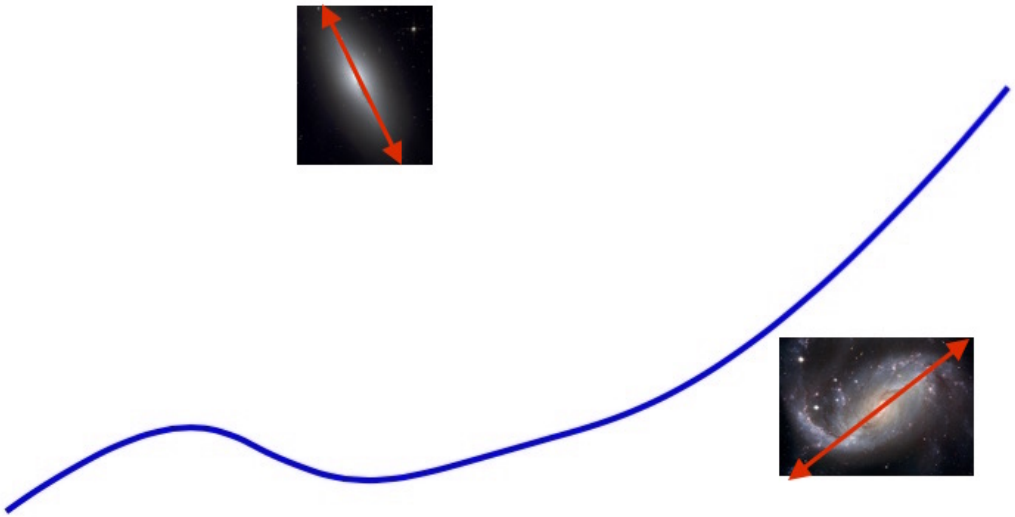
The Importance of Filaments

- A galaxy's brightness, size, and mass are associated with the distance to filaments.



The Importance of Filaments

- A galaxy's brightness, size, and mass are associated with the distance to filaments.
- A galaxy's alignment is associated with filaments.



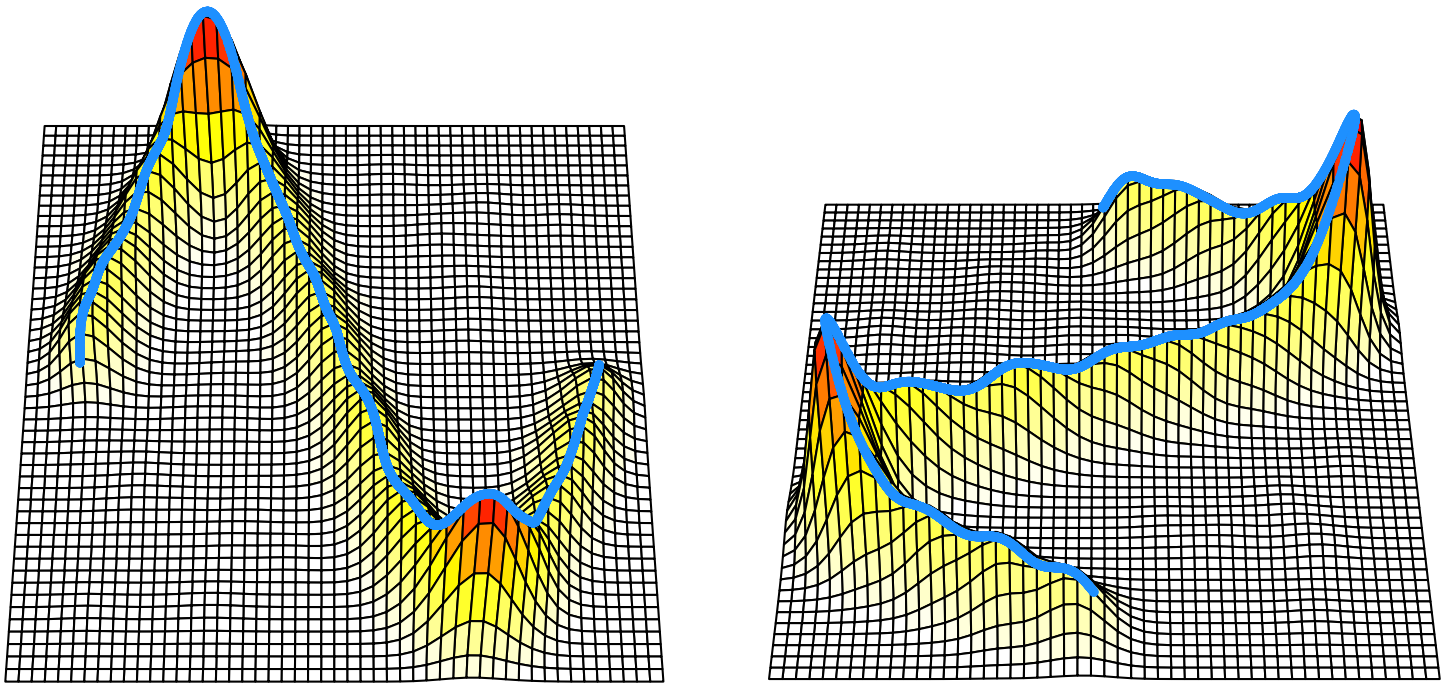
Density Ridges

We formalize the notion of filaments as *density ridges*.

Example: Ridges in Mountains

Credit: Google

Example: Ridges in Smooth Functions



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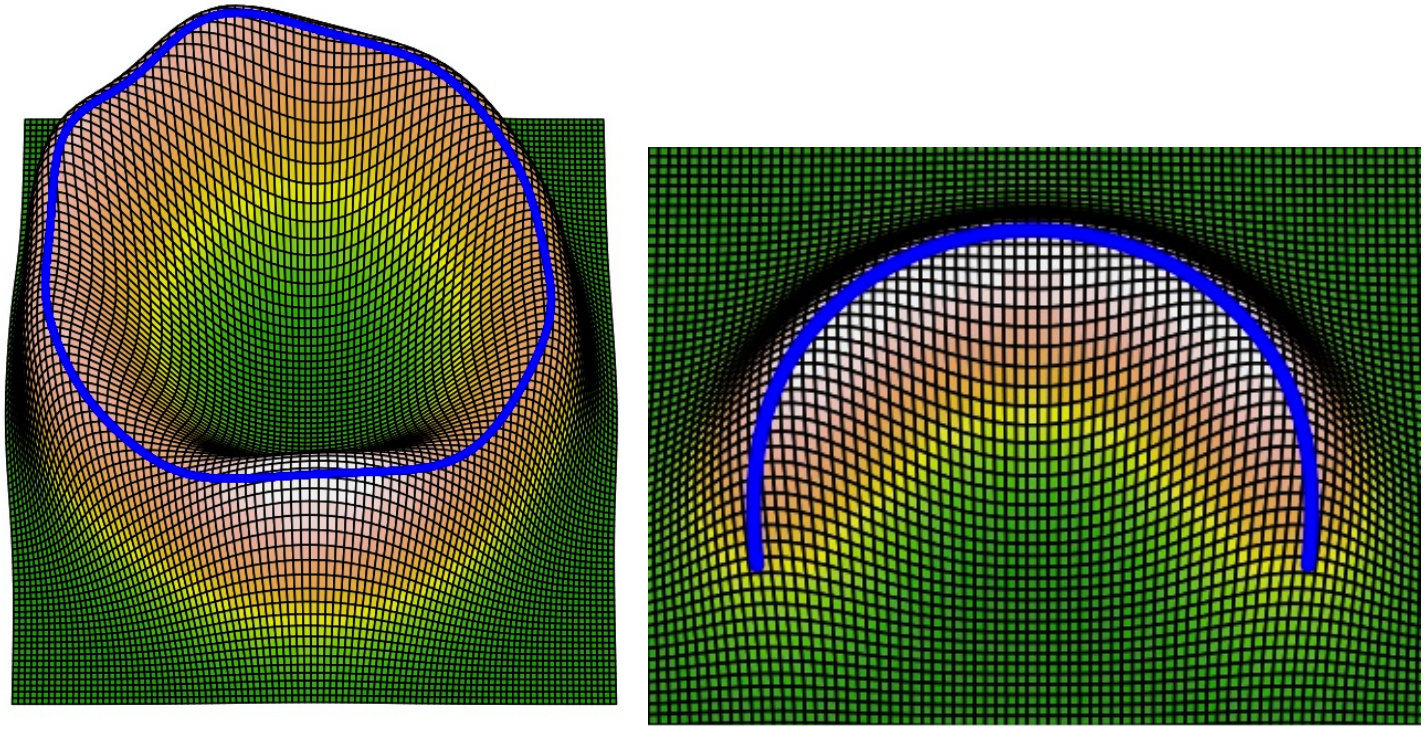
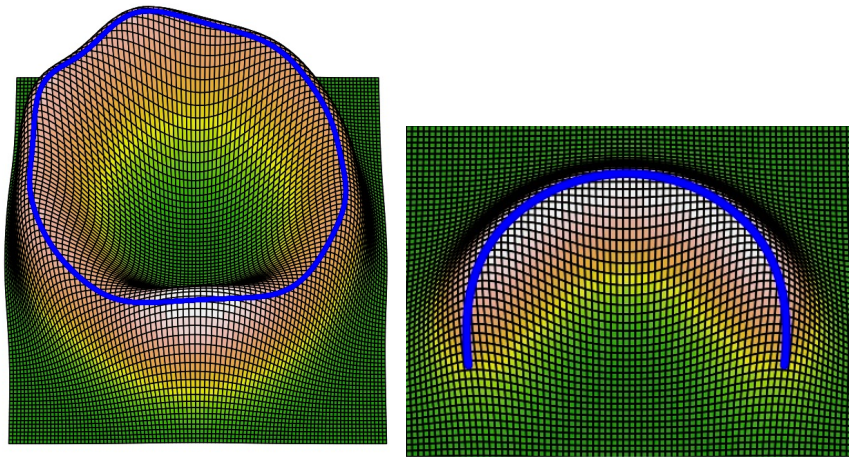
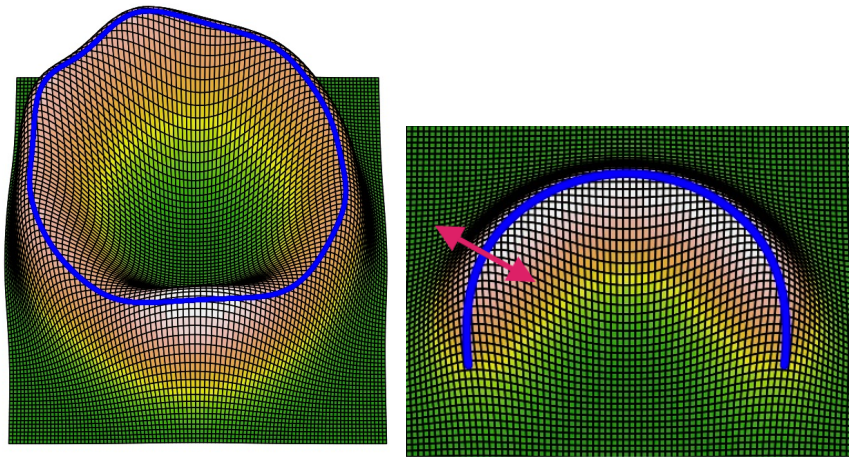


Ridges: Local Modes in Subspace



- A generalized local mode in a specific 'subspace'.

Ridges: Local Modes in Subspace



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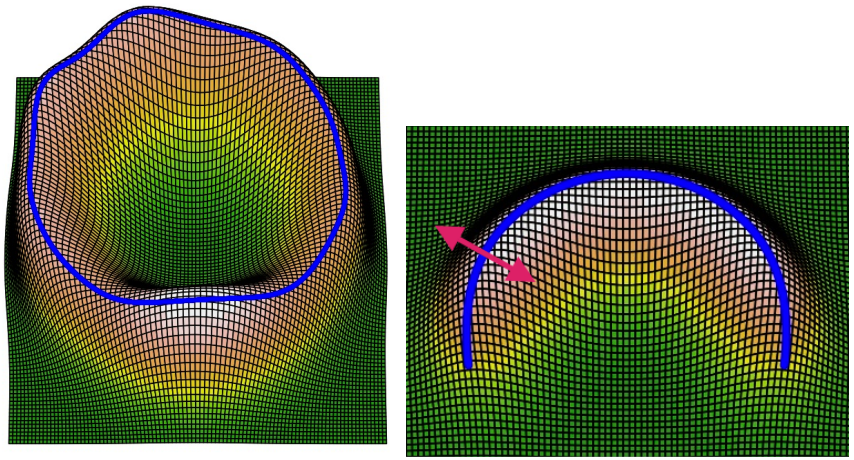
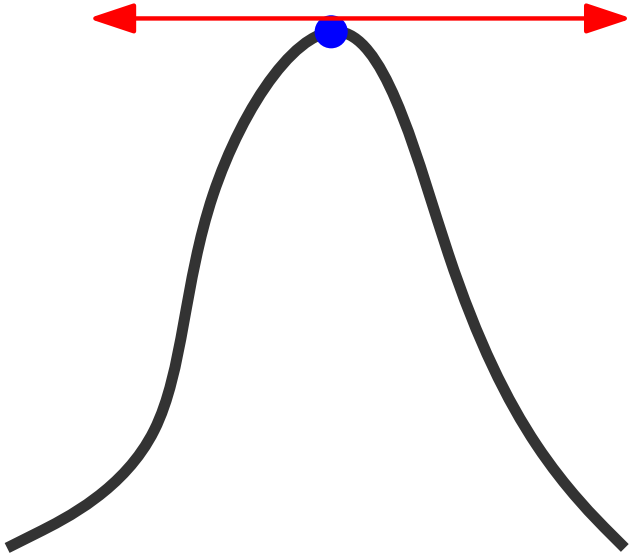
Ridges: Local Modes in Subspace





- A generalized local mode in a specific 'subspace'.

Formal Definition of Density Ridges

- $p : \mathbb{R}^d \mapsto \mathbb{R}$, the density function.
- $(\lambda_j(x), v_j(x))$: jth eigenvalue/vector of $H(x) = \nabla\nabla p(x)$.
- $V(x) = [v_2(x), \cdots, v_d(x)]$: matrix of the 2nd eigenvector to the last eigenvector.
- $V(x)V(x)^T$: a projection.
- Ridges:

$$R = \text{Ridge}(p) = \{x : V(x)V(x)^T \nabla p(x) = 0, \lambda_2(x) < 0\}.$$

- Local modes:

$$\text{Mode}(p) = \{x : \nabla p(x) = 0, \lambda_1(x) < 0\}.$$

Dimension of Ridges

The dimension of a ridge is 1.

This is because ridges are points satisfying $V(x)V(x)^T \nabla p(x) = 0$.

$V(x)V(x)^T$ has rank $d - 1$, so there are $d - 1$ effective constraints.

By the Implicit Function Theorem, ridges have dimension 1.

Estimator and Algorithm

We use the plug-in estimate:

$$\widehat{R}_n = \text{Ridge}(\widehat{p}_n),$$

where $\widehat{p}_n(x) = \frac{1}{nh^d} \sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)$ is the kernel density estimator (KDE) and $X_1, \cdots, X_n$ are the locations of galaxies.

[1]Ozertem, Umut, and Deniz Erdogmus. "Locally defined principal curves and surfaces." JMLR (2011).

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- In general, finding ridges from a given function is hard.
- The Subspace Constraint Mean Shift[1] (SCMS) algorithm allows us to find $\widehat{R}_n$, ridges of the KDE.

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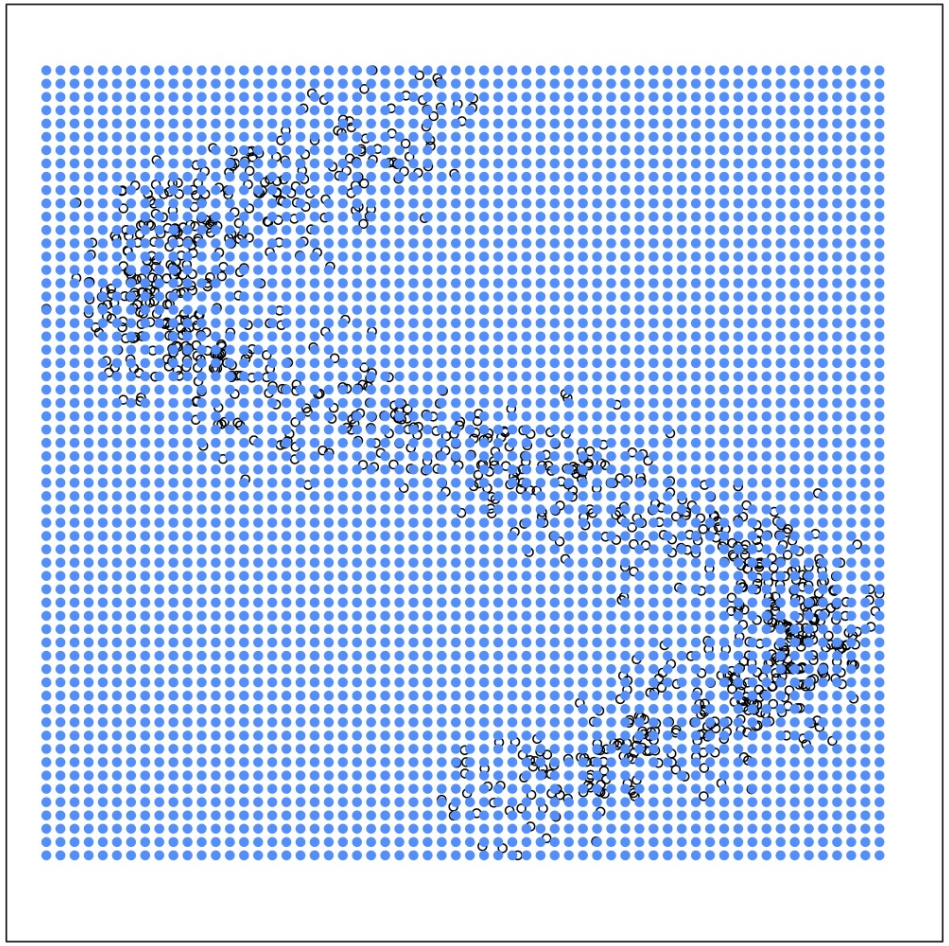
SCMS: Ridge Recovery Algorithm



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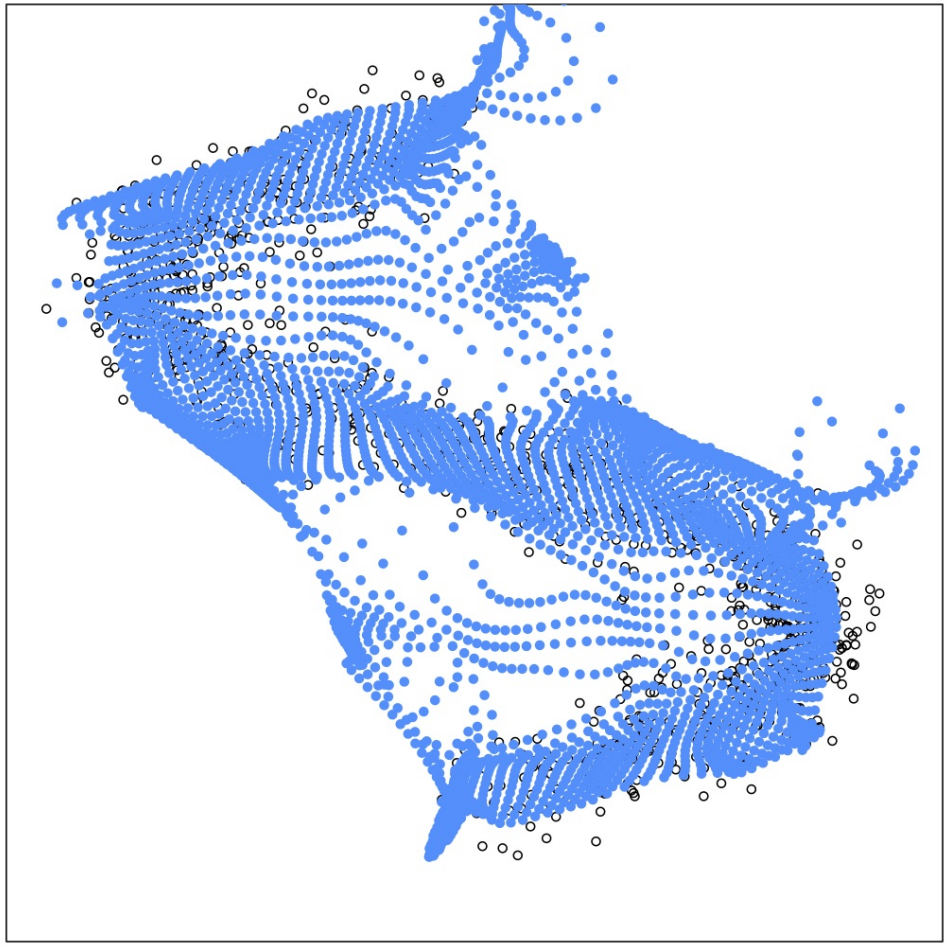

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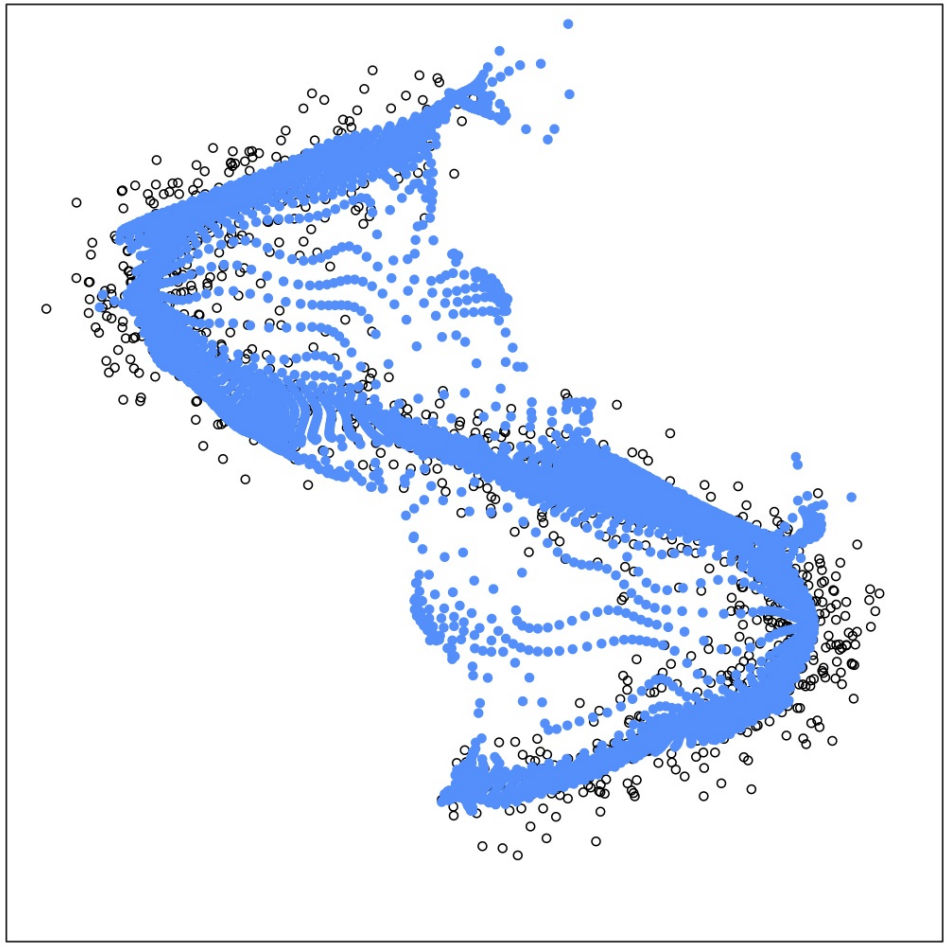

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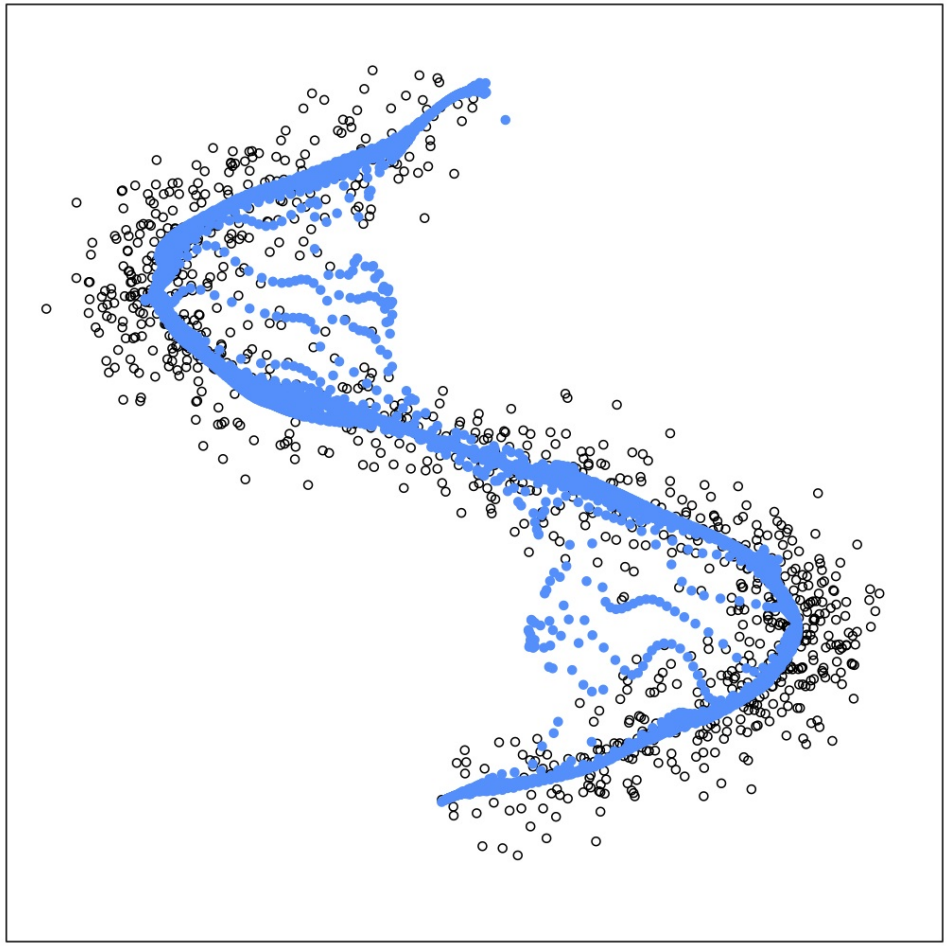

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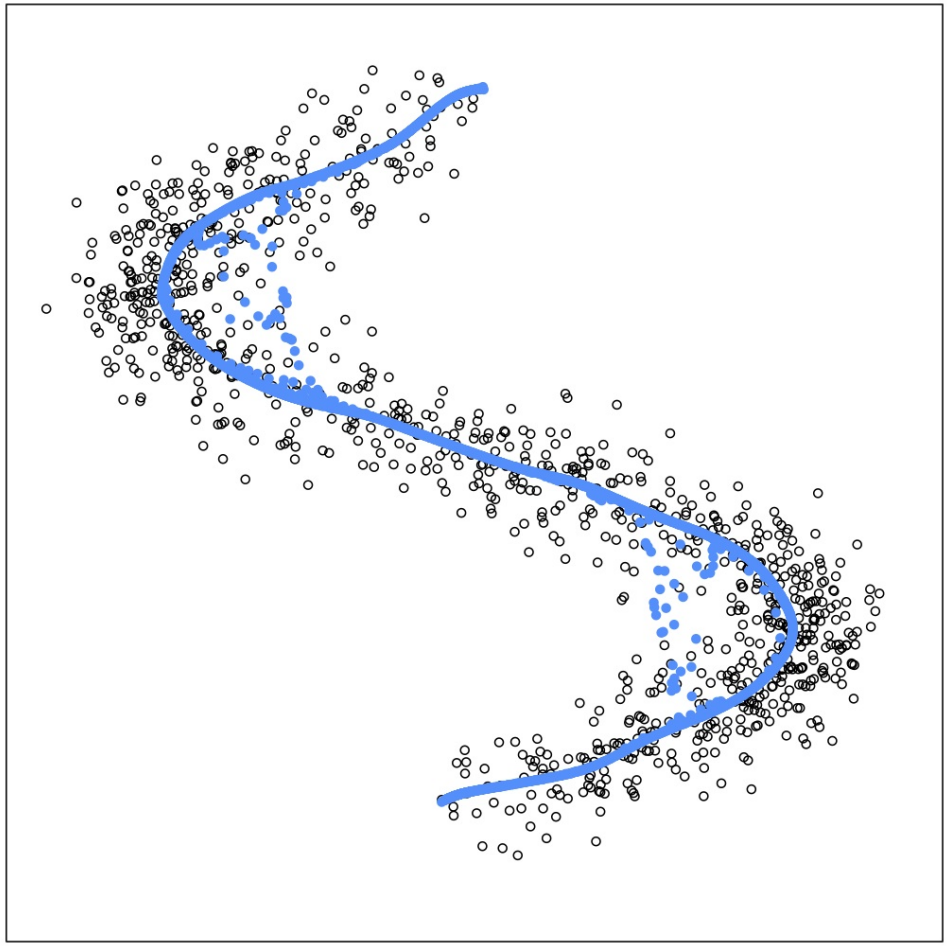

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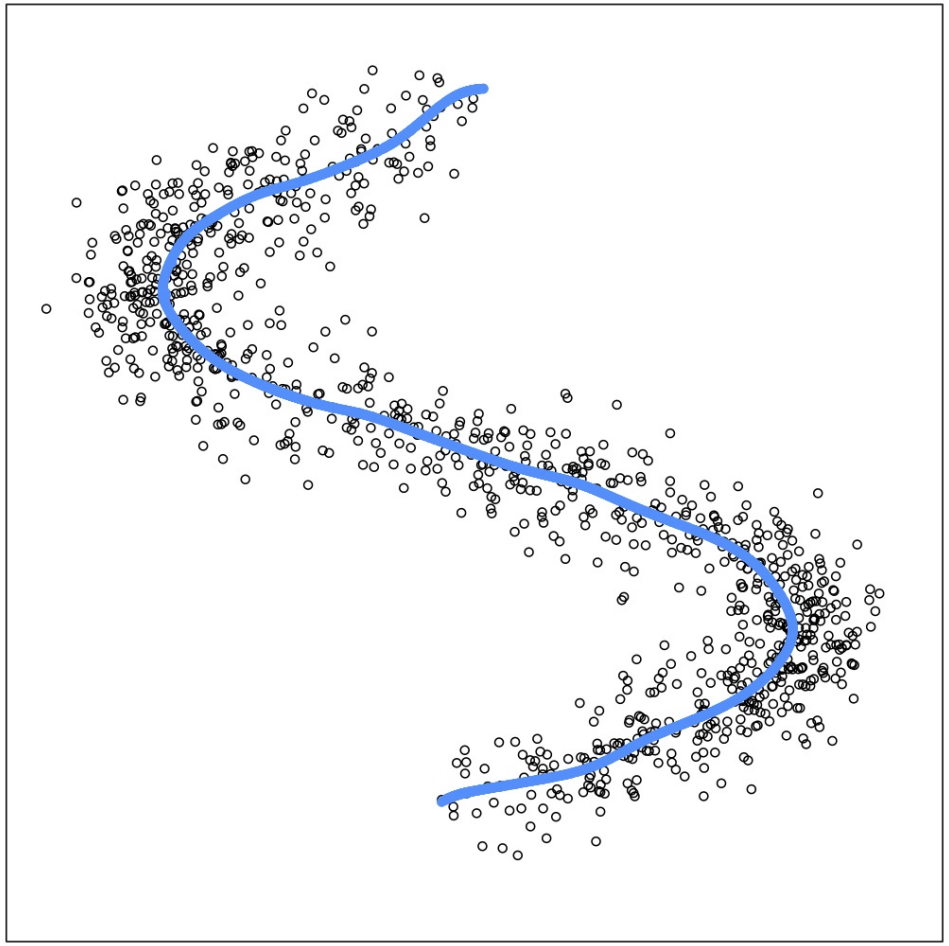

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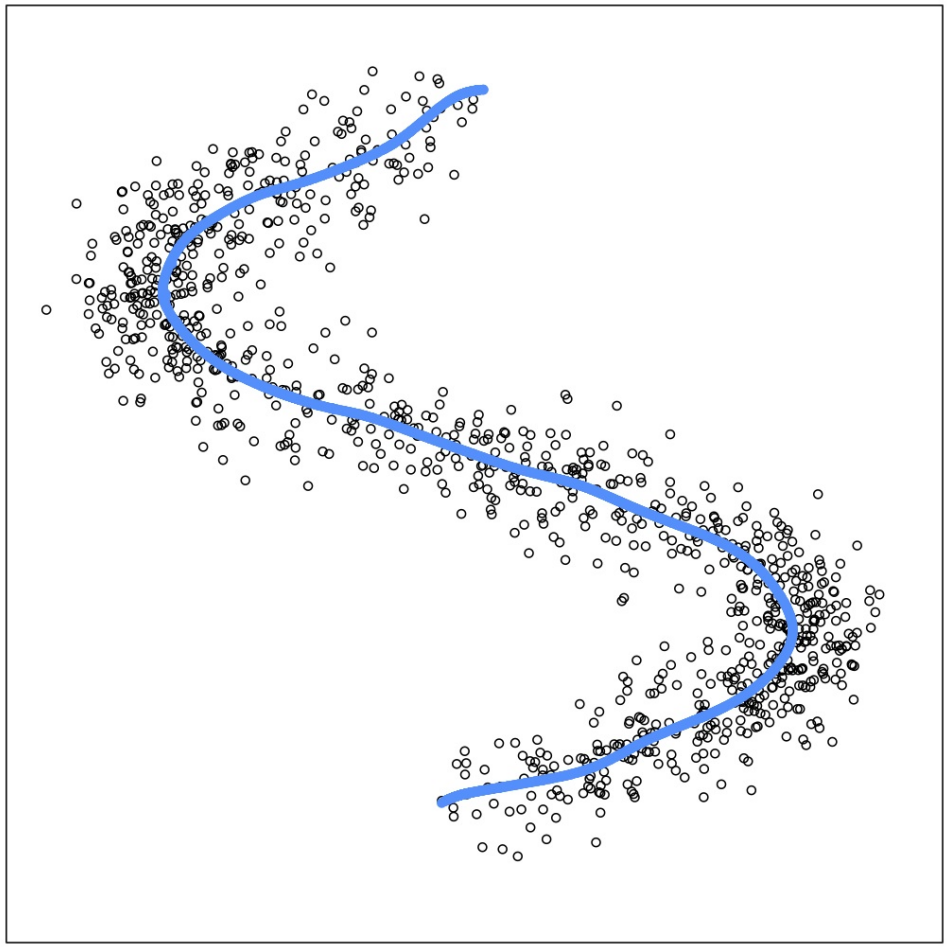
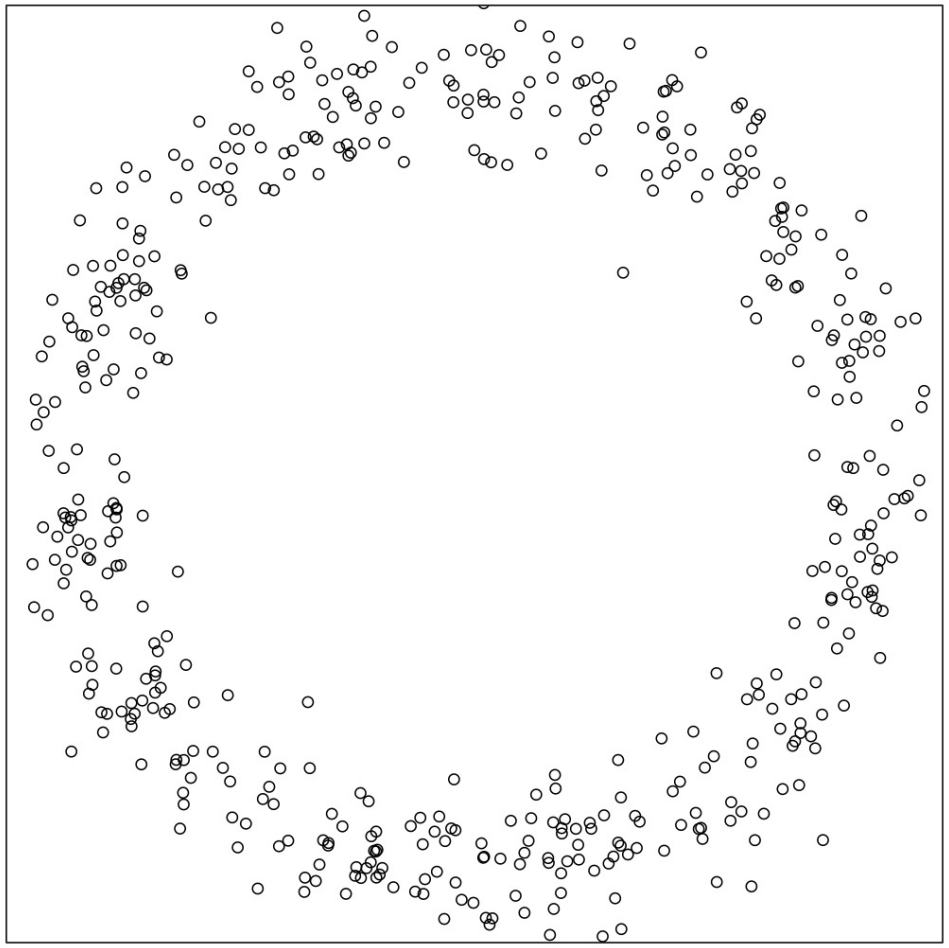


SCMS moves blue mesh points by gradient ascent and a projection.

Example for Estimated Density Ridges



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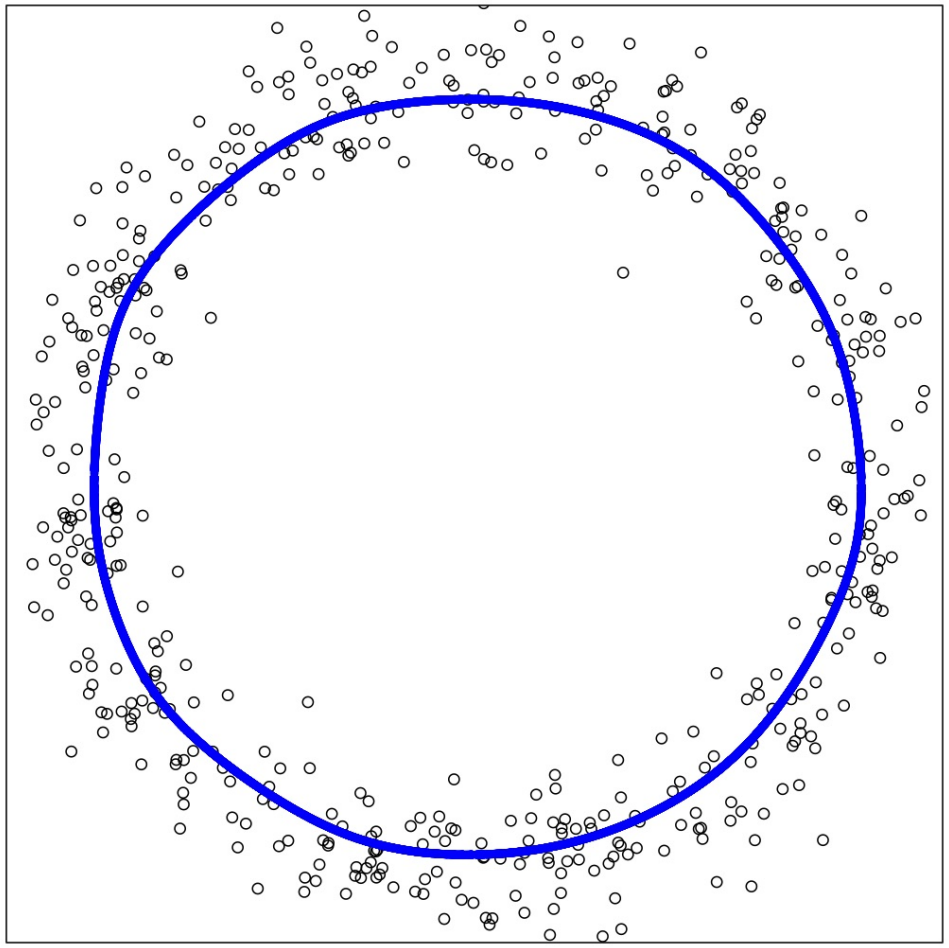

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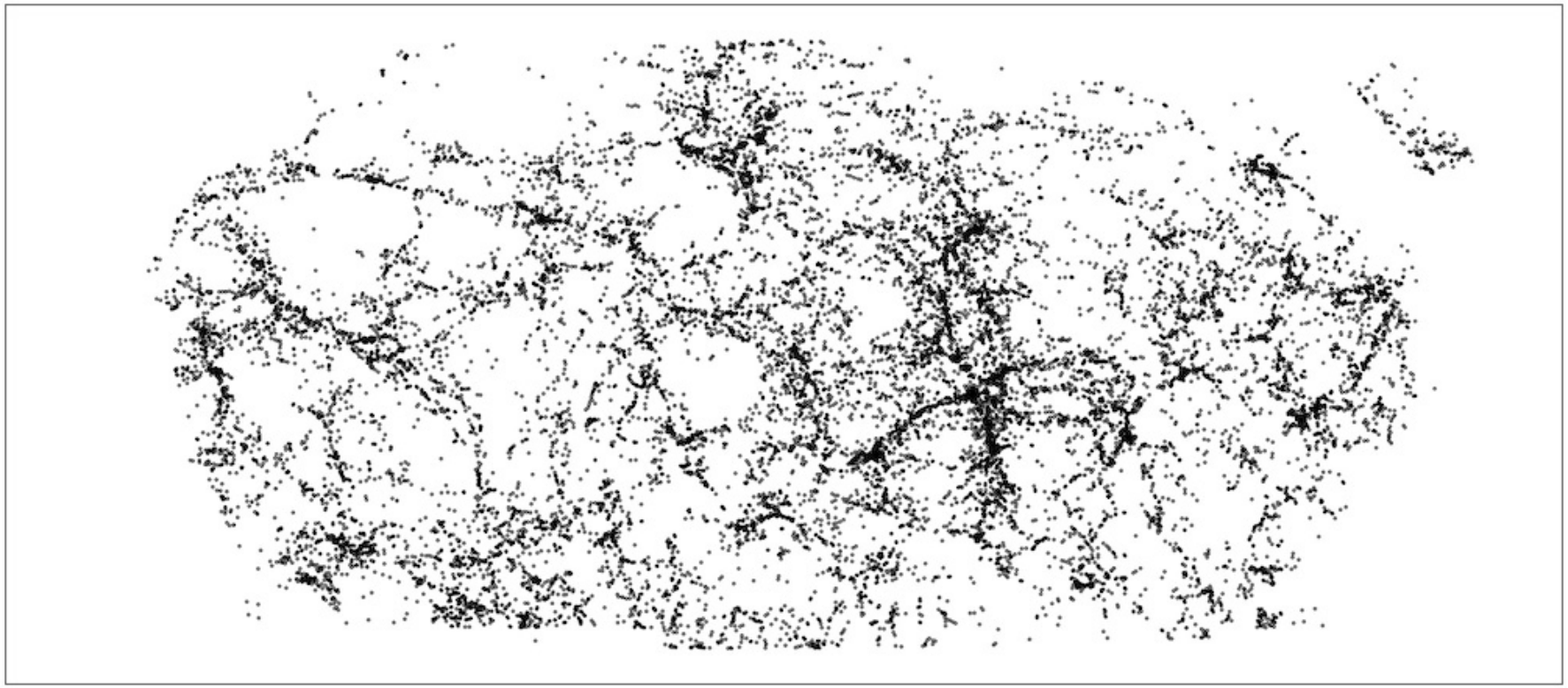

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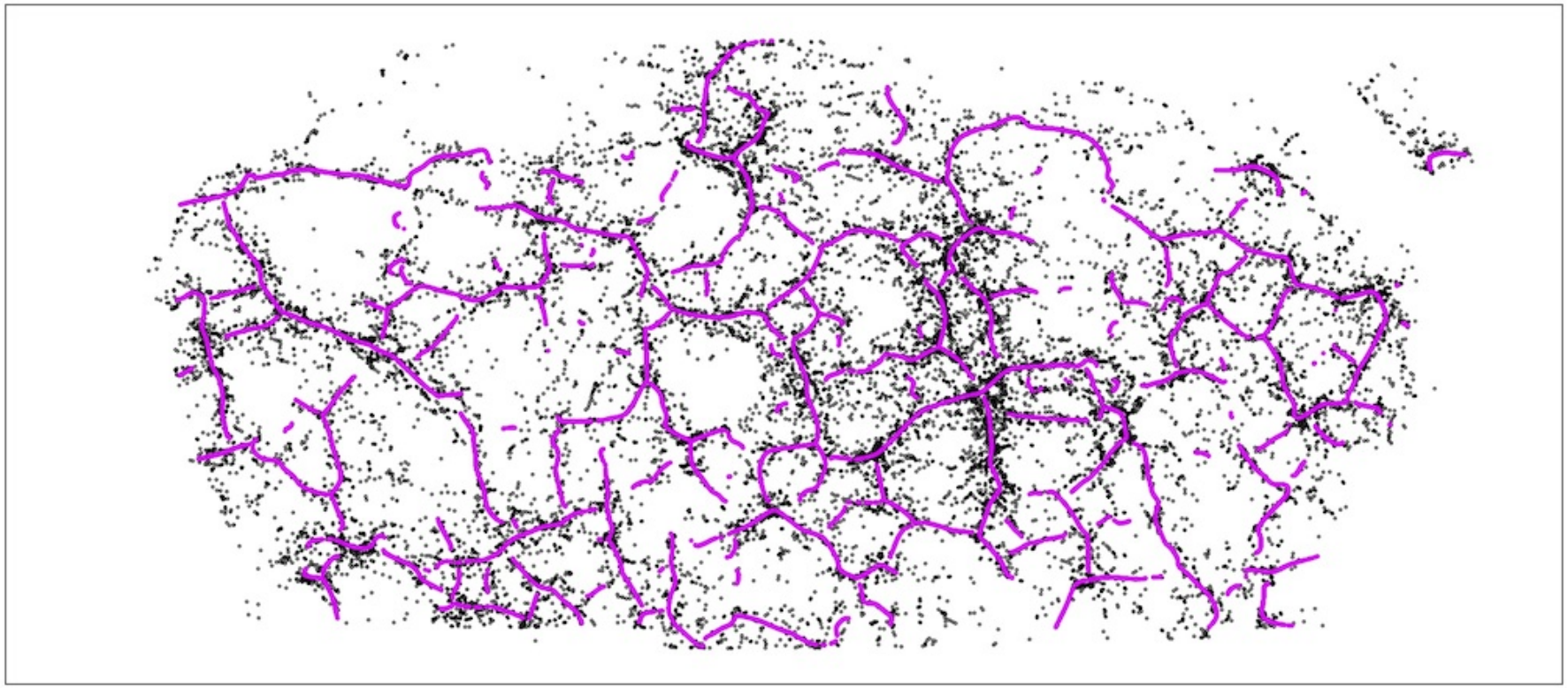
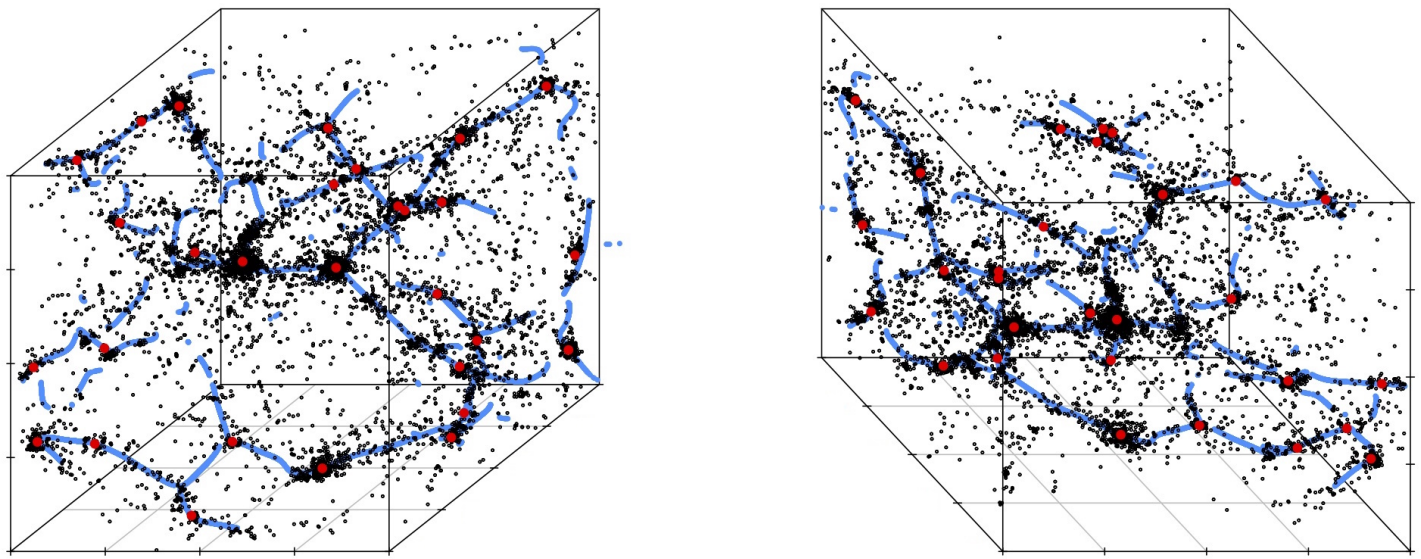


3D Example for Estimated Ridges



Blue curves: density ridges.

Red points: density local modes.

Formal definition of the SCMS algorithm

- Starting at an initial point $x^{(0)}$, the SCMS algorithm generates a sequence of points $x^{(1)}, x^{(2)}, \cdots$ via the following updating procedure:

$$x^{(t+1)} = x^{(t)} + \eta \widehat{V}(x^{(t)}) \widehat{V}(x^{(t)})^T \nabla \widehat{p}_n(x^{(t)})$$

  for $t = 0, 1, 2, 3, \cdots$.
- The tuning parameter $\eta > 0$ is the step size.

Convergence of the SCMS algorithm

- Let $x^{(\infty)} \in \widehat{R}_n$ be its destination.

Theorem (Linear convergence of SCMS)

Under suitable conditions and $\|x^{(0)} - x^{(\infty)}\|_2 \leq r_0$, *we have*

$$\|x^{(t)} - x^{(\infty)}\|_2 \leq \Gamma^t \|x^{(0)} - x^{(\infty)}\|_2,$$

where $\Gamma \in (0, 1)$.

- We provide an explicit description of Γ, r_0 in our paper.
- Technical challenge: the projection matrix $\widehat{V}(x^{(t)})\widehat{V}(x^{(t)})^T$ also depends on the current location $x^{(t)}$, so we have to bound this difference as well.

Asymptotic Theory

To study the uncertainty of the ridge estimator, we use *the Hausdorff distance* to quantify the errors: $\mathsf{Haus}(\widehat{R}_n, R)$.

$\longrightarrow$Hausdorff distance is like an L_∞ distance for sets.

Key observation:

$$\begin{aligned}\mathsf{Haus}(\widehat{R}_n, R) &\approx \sup\{\text{Empirical process on } R\} \\ &\approx \sup\{\text{Gaussian process on } R\}.\end{aligned}$$

Theorem

Under regularity conditions and $\frac{\log n}{nh^{d+8}} \to 0$, *there exists a Gaussian process* $\mathbb{B}_n$ *defined on a certain function space* $\mathcal{F}$ *such that*

$$\sup_t \left| \mathbb{P}\left(\sqrt{nh^{d+2}}\mathsf{Haus}(\widehat{R}_n, R) < t\right) - \mathbb{P}\left(\sup_{f\in\mathcal{F}} |\mathbb{B}_n(f)| < t\right)\right| = O\left(\left(\frac{\log^7 n}{nh^{d+2}}\right)^{1/8}\right).$$

Confidence Set: $\oplus$ operation

We define $A \oplus r = \{x : d(x, A) \leq r\}$.

Then we have the following inclusion property:

$$A \subset B \oplus \mathsf{Haus}(A, B), \quad B \subset A \oplus \mathsf{Haus}(A, B).$$

This also implies

$$R \subset \widehat{R}_n \oplus \mathsf{Haus}(\widehat{R}_n, R).$$

Bootstrap Consistency

Let $\widehat{R}_n^*$ be the bootstrap ridges. We proved that

$$\begin{aligned}\mathsf{Haus}(\widehat{R}_n^*, \widehat{R}_n) &\approx \sup\{\text{Gaussian process on } \widehat{R}_n\} \\ &\approx \sup\{\text{Gaussian process on } R\} \\ &\approx \mathsf{Haus}(\widehat{R}_n, R).\end{aligned}$$

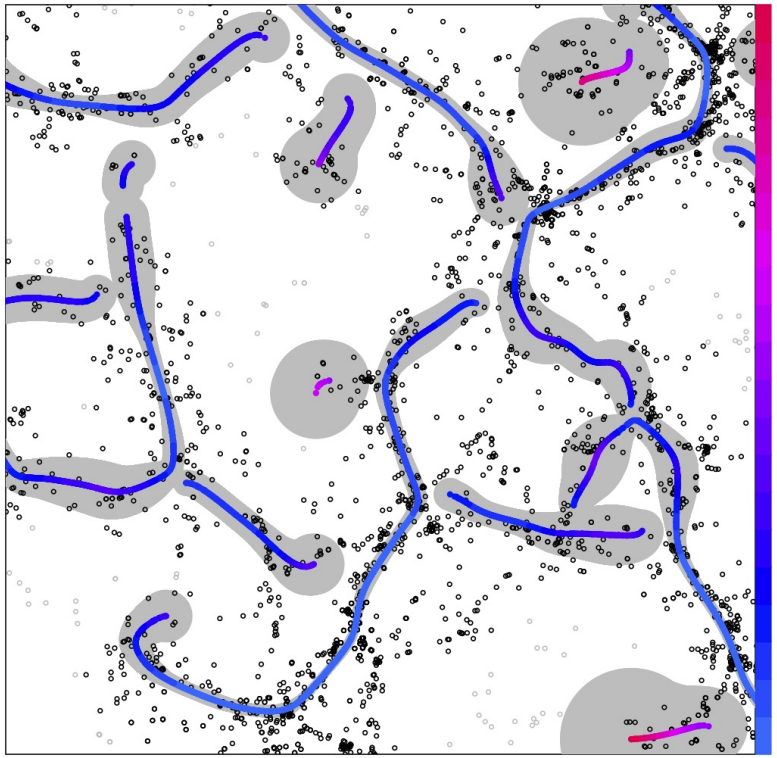
Theorem

Under regularity conditions and $\frac{\log n}{nh^{d+8}} \to 0$,

$$\mathbb{P}\left(R \subset \widehat{R}_n \oplus \widehat{t}_{1-\alpha}\right) = 1 - \alpha + O\left(\left(\frac{\log^7 n}{nh^{d+2}}\right)^{1/8}\right),$$

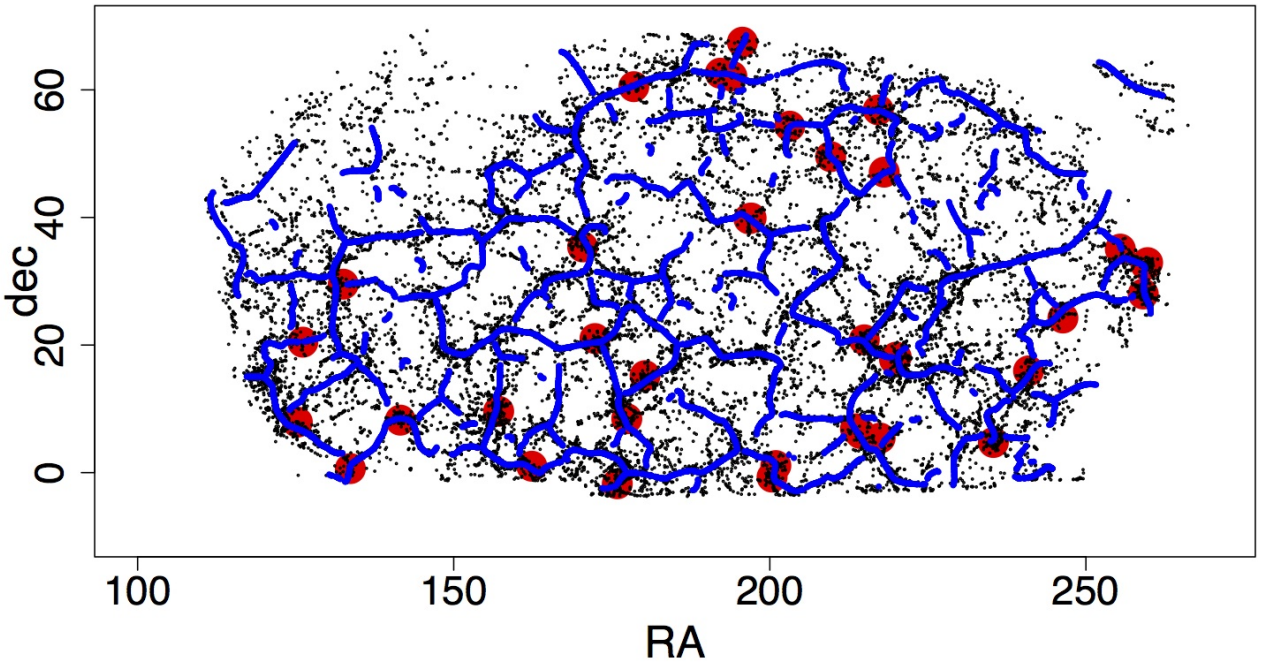
where $\widehat{t}_{1-\alpha}$ = *is the* $1-\alpha$ *quantile of* $\mathsf{Haus}(\widehat{R}_n^*, \widehat{R}_n)$.

Uncertainty of Ridges



SDSS: Comparing to Clusters

- Blue: filaments. Red: galaxy clusters (redMaPPer).



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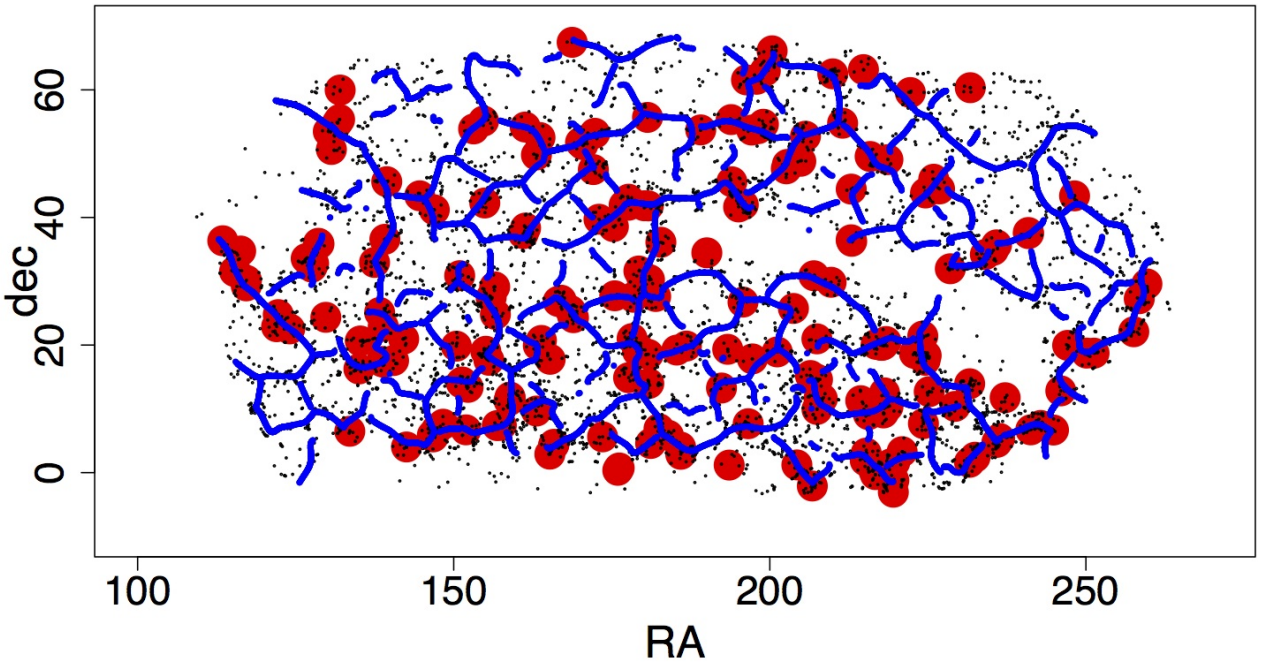

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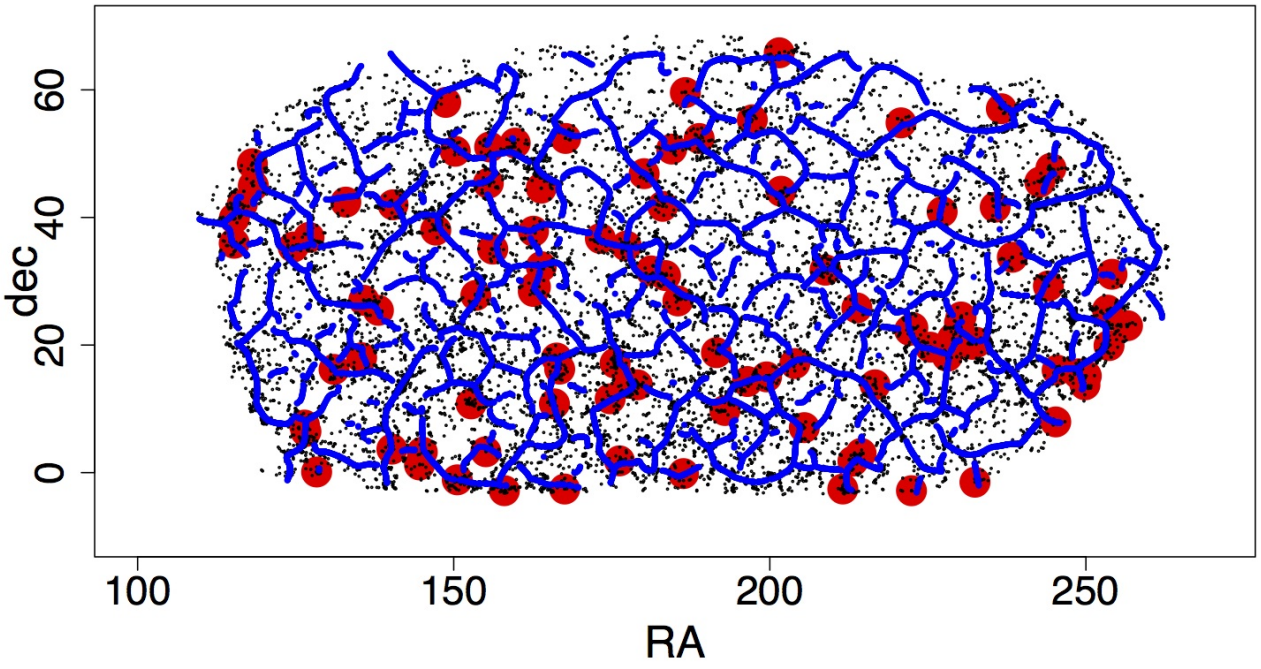
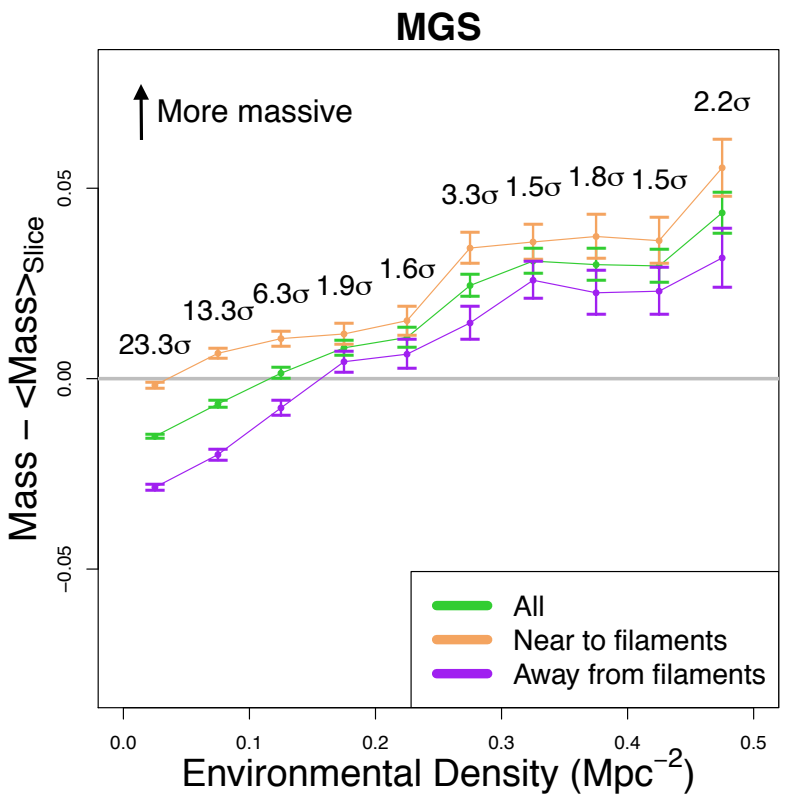


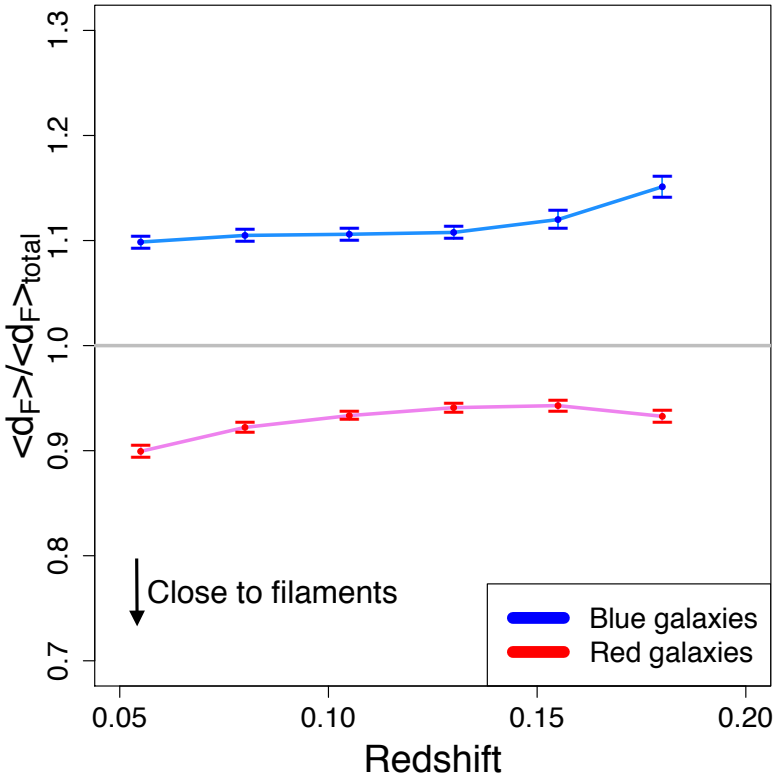
SDSS: Filament Effects VS Environments

Do filaments have an extra effect other than environments?

⟶Yes!



SDSS: Color



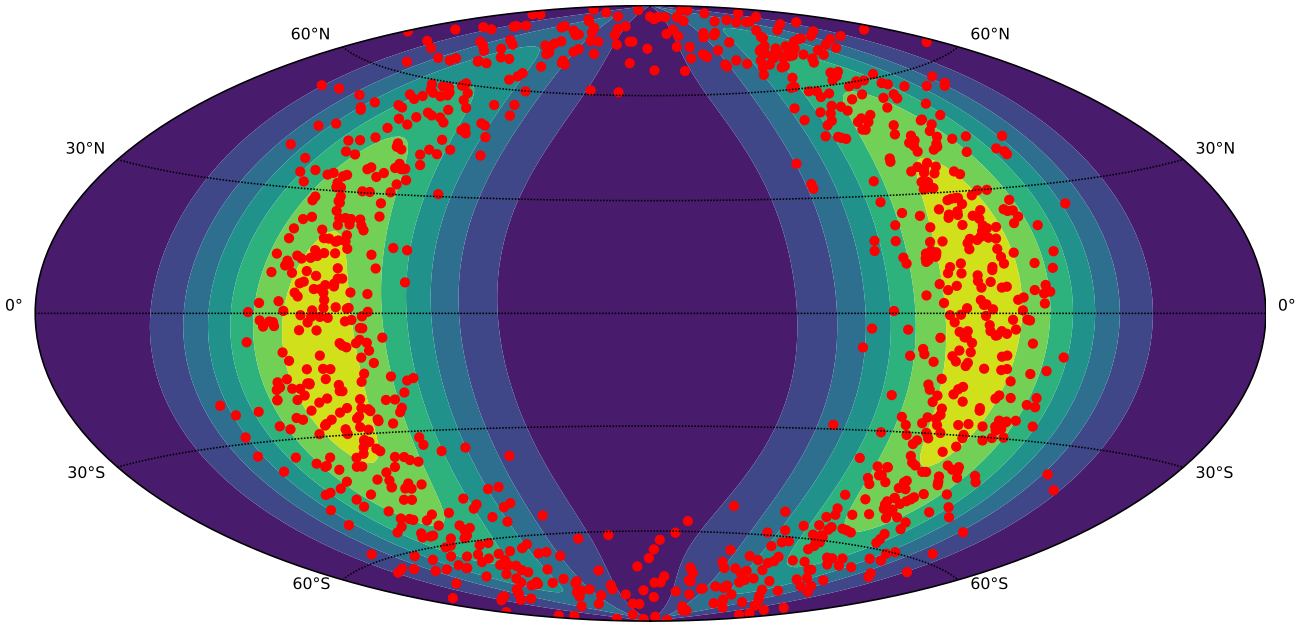
Similar pattern also appears for other galaxy properties such as brightness, size, and age.

Accounting for the spherical geometry

Directional data

- While the above results seem to be good, it has a severe problem: our data (locations of galaxies) is not in Euclidean space.
- In particular, we use (RA, dec) to represent the location of a galaxy.
- (RA, dec) are spherical coordinate!
- The Euclidean ridge finding algorithm may lead to a severe bias.

Failure of usual SCMS



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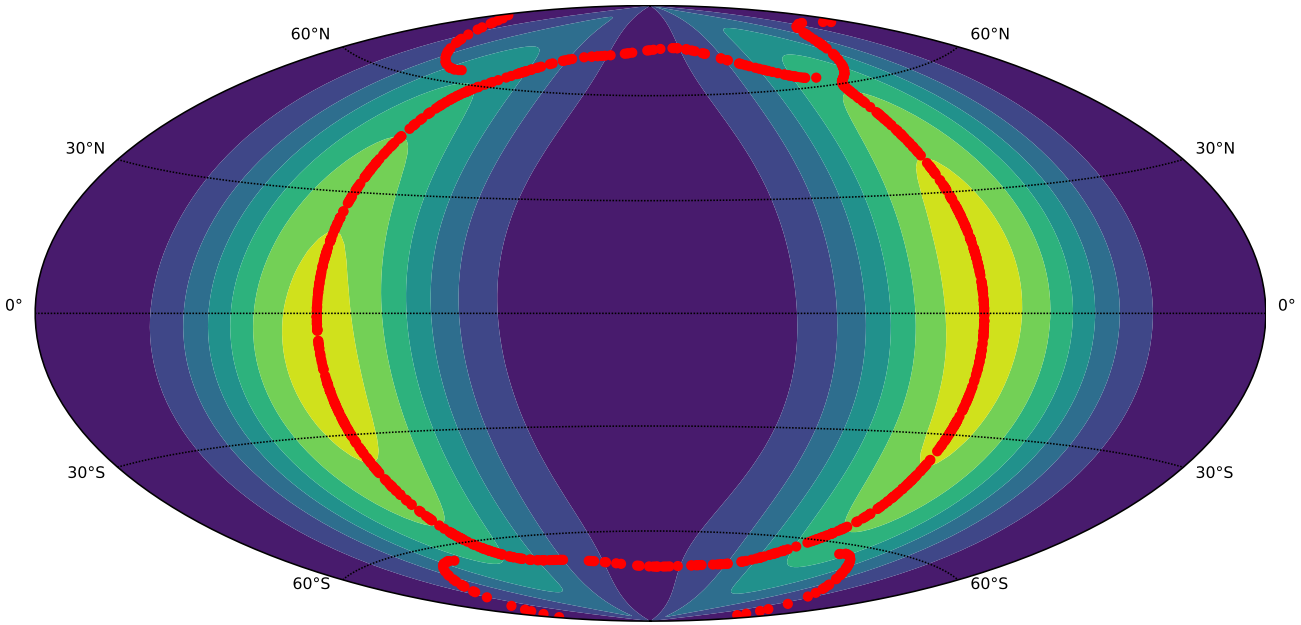

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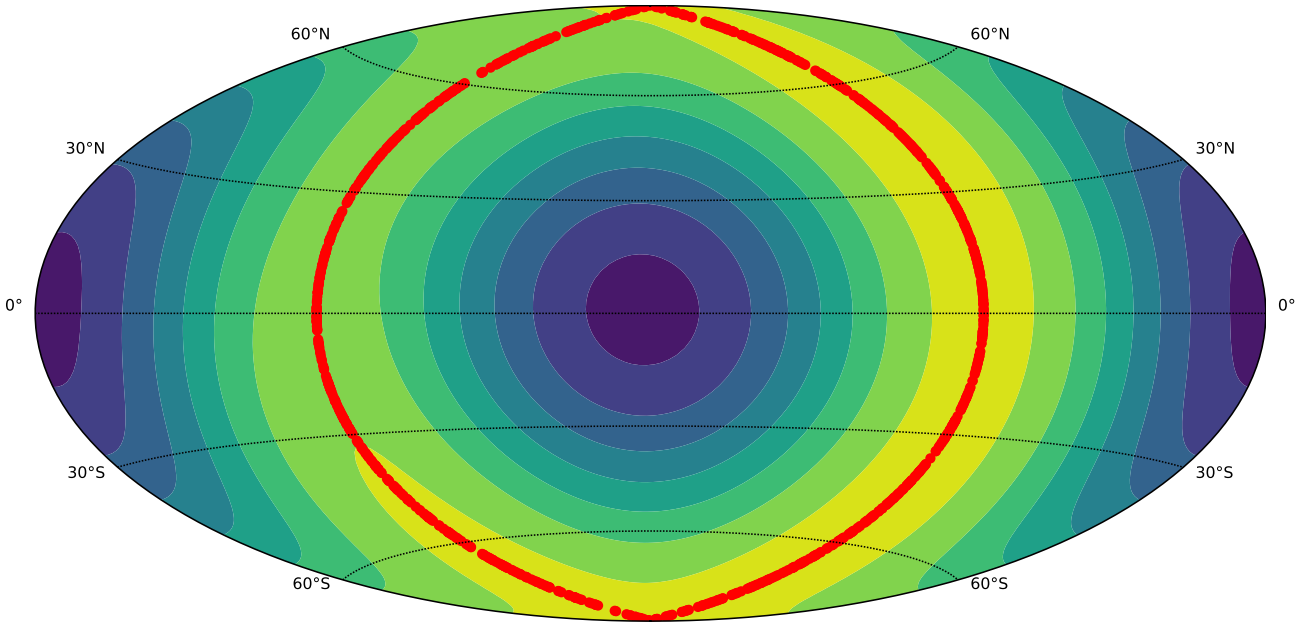


Directional ridges - 1

- Let $X_1, \cdots, X_n \in \Omega_q$, where $\Omega_q = \{x \in \mathbb{R}^{q+1} : \|x\|_2 = 1\}$ be the directional data on q-dimensional sphere.
- To define ridges on Ω_q, we need to use gradient on a Riemannian manifold.
- Luckily, in this case, we have a simple representation of the gradient on Riemannian manifold grad using the usual gradient operator ∇ (in $(q+1)$-dimension):

$$\mathsf{grad} f(x) = (I_{q+1} - xx^T)\nabla f(x),$$

  where $x \in \mathbb{R}^{q+1}$ and $I_{q+1} = \mathsf{diag}(1, 1, \cdots, 1) \in \mathbb{R}^{(q+1)\times(q+1)}$.
- When $x \in \Omega_q$, this will lead to the gradient on a Riemannian manifold.

Directional ridges - 2

- With the above representation, the Hessian on Riemannian manifold can be expressed as

$$\mathcal{H} f(x) = (I_{q+1} - xx^T)\nabla\nabla f(x)(I_{q+1} - xx^T)$$

  when $x \in \Omega_q$.
- The directional ridges are then defined as

$$\underline{R} = \text{Ridge}(p) = \{x : \underline{V}(x)\underline{V}(x)^T \nabla p(x) = 0, \underline{\lambda}_2(x) < 0\},$$

  where $\underline{V}(x)$ is the matrix of the smallest (q-1) eigenvectors and $\underline{\lambda}_2(x)$ is the second largest eigenvalue of $\mathcal{H} p(x)$.

Directional ridges - 3

- In practice, we estimate p by the directional KDE:

$$\widehat{p}_{\mathsf{dir}}(x) = \frac{c_{L,q}(h)}{n} \sum_{i=1}^{n} L\left(\frac{1 - x^T X_i}{h^2}\right),$$

  where $c_{L,q}(h) = O(h^{-q})$ is the normalizing constant and L is the directional kernel.
- A popular choice is the von-Mises kernel, i.e., $L(r) = e^{-r}$.
- This leads to $\widehat{\mathscr{H} f}(x)$ and $\underline{\widehat{V}}(x)$ and $\underline{\widehat{\lambda}}_2(x)$ and $\underline{\widehat{R}}$.

Directional SCMS

- The SCMS algorithm can be generalized to a directional SCMS with some modifications.
- We showed that the directional SCMS can be expressed as the following fixed-point iteration (starting at $x^{(0)}$):

$$x^{(t+1)} = \frac{\underline{\widehat{V}}(x^{(t)})\underline{\widehat{V}}(x^{(t)})^T \nabla \widehat{p}_{\mathsf{dir}}(x^{(t)}) + \|\nabla \widehat{p}_{\mathsf{dir}}(x^{(t)})\|_2 \cdot x^{(t)}}{\|\underline{\widehat{V}}(x^{(t)})\underline{\widehat{V}}(x^{(t)})^T \nabla \widehat{p}_{\mathsf{dir}}(x^{(t)}) + \|\nabla \widehat{p}_{\mathsf{dir}}(x^{(t)})\|_2 \cdot x^{(t)}\|_2},$$

for $t = 0, 1, 2, 3, \cdots$.

Convergence of the directional SCMS algorithm

- Let $x^{(0)}$ be an initial point of the SCMS on Ω_q and let $x^{(\infty)} \in \underline{\widehat{R}}$ be its destination.

Theorem (Linear convergence of directional SCMS)

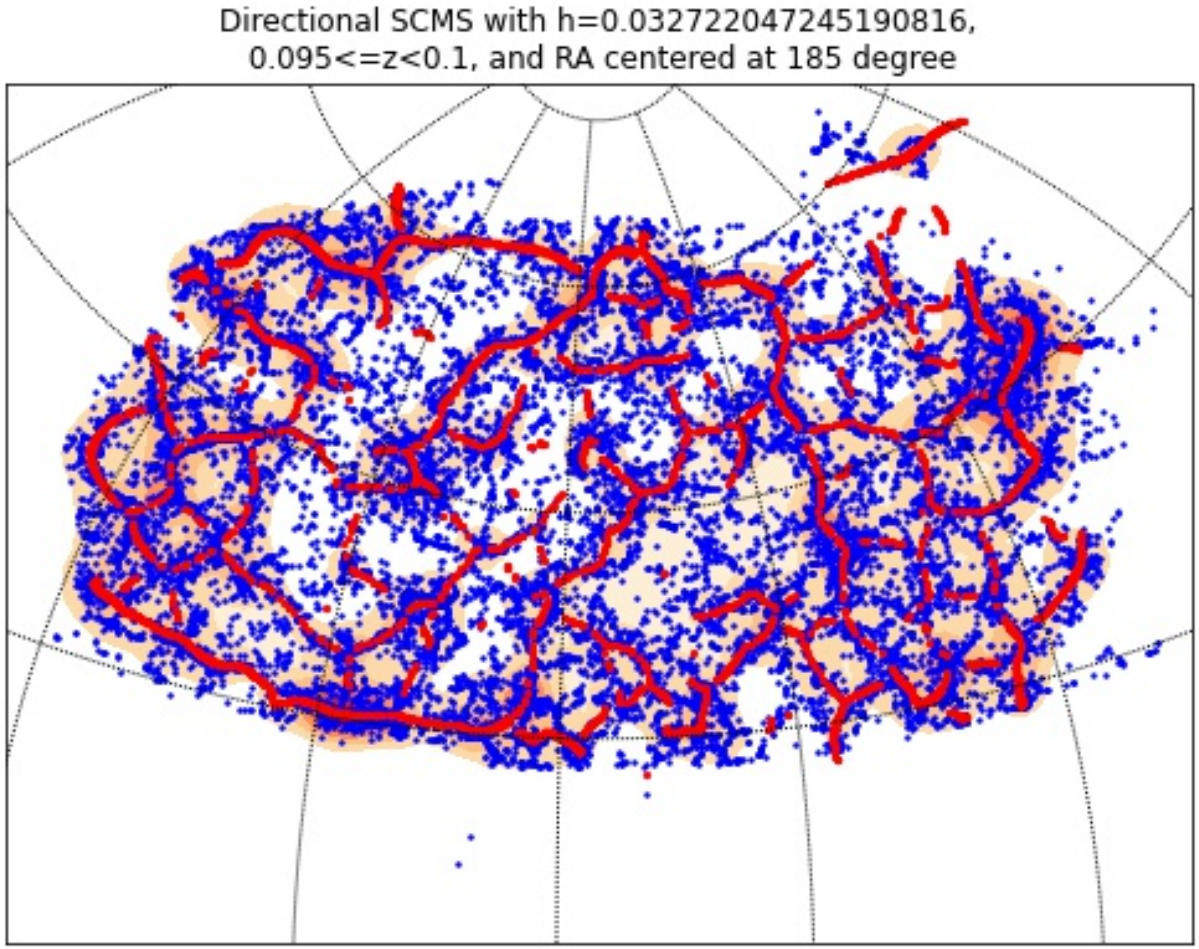
Under suitable conditions and $\|x^{(t)} - x^{(\infty)}\|_2 \leq r_{\mathsf{dir}}$, *we have*

$$\|x^{(t)} - x^{(\infty)}\|_2 \leq \Gamma_{\mathsf{dir}}^t \|x^{(0)} - x^{(\infty)}\|_2,$$

where $\Gamma_{\mathsf{dir}} \in (0, 1)$.

- We provide bounds on Γ_{dir} and r_{dir} in the paper.

Applying to the SDSS data



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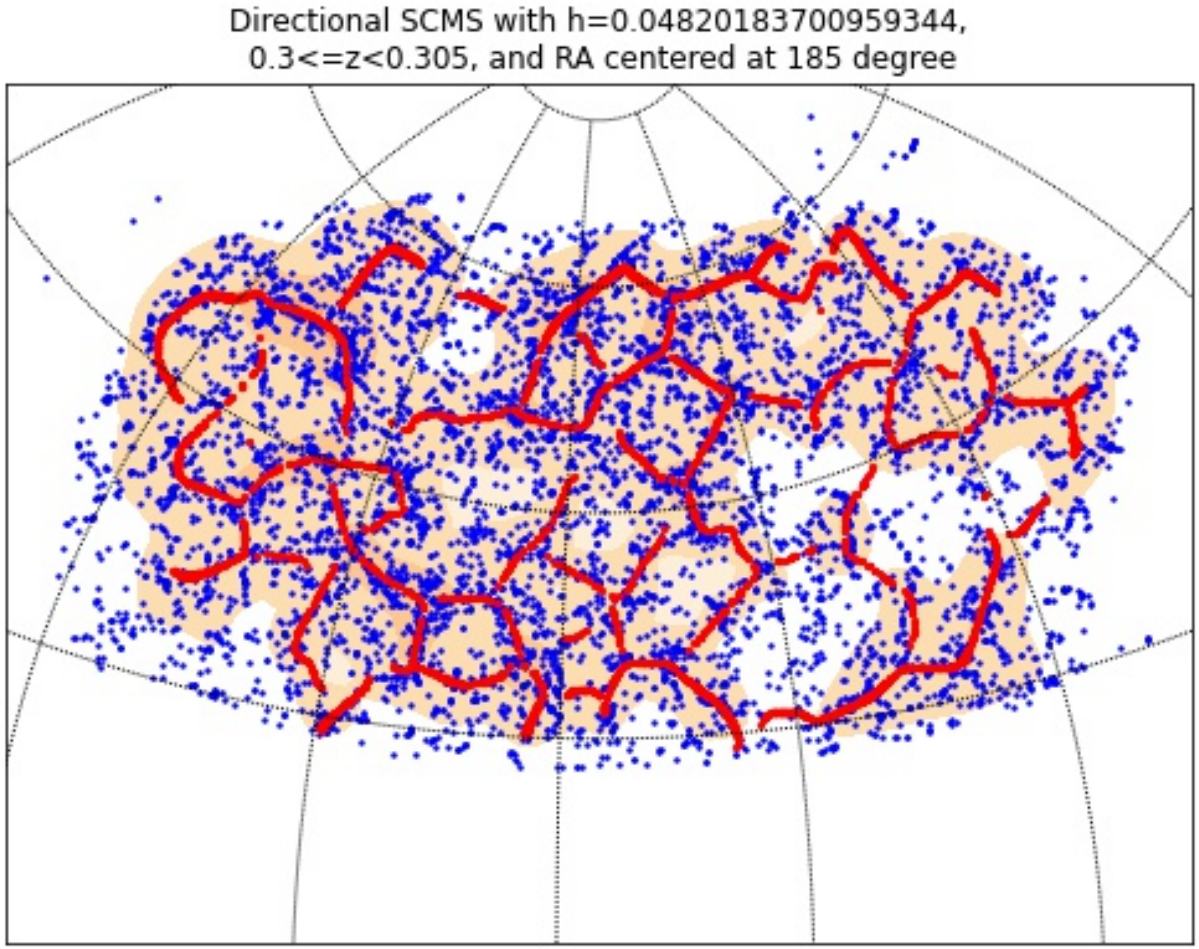

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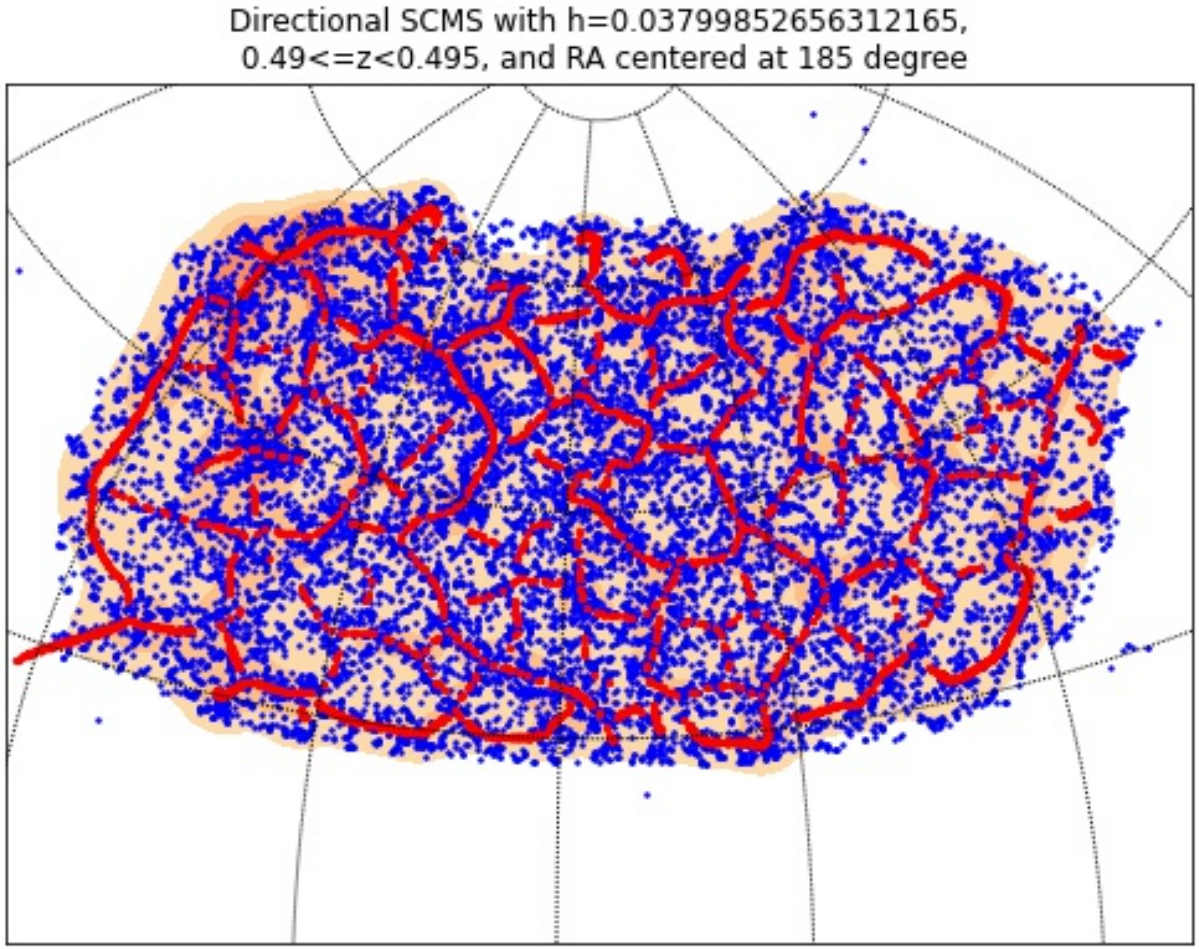


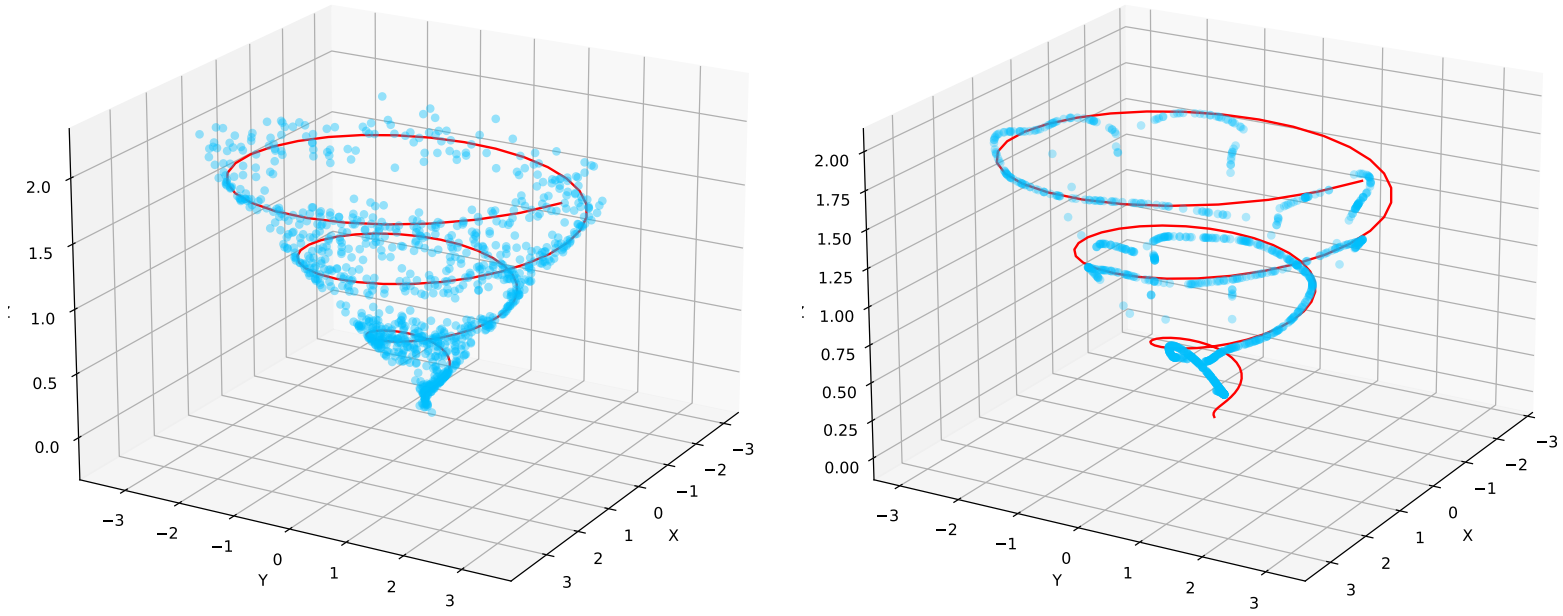
Incorporating the redshift

Incorporating the redshift information

- All the above approach is based on the idea of 'slicing the Universe'.
- Namely, we take slices based on redshift and find filaments in each slice.
- How to incorporate the information from redshift is a key problem.

Failure of a naive idea

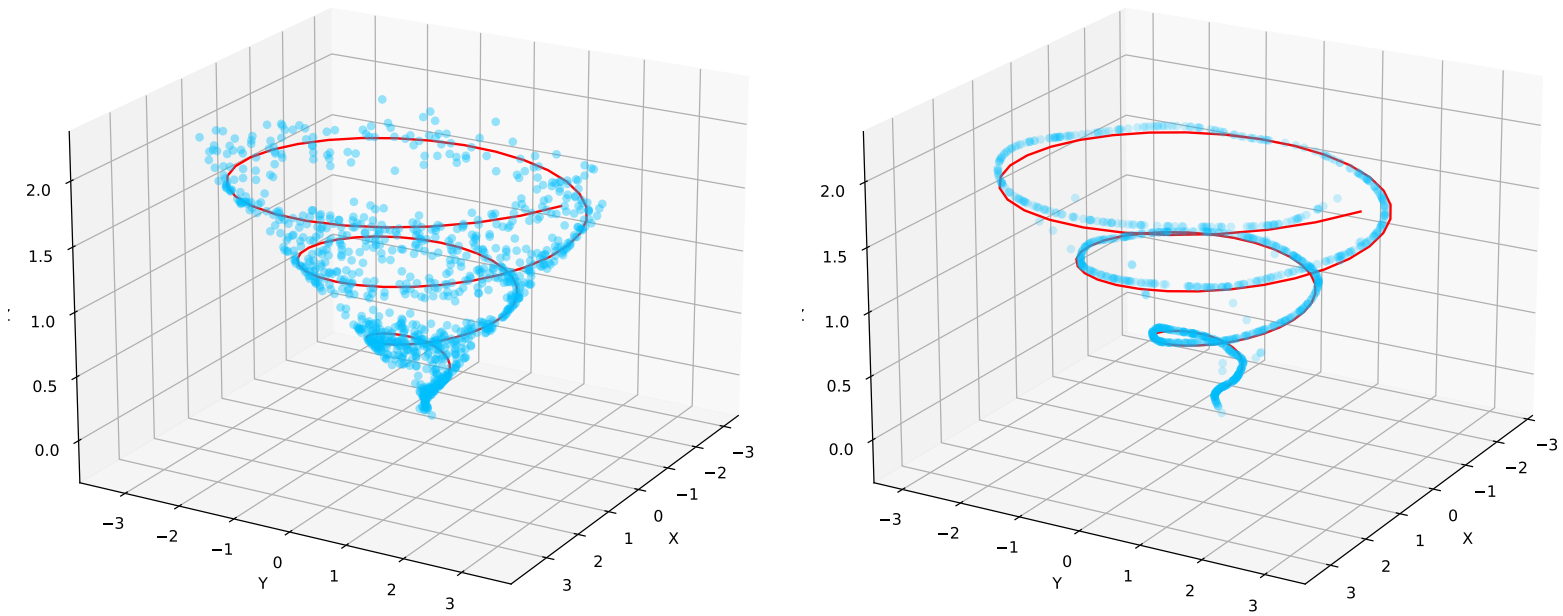
- Naively, one may think that we can convert (RA, dec, z) into 3-dimensional Cartesian coordinate and apply the 3D ridge finding algorithm.
- This idea may lead to unstable results. See the following simulation:



$\Omega_2 \times \mathbb{R}$ space

- To incorporate the redshift, we consider the product space $\Omega_2 \times \mathbb{R}$.
- Ω_2 is the 2-sphere, which describes the angular position (RA, dec).
- $\mathbb{R}$ is the 1-dimensional Euclidean space, which describes the redshift z.
- We attempt to find ridges in $\Omega_2 \times \mathbb{R}$.

Filament findings in $\Omega_2 \times \mathbb{R}$



- The right panel is the result from our directional-linear SCMS, which recover the true filament (red curve).

Density estimation in $\Omega_2 \times \mathbb{R}$

- The idea is to estimate the density in the product space directly.
- Let $x \in \Omega_2$ denotes the angular coordinate and $z \in \mathbb{R}$ denotes the redshift.
- Our data will be $(X_1, Z_1), \cdots, (X_n, Z_n) \in \Omega_2 \times \mathbb{R}$.
- We estimate the density using the product kernel:

$$\widehat{p}_{\mathsf{DL}}(x, z) = \frac{c_{L,2}(h_x)}{nh_z} \sum_{i=1}^{n} L\left(\frac{1 - x^T X_i}{h_x^2}\right) K\left(\frac{Z_i - z}{h_z}\right),$$

where $L(y) = e^{-y}$ and $K(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} y^2)$ are a directional and Gaussian kernel.

Idea: Mean-Shift in $\Omega_2 \times \mathbb{R}$

- We show that a *gradient ascent* of $\widehat{p}_{\mathsf{DL}}(x, z)$ with a suitable step size can be written as follows.
- Starting at $x^{(t)}, z^{(t)}$, we compute

$$\widetilde{x}^{(t+1)} = \frac{\sum_{i=1}^{n} X_i L\left(\frac{1-x^{(t)T}X_i}{h_x^2}\right) K\left(\frac{Z_i - z^{(t)}}{h_z}\right)}{\sum_{i=1}^{n} L\left(\frac{1-x^{(t)T}X_i}{h_x^2}\right) K\left(\frac{Z_i - z^{(t)}}{h_z}\right)},$$

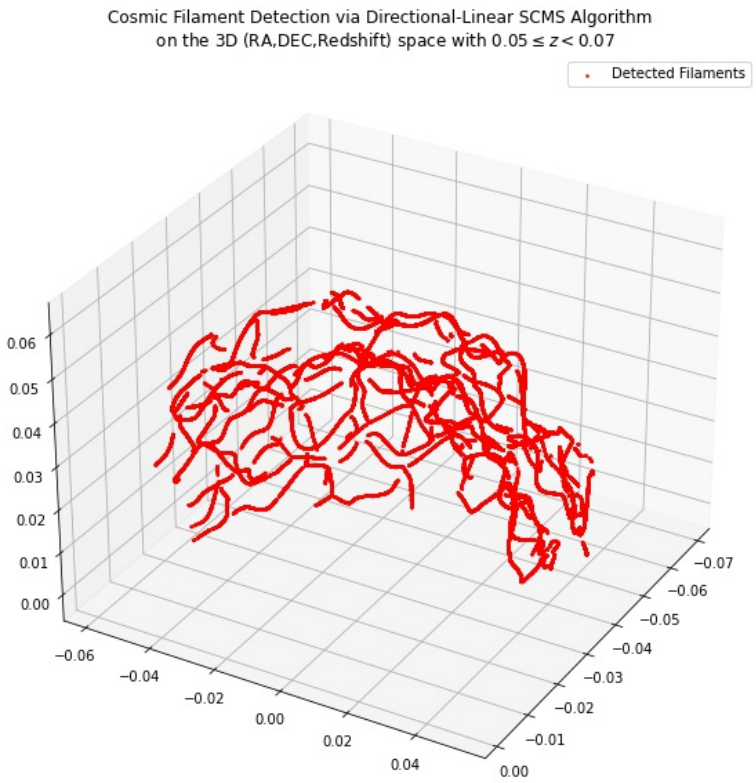
and update

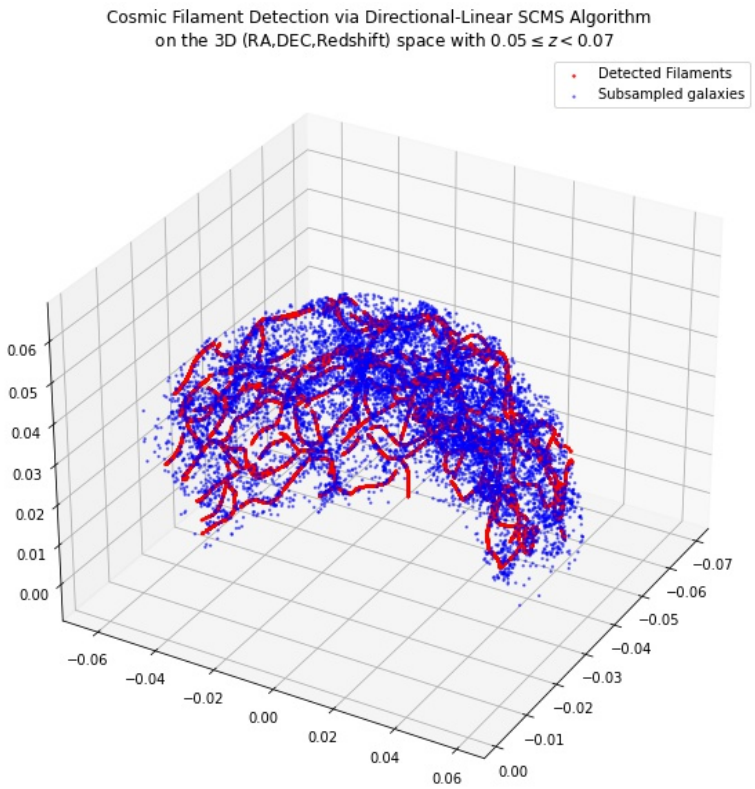
$$x^{(t+1)} = \frac{\widetilde{x}^{(t+1)}}{\|\widetilde{x}^{(t+1)}\|}.$$

Also, the location $z^{(t)}$ is updated to

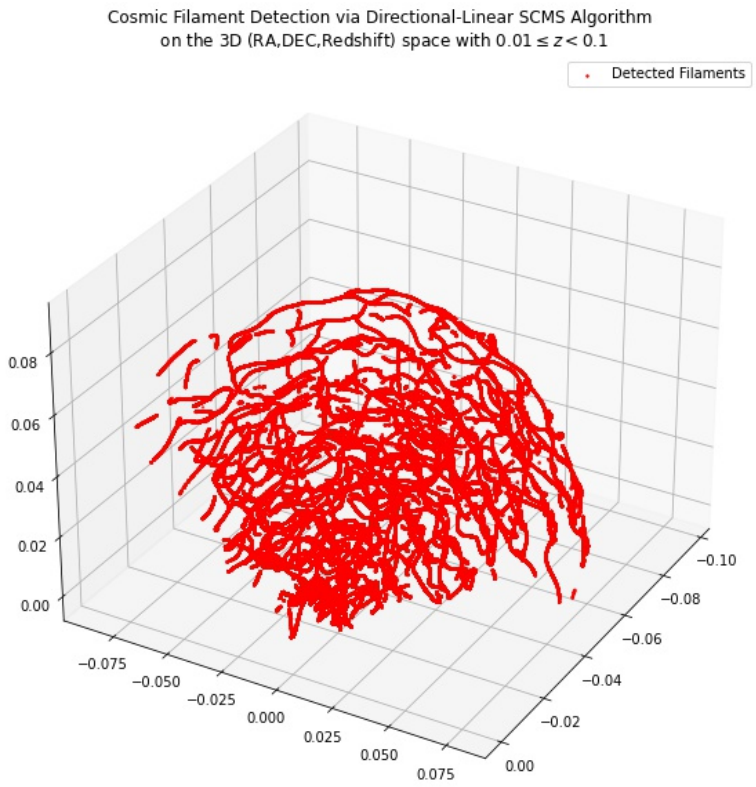
$$z^{(t+1)} = \frac{\sum_{i=1}^{n} Z_i L\left(\frac{1-x^{(t+1)T}X_i}{h_x^2}\right) K\left(\frac{Z_i - z^{(t)}}{h_z}\right)}{\sum_{i=1}^{n} L\left(\frac{1-x^{(t)T}X_i}{h_x^2}\right) K\left(\frac{Z_i - z^{(t)}}{h_z}\right)}.$$

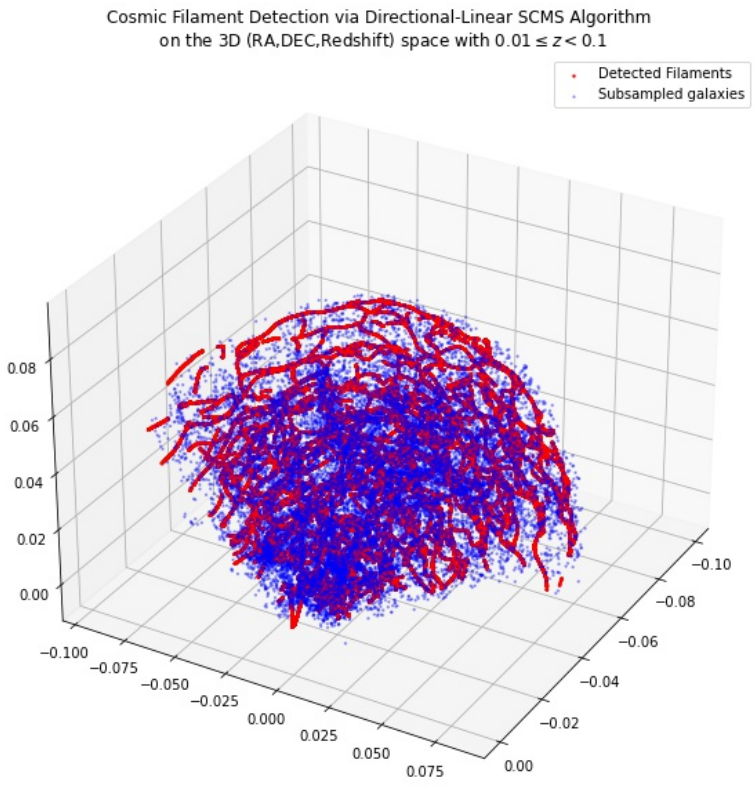
Directional-Euclidean SCMS on SDSS





Directional-Euclidean SCMS on SDSS





Conclusion and future direction

- We have generalized the usual ridge finding problem into directional x Euclidean data, which is better suited for Astronomy data.
- We proved both statistical and computational learning theory of our algorithm.
- Currently, we are applying this to the whole SDSS data and analyzing how filaments affect nearby galaxies.
- We are also designing a weighted approach that each galaxy is weighted differently (e.g., by its stellar mass).

Thank You!

More details can be found in
`http://faculty.washington.edu/yenchic`.

References

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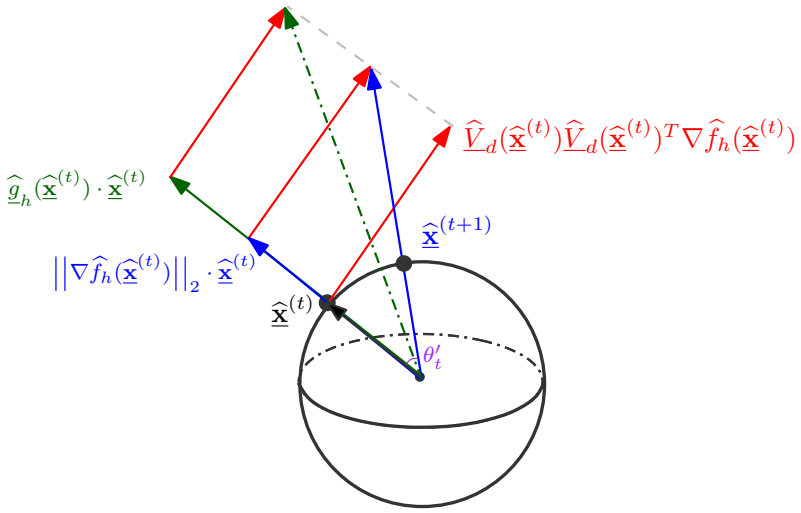
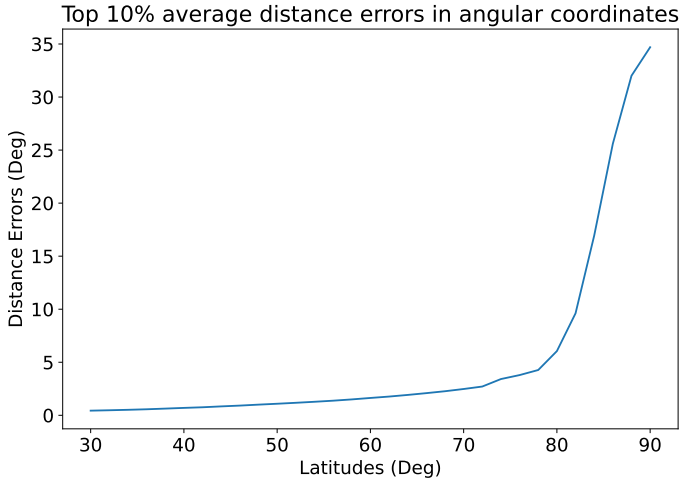
Directional SCMS



$$x^{(t+1)} = \frac{\underline{\widehat{V}}(x^{(t)})\underline{\widehat{V}}(x^{(t)})^T\nabla\widehat{p}_{\mathsf{dir}}(x^{(t)}) + ||\nabla\widehat{p}_{\mathsf{dir}}(x^{(t)})||_2\cdot x^{(t)}}{||\underline{\widehat{V}}(x^{(t)})\underline{\widehat{V}}(x^{(t)})^T\nabla\widehat{p}_{\mathsf{dir}}(x^{(t)}) + ||\nabla\widehat{p}_{\mathsf{dir}}(x^{(t)})||_2\cdot x^{(t)}||_2},$$

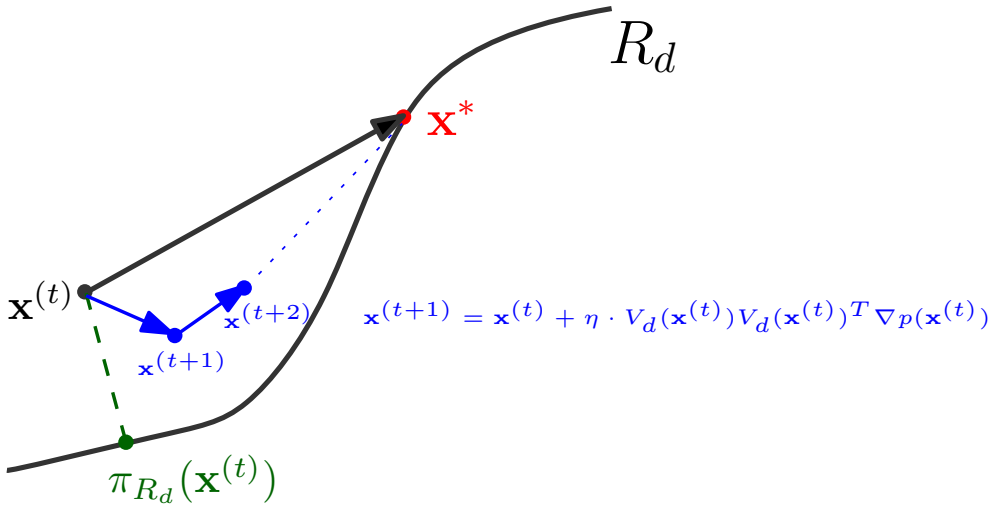
Note: $\widehat{p}_{\mathsf{dir}} = \widehat{f}_h$.

Comparison: Euclidean ridges vs directional ridges



We apply both Euclidean and directional ridge finding algorithms and study the errors of Euclidean ridges as a function of latitude.

Linear convergence: high-level idea



The projection matrix makes the algorithm not a conventional gradient ascent.

A key step to the proof is to bound the projection $(I_{q+1} - V_d(x^{(t)})V_d(x^{(t)})^T)(x^{(t)} - x^*)$ to be $O(\|x^{(t)} - x^*\|^2)$.

We use this decomposition to achieve that.