# Analyzing GPS data using density ranking

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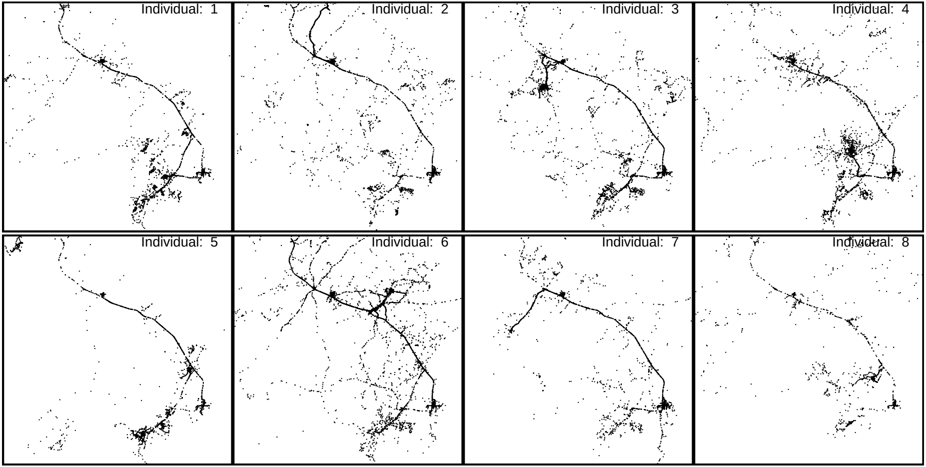
# A Motivating Example: GPS data

- GPS technology provides a new way of collecting mobility patterns of humans and animals.
- GPS data is very rich, but also very complex.
- Here we will focus on a simple case, assuming that we only have access to the longitude and latitude information.

# Real Person Datasets

- This data is about 10 real person's GPS records from Chen and Dobra (2017).
- All these participants share the same work place.
- The ages of the study participants were between 34 and 48 years.
- Each person has around $3,500$ to $8,500$ GPS records during the 6 months study period.

# GPS Data: Real People



# African Animal Datasets

- This data is from the Movebank Data Repository[1] and was analyzed in Abrahms et al. (2017).
- Here we compare 4 different types of animals: elephants, jackals, vultures, and zebras.
- In this data, we have 8 elephants, 15 jackals, 10 vultures, and 9 zebras.
- Each animal has a set of GPS records with record size ranging from $1,000$ to $10,000$.

[1] https://www.datarepository.movebank.org/

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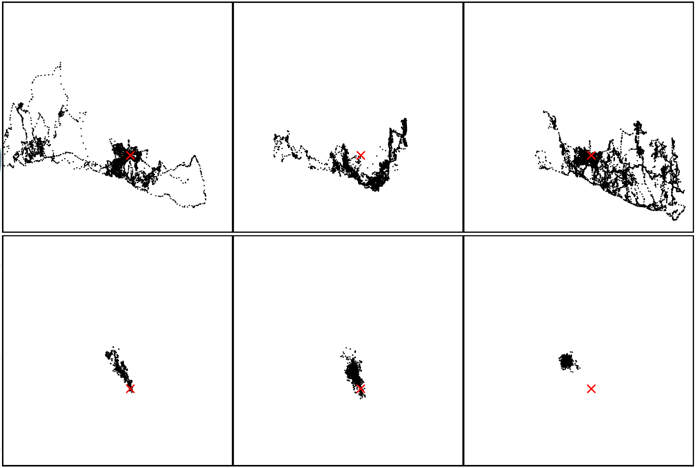

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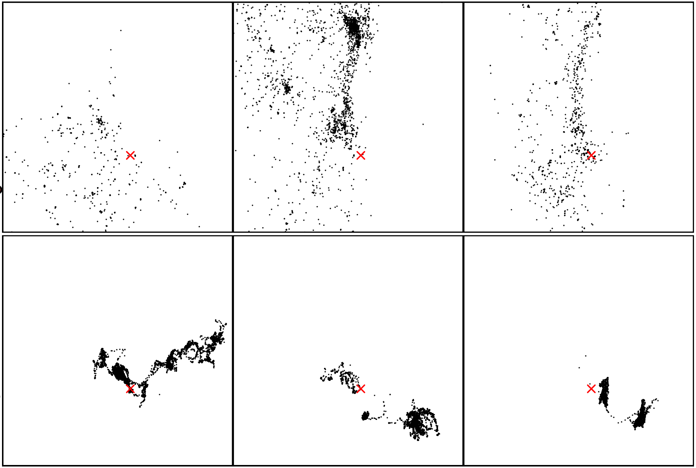
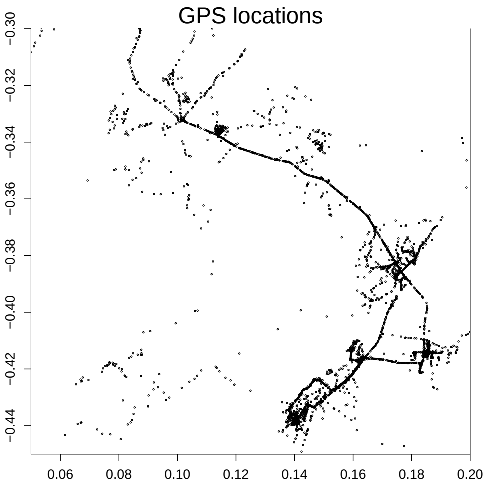


# Kernel Density Estimator

- Kernel Density Estimator (KDE) is one of the most popular method for density estimator.
- When we are given a set of point cloud, it is a natural way to use KDE or other density estimate to analyze the data.
- However, this idea may fail for GPS data.

# Failure of KDE: an Example of GPS dataset



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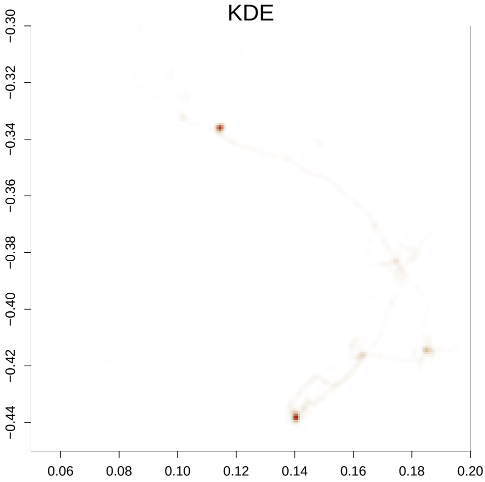

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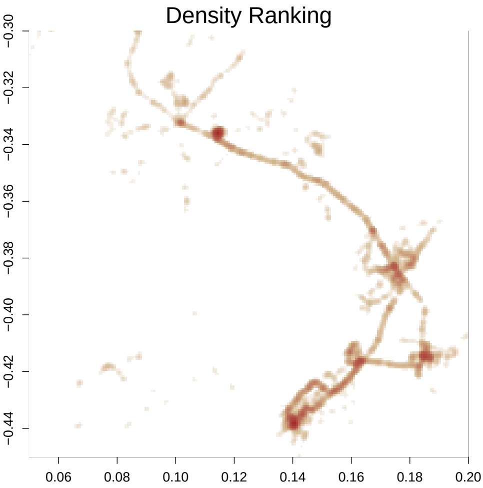

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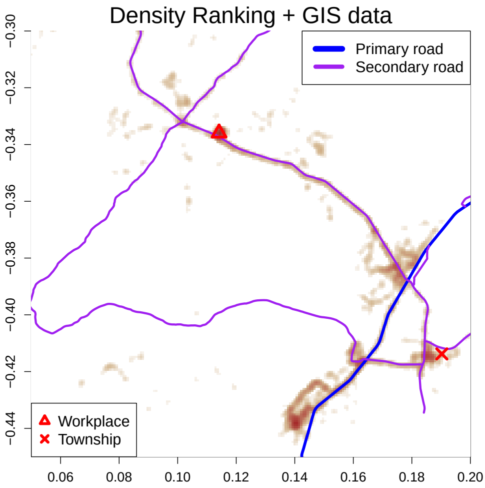

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- Namely, our probability distribution function is singular.
- However, density ranking still works!

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- The density ranking at point $x$ is

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\begin{aligned}
\widehat{\alpha}(x) &= \frac{1}{n}\sum_{i=1}^{n} I\left(\widehat{p}(x) \geq \widehat{p}(X_i)\right) \\
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  where $\widehat{p}$ is the KDE.
- Namely, $\widehat{\alpha}(x) = 0.3$ implies that the (estimated) density of point $x$ is above the (estimated) density of 30% of all observations.

# Property of Density Ranking

- For an observation $X_{\max}$ with $\widehat{\alpha}(X_{\max}) = 1$, then it means

$$\widehat{p}(X_{\max}) = \max\left\{\widehat{p}(X_1), \cdots, \widehat{p}(X_n)\right\}.$$

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- Moreover, for any pairs of data points $X_i, X_j$,

$$\begin{aligned}
\widehat{p}(X_i) > \widehat{p}(X_j) &\Longleftrightarrow \widehat{\alpha}(X_i) > \widehat{\alpha}(X_j)\\
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- But GPS data may not have a well-defined PDF.

# A statistical model for GPS dataset -1

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- Ignoring time label, the GPS records can be viewed as

$$X_1, \cdots, X_n \sim P_{\mathsf{GPS}},$$

  where $P_{\mathsf{GPS}}$ is a probability distribution.
- Because of the natural of GPS records, we can decompose $P_{\mathsf{GPS}}$ as

$$P_{\mathsf{GPS}}(x) = \pi_0 P_0(x) + \pi_1 P_1(x) + \pi_2 P_2(x),$$

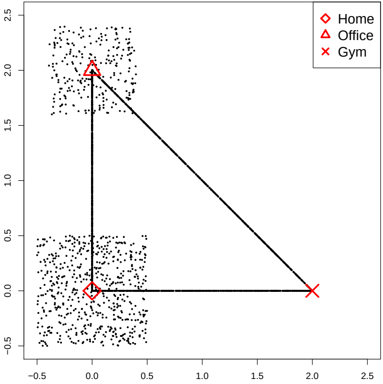
  where $P_0(x)$ is a distribution of point mass, and $P_1(x)$ is a distribution of a $1D$ density function, and $P_2(x)$ is a distribution of a $2D$ density function, and $\pi_0 + \pi_1 + \pi_2 = 1$ with $\pi_j \geq 0$ are proportions.

# A statistical model for GPS dataset - 2

$$P_{\mathsf{GPS}}(x) = \pi_0 P_0(x) + \pi_1 P_1(x) + \pi_2 P_2(x).$$

- $P_0(x)$: a distribution that puts probability on the anchor/key locations.
- $P_1(x)$: a distribution describing the path/road that the individual regularly takes.
- $P_2(x)$: a distribution describing the activity on an open space.

# A simulated GPS data




$\pi_0 = 0.6, \pi_1 = 0.3, \pi_2 = 0.1.$

$P_0(x) = 0.5\delta_{0,0}(x) + 0.3\delta_{0,2}(x) + 0.2\delta_{2,0}(x).$

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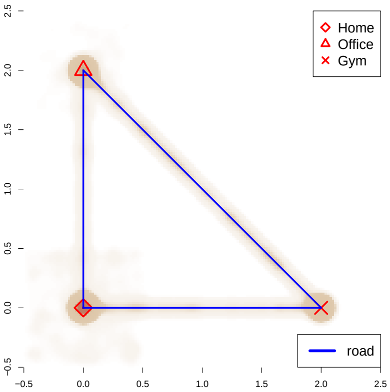


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- Let $C_d$ be the volume of a $d$ dimensional unit ball and $B(x,r) = \{y : \|x-y\| \leq r\}$.
- For any integer $s$, we define

$$\mathcal{H}_s(x) = \lim_{r \to 0} \frac{P(B(x,r))}{C_s r^s}.$$

# Density Ranking in Singular Measures - 2

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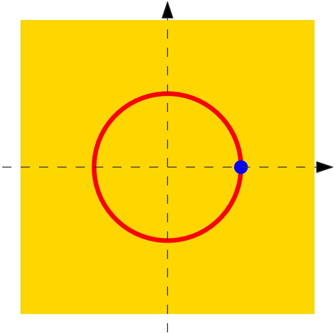
- For a point $x$, we then define

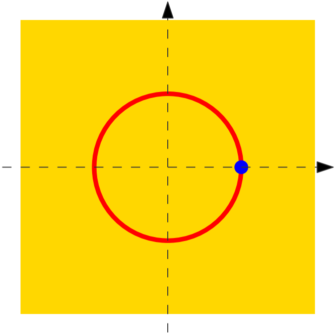
$$\tau(x) = \max\{s \le d : \mathscr{H}_s(x) < \infty\}, \quad \rho(x) = \mathscr{H}_{\tau(x)}(x).$$

# Hausdorff Density: Example - 1

- Assume the distribution function $P$ is a mixture of a $2D$ uniform distribution within $[-1, 1]^2$, a $1D$ uniform distribution over the ring $\{(x, y) : x^2 + y^2 = 0.5^2\}$, and a point mass at $(0.5, 0)$, then the support can be partitioned as follows:



# Geometric Hausdorff: Example - 2



- Orange region: $\tau(x) = 2 \Leftrightarrow$ contribution of $P_2(x)$.
- Red region: $\tau(x) = 1 \Leftrightarrow$ contribution of $P_1(x)$.
- Blue region: $\tau(x) = 0 \Leftrightarrow$ contribution of $P_0(x)$.

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- For two points $x_1, x_2$, we define an ordering such that $x_1 \succ_{\tau,\rho} x_2$ if

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- Namely, we first compare the dimension of the two points, the lower dimensional structure wins. If they are on regions of the same dimension, we then compare the density of that dimension.

# Constructing Density Ranking using Hausdorff Density

- Using the ordering $\succ_{\tau,\rho}$, we then define the population density ranking as

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- When the PDF exists, the ordering $\succ_{\tau,\rho}$ equals to $\succ_{d,p}$ so

$$\alpha(x) = P(x \succeq_{d,p} X_1) = P(p(x) \geq p(X_1)),$$

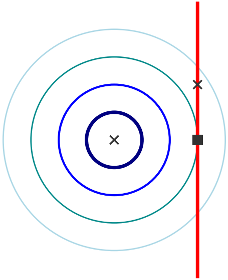
which recovers the definition in the simple case.

# Dimensional Critical Points - 1

- In singular measure, there is a new type of critical points. We call them the *dimensional critical points*.
- These critical points contribute to the change of topology of level sets as the usual critical points but they cannot be defined by setting gradient to be 0.

# Dimensional Critical Points

- The box in the following figure is a dimensional critical point.
- Note: this is a mixture of 2D distribution and a 1D distribution on the black line (maximum value occurs at the cross).



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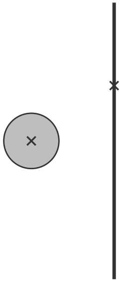

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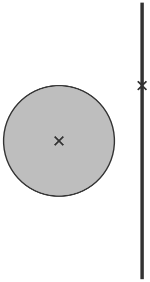

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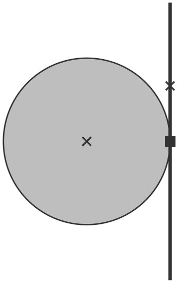

# Convergence under Singular Measure: Density Ranking - 1

- When $P$ is a singular distribution and satisfies certain regularity conditions,

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- Intuition of convergence: as $h \to 0$, the KDE

$$\widehat{p}(x) = \frac{1}{nh^d} \sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right)$$

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- However, the speed of diverging depends on $\tau(x)$. The smaller $\tau(x)$, the faster (actually the diverging rate is $O(h^{\tau(x)-d})$).
- So eventually, we can separate different dimensional structures.

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- Example of non-convergence of supreme norm: consider a sequence of points on a higher dimensional space but moving toward a lower dimensional structure within distance $\frac{h}{2}$.
- Interestingly, we can still prove that some topological features (local modes, level sets, cluster trees, persistent diagrams) are converging.

# Convergence under GPS model

$$P_{\mathsf{GPS}}(x) = \pi_0 P_0(x) + \pi_1 P_1(x) + \pi_2 P_2(x).$$

Anchor locations: $\mathscr{A} = \mathsf{supp}(P_0)$.
Roads: $\mathscr{R} = \mathsf{supp}(P_1)$.

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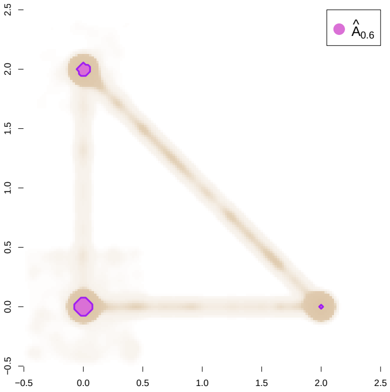


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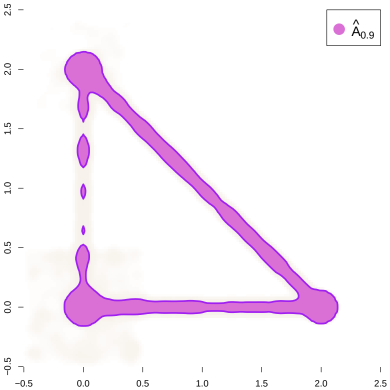


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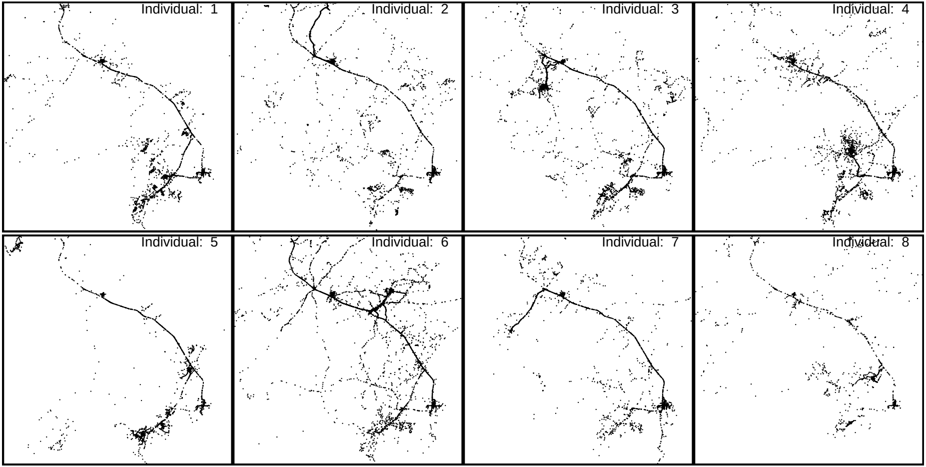
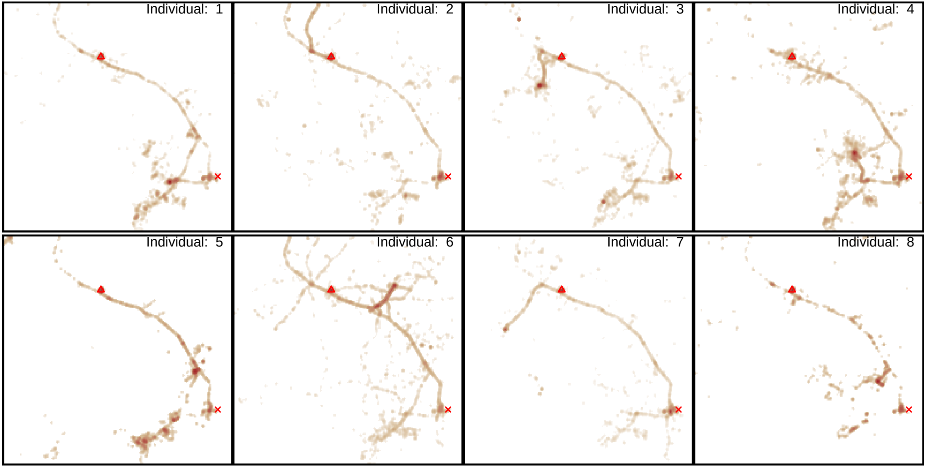
# Application of Density Ranking: GPS dataset - 1



# Application of Density Ranking: GPS dataset - 2



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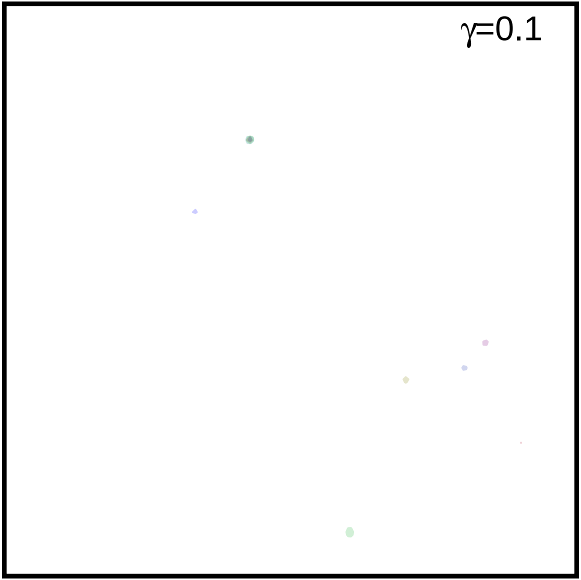

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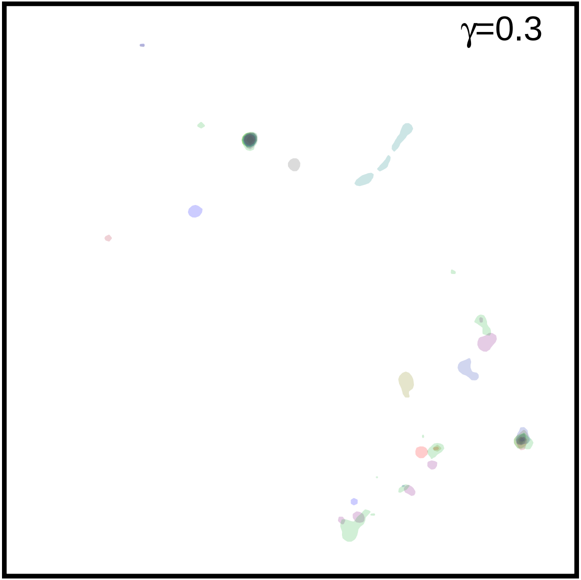

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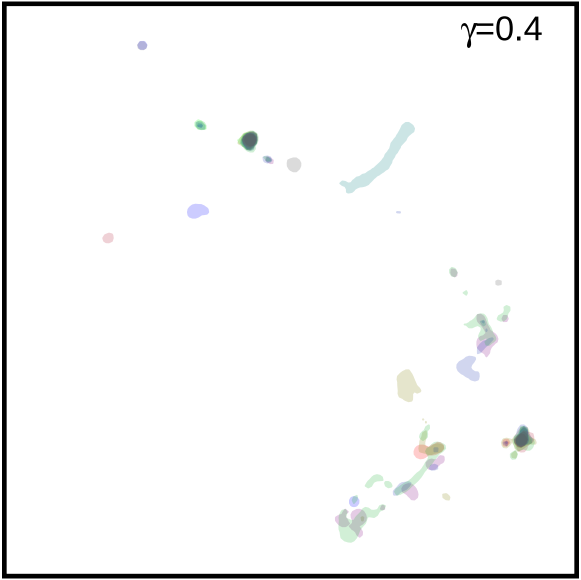

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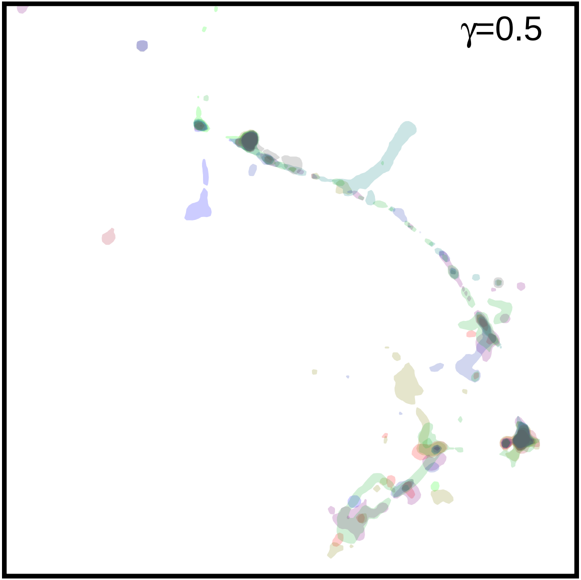

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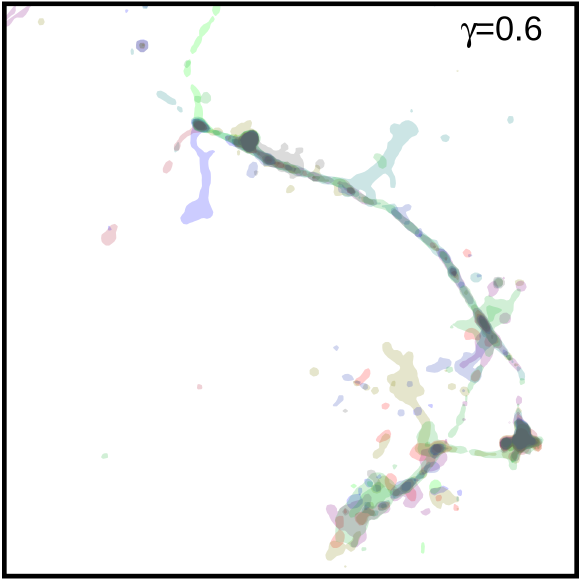

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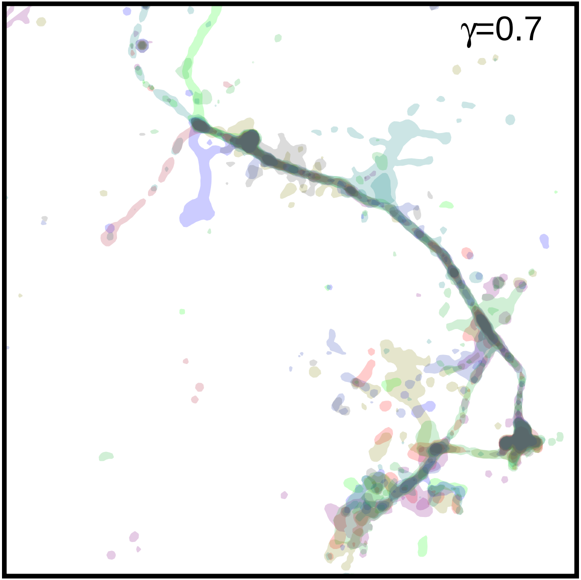

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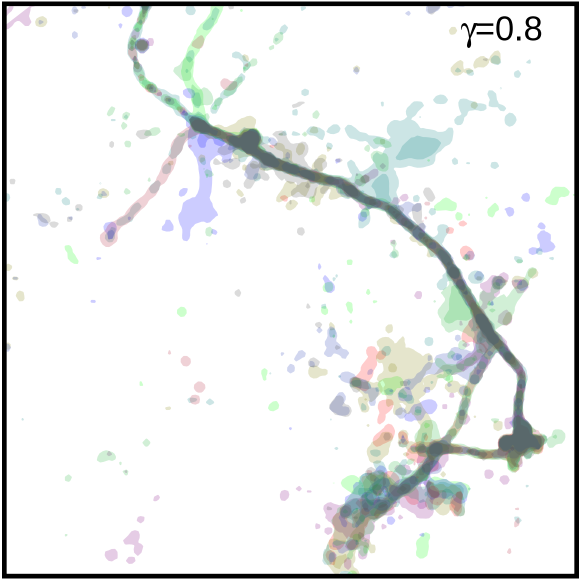

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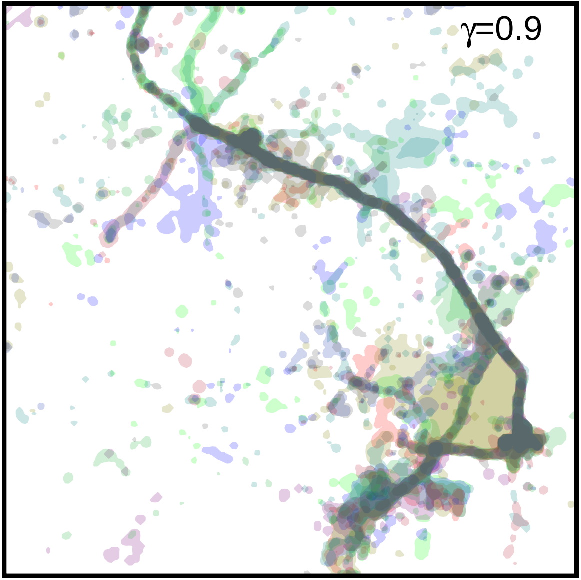

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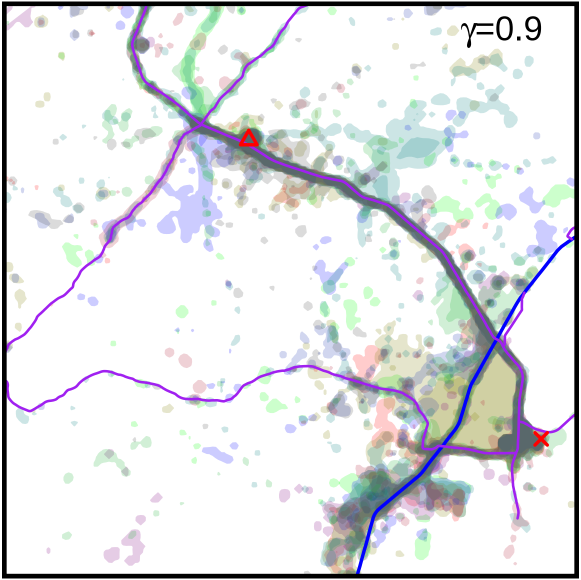

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  - When we have many individuals, this approach might not work (too many contours).
  - We often need to choose a level $\gamma$ to show the plot but which level to be chosen is unclear.
- Here we introduce a few curves to summarize geometric and topological features of density ranking.

# Mass-Volume Curve

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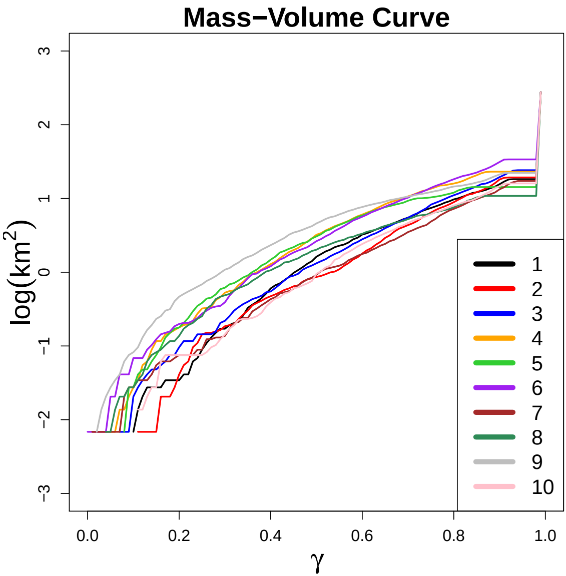
- In practice, we often plot $\gamma$ versus $\log \mathsf{Vol}(\widehat{A}_\gamma)$.

# Mass-Volume Curve: Example



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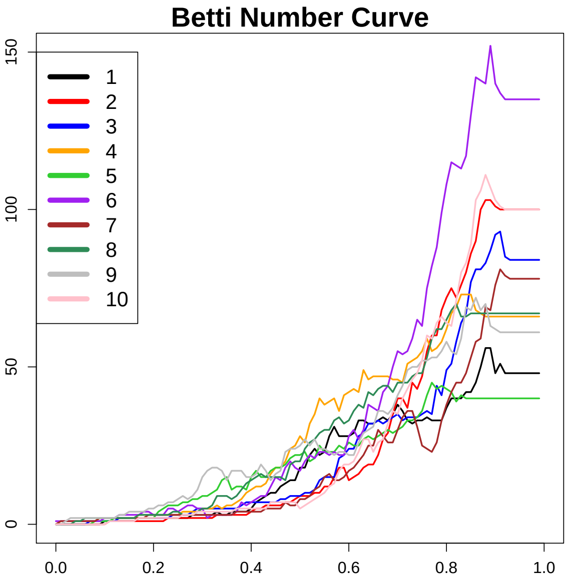
- Note that the number of connected component is called the 0th order Betti number (0th order topological structure); one can generalize this idea to higher order topological structures.

# Betti Number Curve: Example



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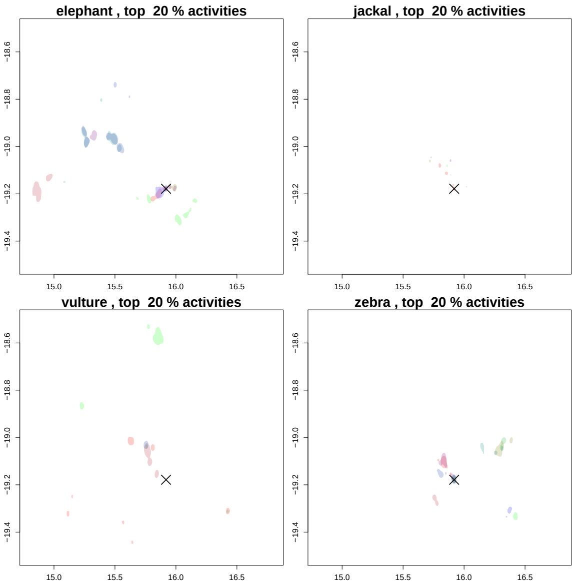

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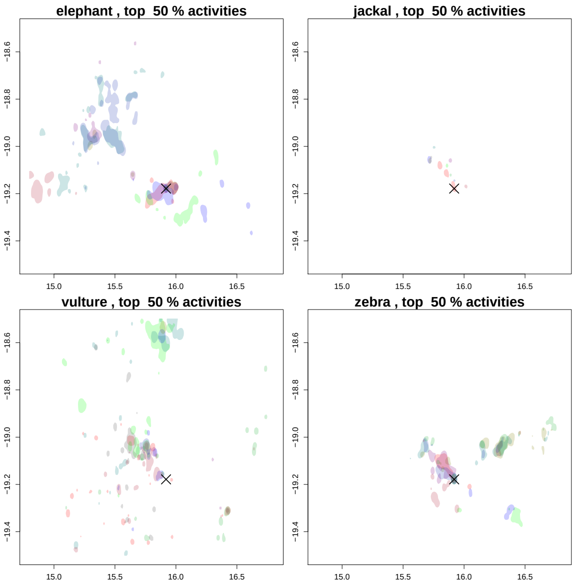

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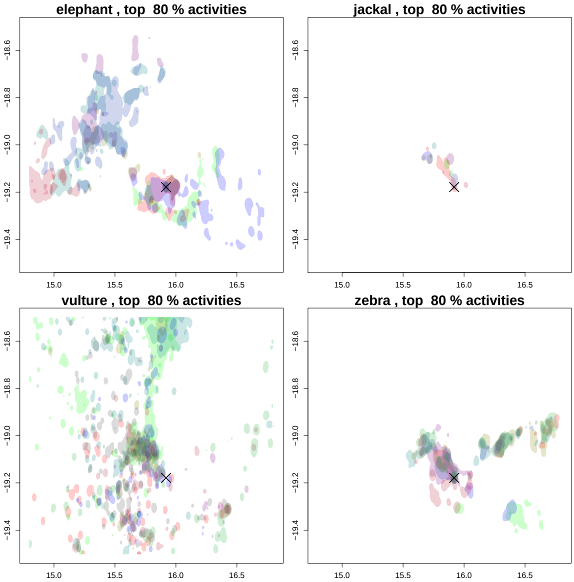
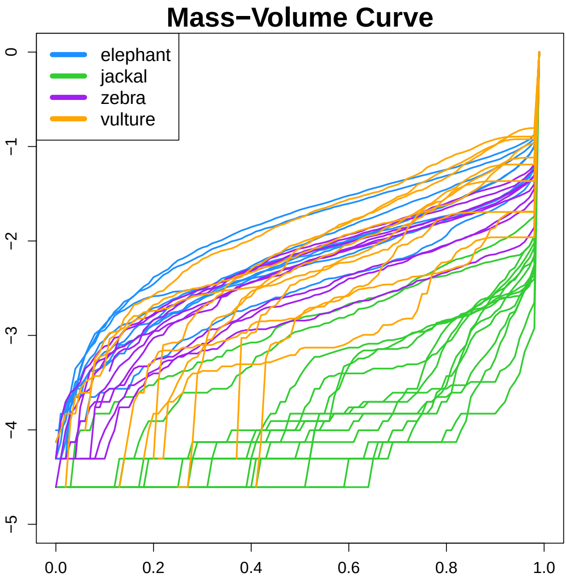


# Other Summary Approaches: Mass-Volume Curve



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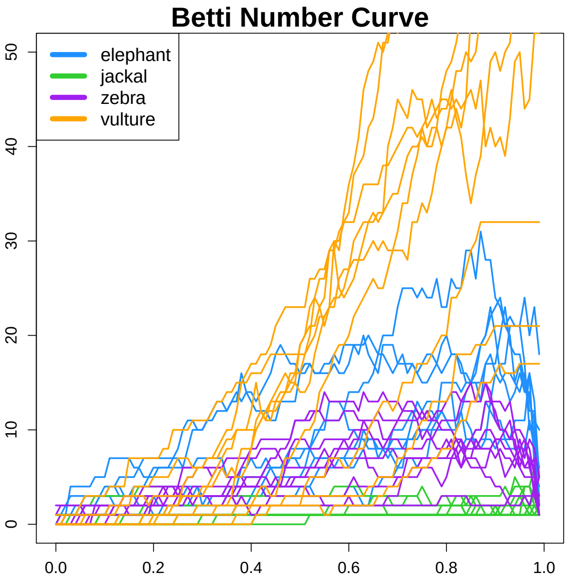

# Conclusion

- When a point cloud is from a singular measure, the traditional density estimator will fail.
- However, the density ranking may still be a well-defined quantity and we can estimate it consistently.
- Using the idea of density ranking, we can analyze complex datasets such as the GPS data.
- Many open questions: generalizing to point processes, modeling the temporal trends, assessing the uncertainty.

# Thank You!

An R script for density ranking:

https://github.com/yenchic/density_ranking

More details can be found in

http://faculty.washington.edu/yenchic/

# References


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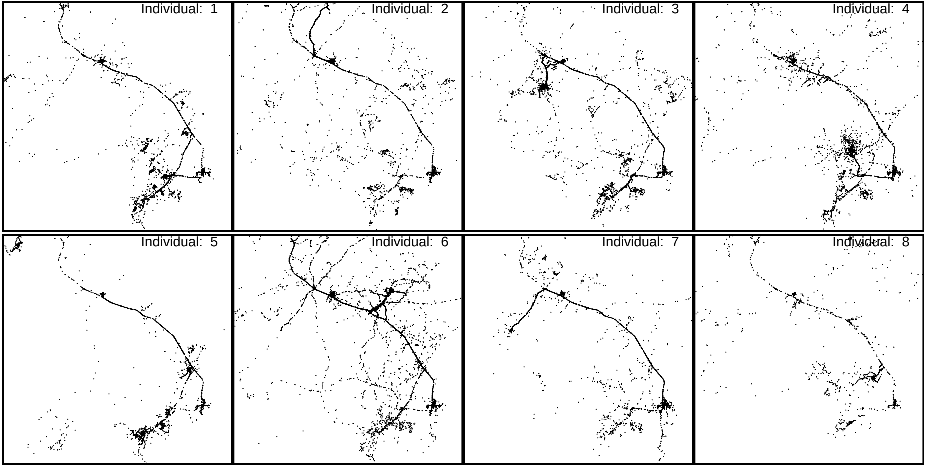
# Real Person Datasets

- This data is about 10 real person's GPS records from Chen and Dobra (2017).
- All these participants share the same work place.
- The ages of the study participants were between 34 and 48 years.
- Each person has around $3,500$ to $8,500$ GPS records.

# Real Persons Datasets: Raw Data



# African Animal Datasets

- This data is from the Movebank Data Repository[2] and was analyzed in Abrahms et al. (2017).
- Here we compare 4 different types of animals: elephants, jackals, vultures, and zebras.
- In this data, we have 8 elephants, 15 jackals, 10 vultures, and 9 zebras.
- Each animal has a set of GPS records with record size ranging from $1,000$ to $10,000$.

[2] https://www.datarepository.movebank.org/

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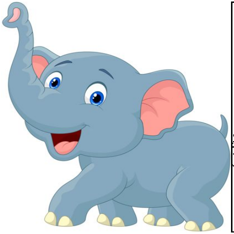

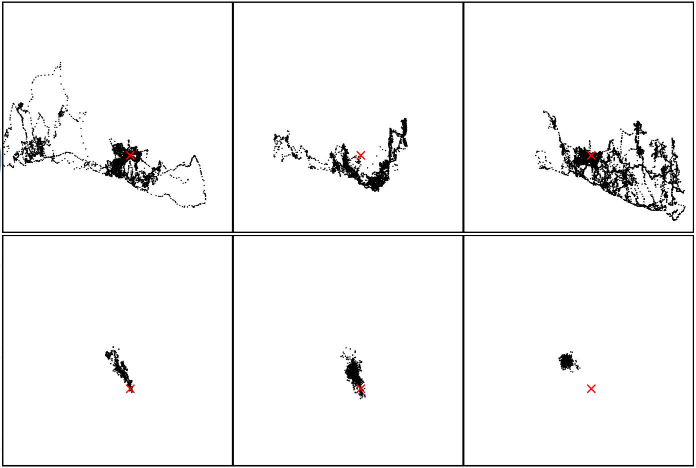

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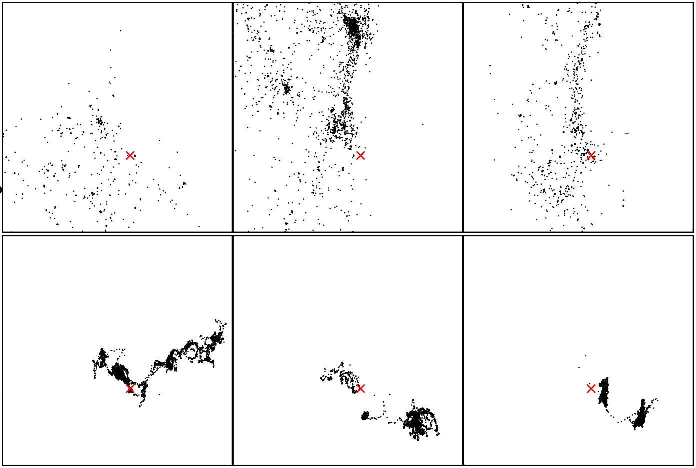

# Assumptions for Regular Distributions

**(R1)** The density function $p$ has a compact support $\mathbb{K}$.

**(R2)** The density function is a Morse function and is in $\mathbf{BC}^3$.

**(K1)** The kernel function $K$ is in $\mathbf{BC}^2$ and integrable.

**(K2)** $K$ satisfies the VC-type class condition.

**(K2)** Let

$$\mathcal{K}_r = \left\{ y \mapsto K^{(\alpha)}\left(\frac{x-y}{h}\right) : x \in \mathbb{R}^d, |\alpha| = r \right\},$$

where $K^{(\alpha)}$ is the $\alpha$-th derivative and let $\mathcal{K}_l^* = \bigcup_{r=0}^{l} \mathcal{K}_r$. We assume that $\mathcal{K}_2^*$ is a VC-type class. i.e. there exists constants $A, v$ and a constant envelope $b_0$ such that

$$\sup_Q N(\mathcal{K}_2^*, \mathcal{L}^2(Q), b_0\epsilon) \leq \left(\frac{A}{\epsilon}\right)^v, \tag{1}$$

where $N(T, d_T, \epsilon)$ is the $\epsilon$-covering number for an semi-metric set $T$ with metric $d_T$ and $\mathcal{L}^2(Q)$ is the $L_2$ norm with respect to the probability measure $Q$.

# Assumptions for Singular Distributions

**(S1)** The support can be partitioned into

$$K = K_0 \bigcup K_1 \bigcup \cdots \bigcup K_d,$$

where $K_\ell = \{x \in \mathbb{K} : \tau(x) = \ell\}$.

**(S2)** There exist $\rho_{\min}, \rho_{\max}$ such that $0 < \rho_{\min} \leq \rho(x) \leq \rho_{\max} < \infty$ for every $x \in \mathbb{K}$.

**(S3)** Restricted to each $\mathbb{K}_\ell$ where $\ell > 0$, $\rho(x)$ is a Morse function.

**(K1')** The kernel function $K$ is in $\mathbf{BC}^2$, integrable, and supported in $[-1, 1]$.

**(K2)** $K$ satisfies the VC-type class condition.

# Estimating a Density Tree (Continue)

- To measure the estimation error, a simple metric is

$$d_\infty(\widehat{T_p}, T_p) = \sup_x \|\widehat{p}_n(x) - p(x)\|,$$

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- Under smoothness conditions and $n \to \infty, h \to 0$,

$$P_n \geq 1 - e^{-nh^{d+4} \cdot C_p},$$

for some constant $C_p$ depending on the density function $p$.

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- Hartigan consistency (Chaudhuri and Dasgupta 2010; Balakrishnan et al. 2013) is another way to measure the consistency of a tree estimator.
- Note: density tree can also be recovered by a kNN approach; see Chaudhuri and Dasgupta (2010) and Chaudhuri et al. (2014) for more details.

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- Example of non-convergence of supreme norm: consider a sequence of points on a higher dimensional space but moving toward a lower dimensional space within distance $\frac{h}{2}$.

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- In addition, the height of each branch of the tree will also converge.

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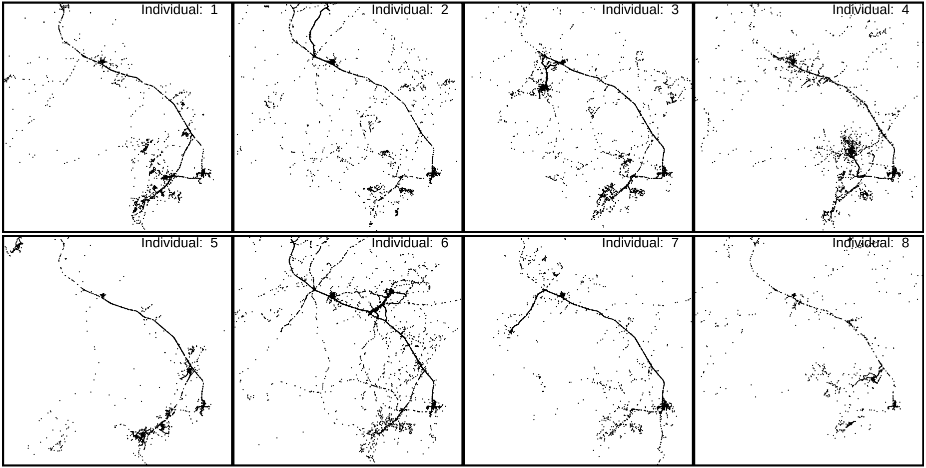
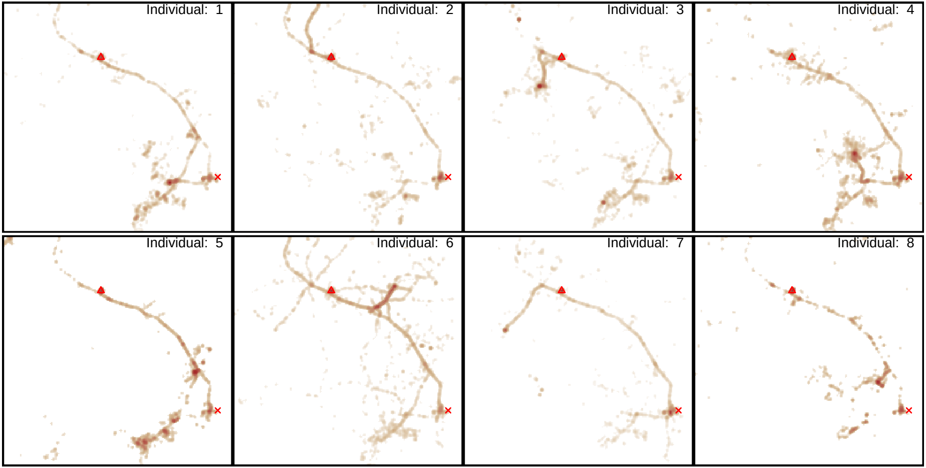



Joint work with Adrian Dobra and Zhihang Dong.

# Application of Density Ranking: GPS dataset - 2

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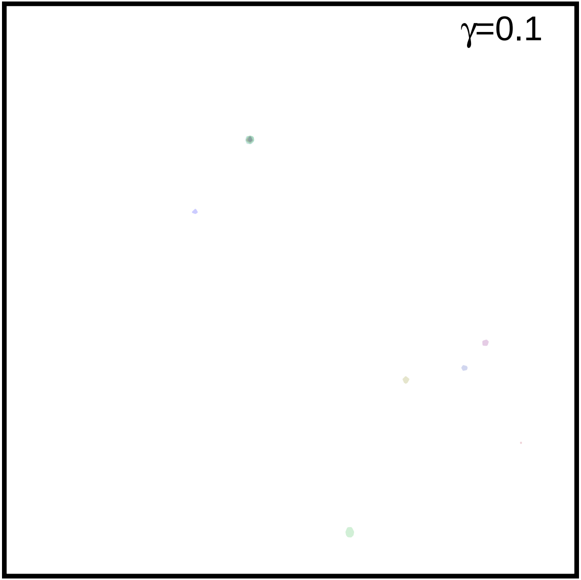
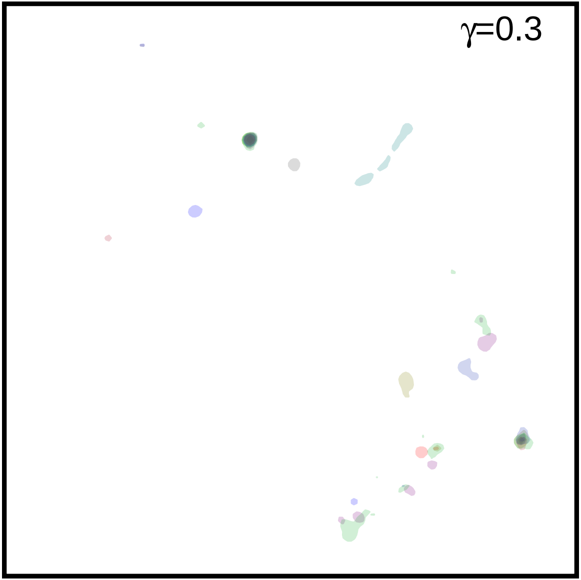
- Namely, $\widehat{A}_{\gamma=0.3}$ is the (top) 30% activity space.

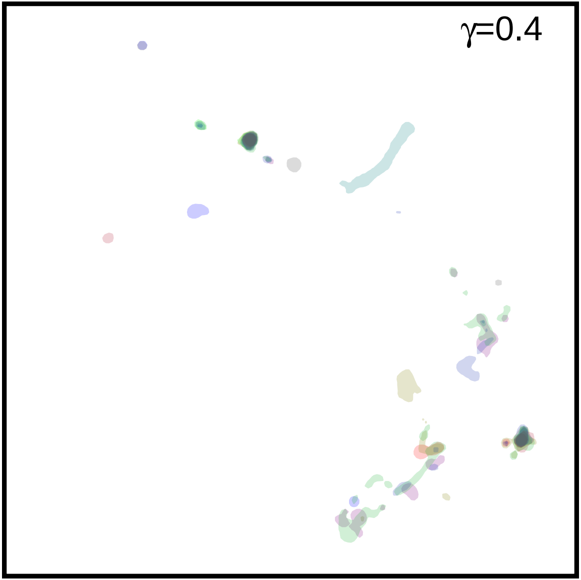

γ=0.1


γ=0.2

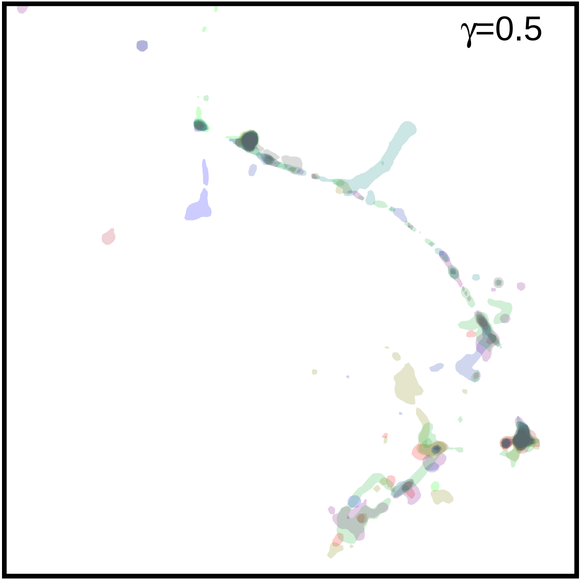
# Level Plots: Example



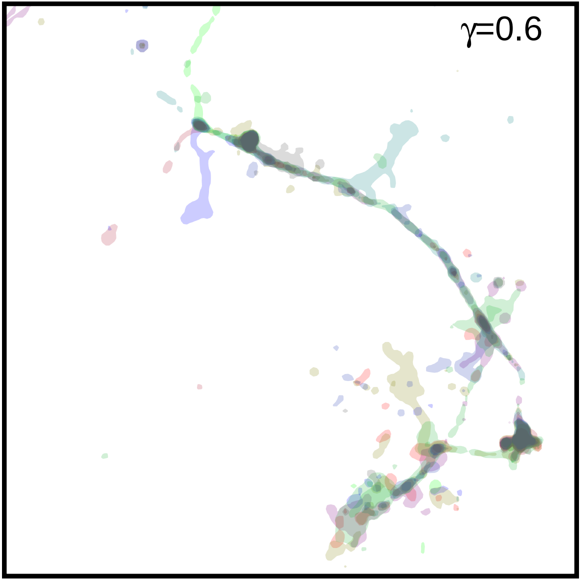
# Level Plots: Example



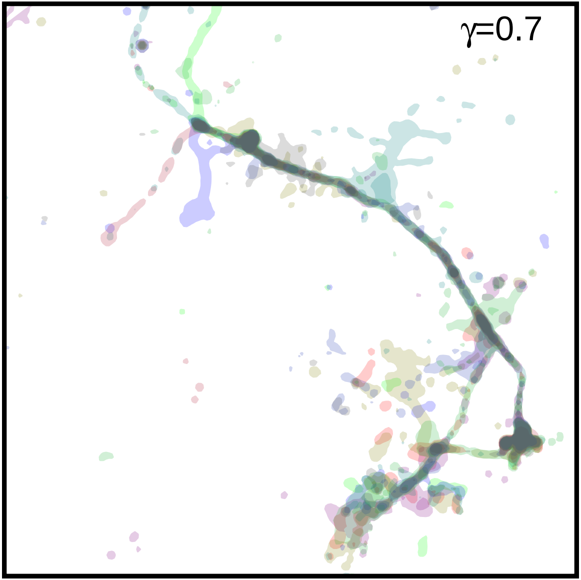
# Level Plots: Example



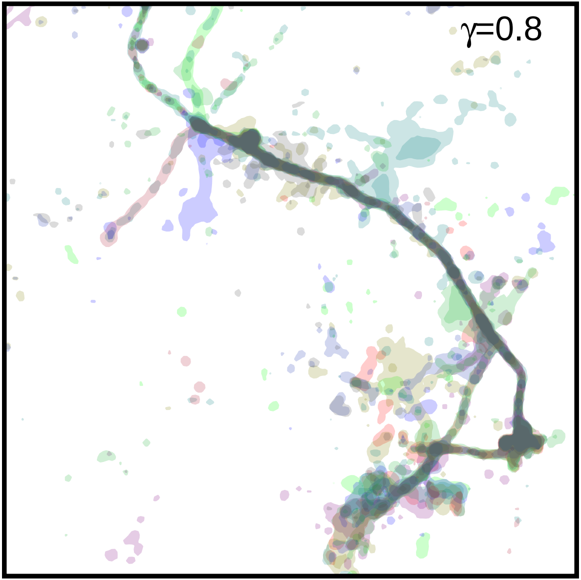
# Level Plots: Example



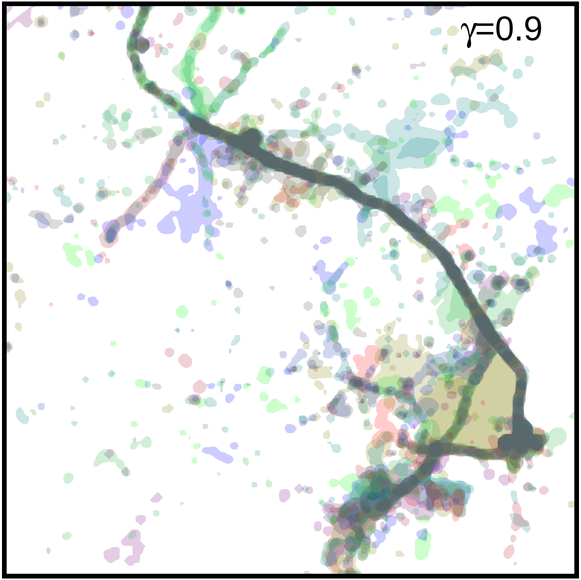
# Level Plots: Example



# Level Plots: Example



# Level Plots: Example



# Level Plots: Example