Density Ranking in Singular Measures

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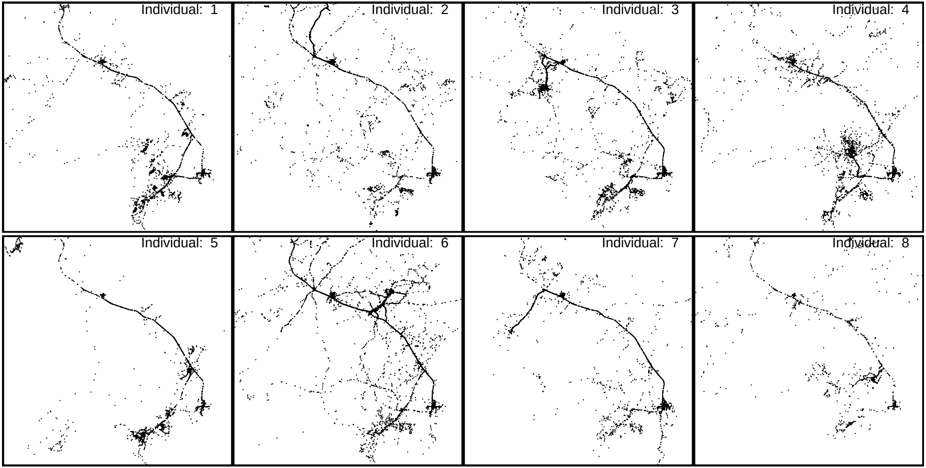
A Motivating Example: GPS data

- GPS technology provides a new way of collecting mobility patterns of humans and animals.
- GPS data is very rich, but also very complex.
- Here we will focus on a simple case, assuming that we only have access to the longitude and latitude information.

GPS Data: Real People



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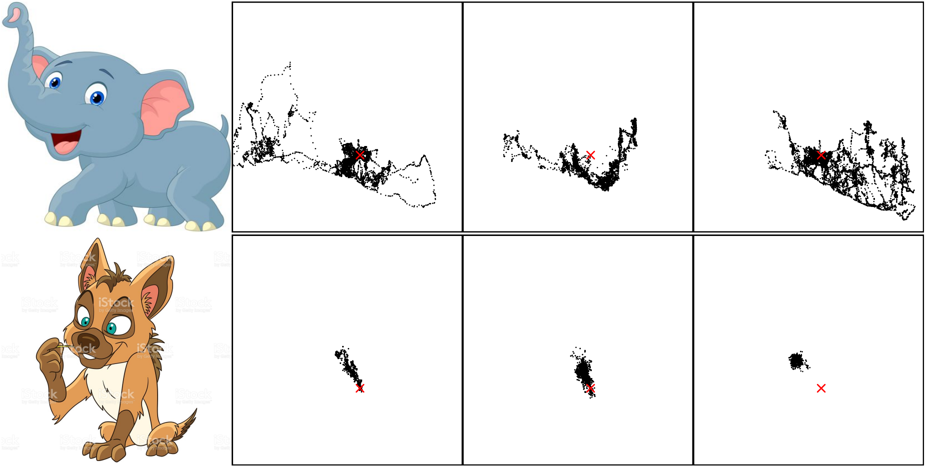

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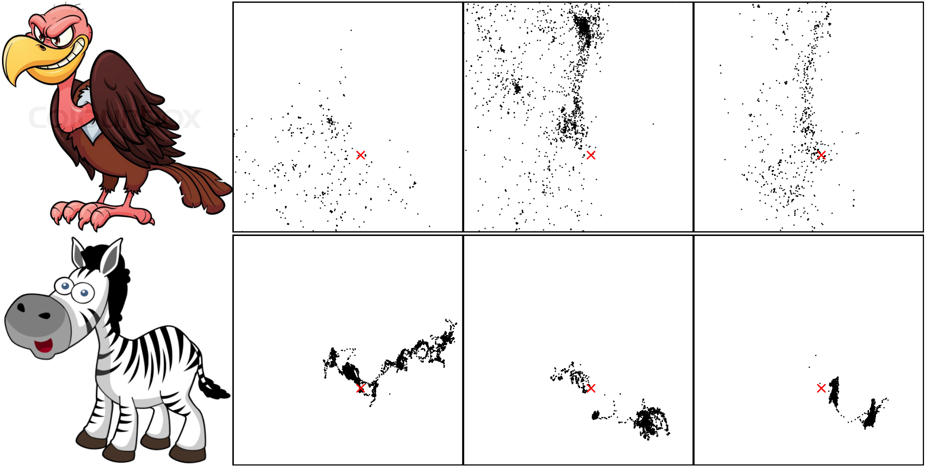
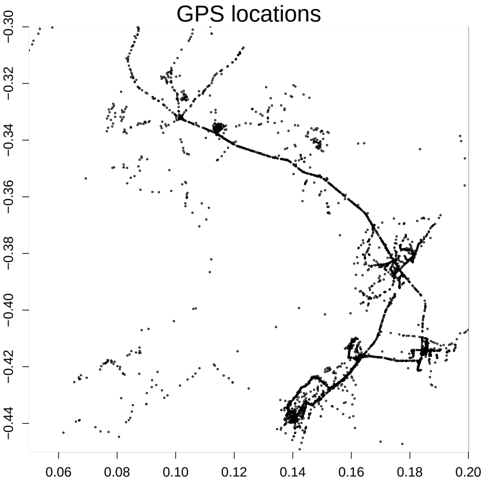


- Kernel Density Estimator (KDE) is one of the most popular method for density estimator.
- When we are given a set of point cloud, it is a natural way to use KDE or other density estimate to analyze the data.
- However, this idea may fail for GPS data.

Failure of KDE: an Example of GPS dataset



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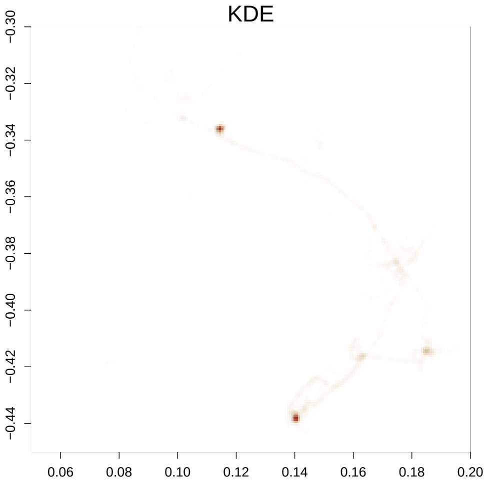

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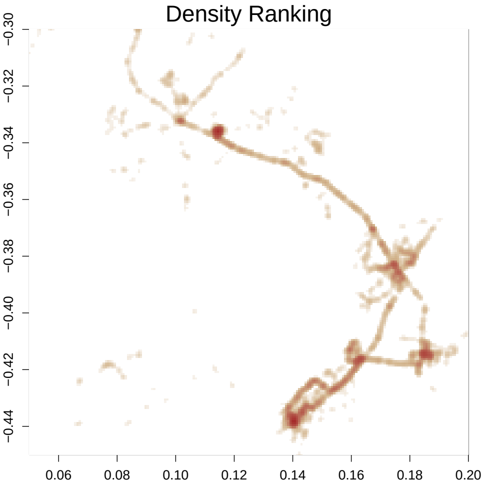

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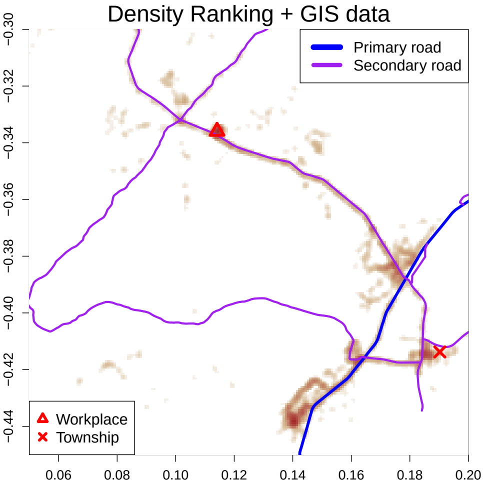

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- Namely, our probability distribution function is singular.
- However, density ranking still works!

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- The density ranking (Chen 2018; Chen and Dobra 2017) is a transformed quantity/function from the KDE.
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  where $\widehat{p}$ is the KDE.
- Namely, $\widehat{\alpha}(x) = 0.3$ implies that the (estimated) density of point x is above the (estimated) density of 30% of all observations.

Property of Density Ranking

- For an observation $X_{\max}$ with $\widehat{\alpha}(X_{\max}) = 1$, then it means

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- If an observation X_ℓ satisfies $\widehat{\alpha}(X_\ell) = 0.25$, this means that the ranking of density at X_ℓ is the 25%.
- Moreover, for any pairs of data points X_i, X_j,

$$\widehat{p}(X_i) > \widehat{p}(X_j) \Longleftrightarrow \widehat{\alpha}(X_i) > \widehat{\alpha}(X_j)$$
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- But GPS data may not have a well-defined PDF.

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- Let C_d be the volume of a d dimensional unit ball and $B(x, r) = \{y : \|x - y\| \leq r\}$.
- For any integer s, we define

$$\mathcal{H}_s(x) = \lim_{r \to 0} \frac{P(B(x, r))}{C_s r^s}.$$

Density Ranking in Singular Measures - 2

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- $\mathcal{H}_s(x)$ occurs in three regimes: 0, ∞, or a number between $(0, \infty)$.

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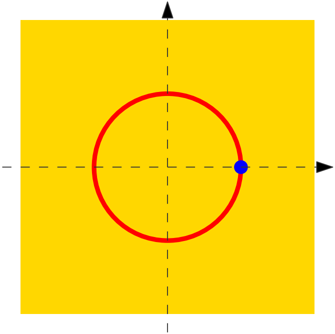
- For a point x, we then define

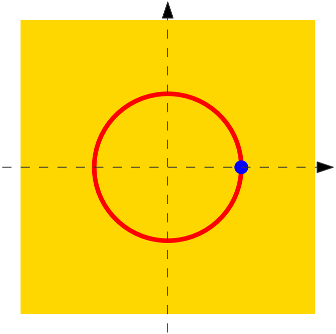
$$\tau(x) = \max\{s \leq d : \mathcal{H}_s(x) < \infty\}, \quad \rho(x) = \mathcal{H}_{\tau(x)}(x).$$

Hausdorff Density: Example - 1

- Assume the distribution function P is a mixture of a $2D$ uniform distribution within $[-1,1]^2$, a $1D$ uniform distribution over the ring $\{(x,y) : x^2 + y^2 = 0.5^2\}$, and a point mass at $(0.5, 0)$, then the support can be partitioned as follows:



Geometric Hausdorff: Example - 2



- Orange region: $\tau(x) = 2$.
- Red region: $\tau(x) = 1$.
- Blue region: $\tau(x) = 0$.

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- We can use τ and ρ to compare any pairs of points and construct a ranking.
- For two points x_1, x_2, we define an ordering such that $x_1 \succ_{\tau,\rho} x_2$ if

$$\tau(x_1) < \tau(x_2), \qquad \text{or} \quad \tau(x_1) = \tau(x_2), \quad \rho(x_1) > \rho(x_2).$$

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- Namely, we first compare the dimension of the two points, the lower dimensional structure wins. If they are on regions of the same dimension, we then compare the density of that dimension.

- Using the ordering $\succ_{\tau,\rho}$, we then define the population density ranking as
$$\alpha(x) = P(x \succeq_{\tau,\rho} X_1)$$

Constructing Density Ranking using Hausdorff Density

- Using the ordering $\succ_{\tau,\rho}$, we then define the population density ranking as

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- When the PDF exists, the ordering $\succ_{\tau,\rho}$ equals to $\succ_{d,p}$ so

$$\alpha(x) = P(x \succeq_{d,p} X_1) = P(p(x) \geq p(X_1)),$$

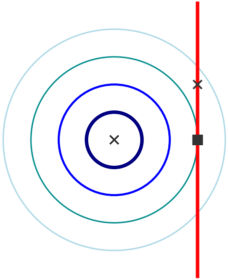
which recovers the definition in the simple case.

Dimensional Critical Points - 1

- In singular measure, there is a new type of critical points. We call them the *dimensional critical points*.
- These critical points contribute to the change of topology of level sets as the usual critical points but they cannot be defined by setting gradient to be 0.

Dimensional Critical Points

- The box in the following figure is a dimensional critical point.
- Note: this is a mixture of 2D distribution and a 1D distribution on the black line (maximum value occurs at the cross).



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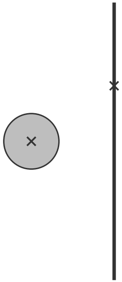

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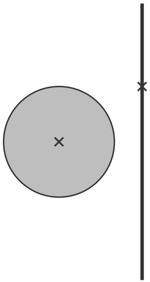

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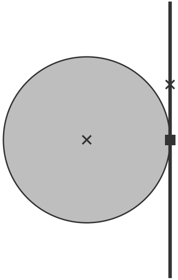

Convergence under Singular Measure: Density Ranking - 1

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$$\widehat{p}(x) = \frac{1}{nh^d} \sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right)$$

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- However, the speed of diverging depends on $\tau(x)$. The smaller $\tau(x)$, the faster (actually the diverging rate is $O(h^{\tau(x)-d})$).
- So eventually, we can separate different dimensional structures.

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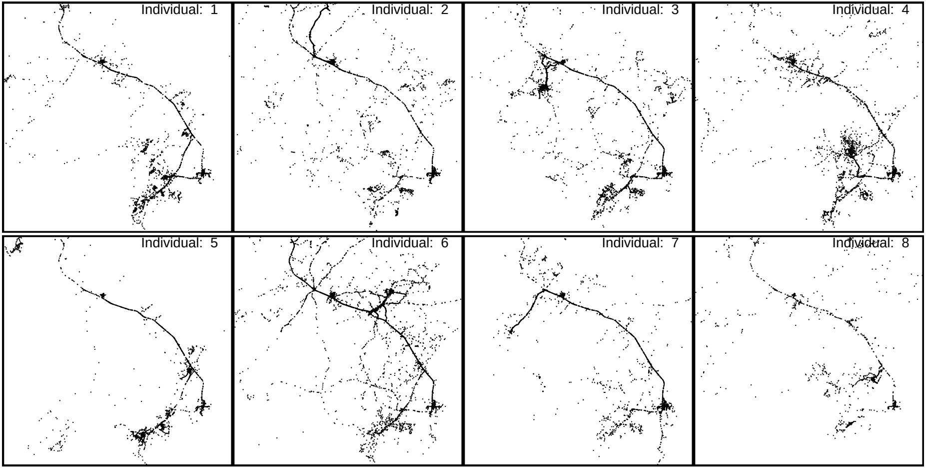
- Example of non-convergence of supreme norm: consider a sequence of points on a higher dimensional space but moving toward a lower dimensional structure within distance $\frac{h}{2}$.
- Interestingly, we can still prove that some topological features (local modes, level sets, cluster trees, persistent diagrams) are converging.


Individual: 1
Individual: 2
Individual: 3
Individual: 4
Individual: 5
Individual: 6
Individual: 7
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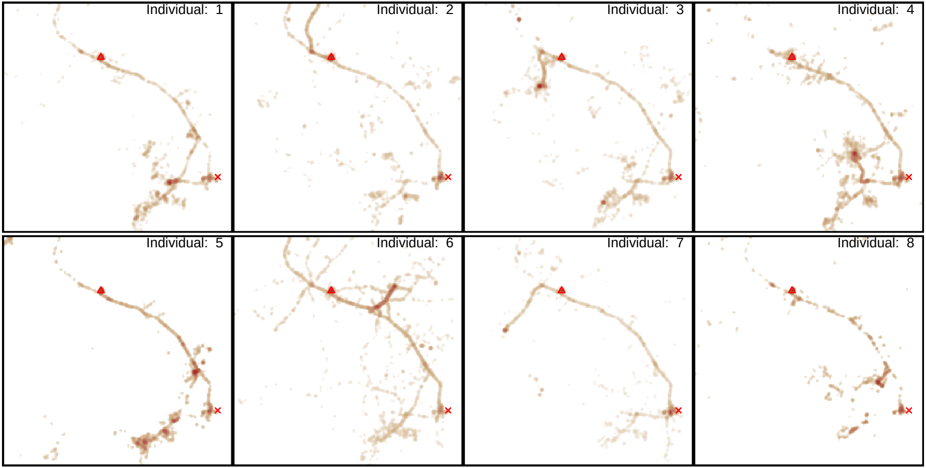
Application of Density Ranking: GPS dataset - 2



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- To compare these density rankings, a simple approach is to overlap the level sets (clusters).
- For a density ranking $\widehat{\alpha}$, let

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- For a density ranking $\widehat{\alpha}$, let

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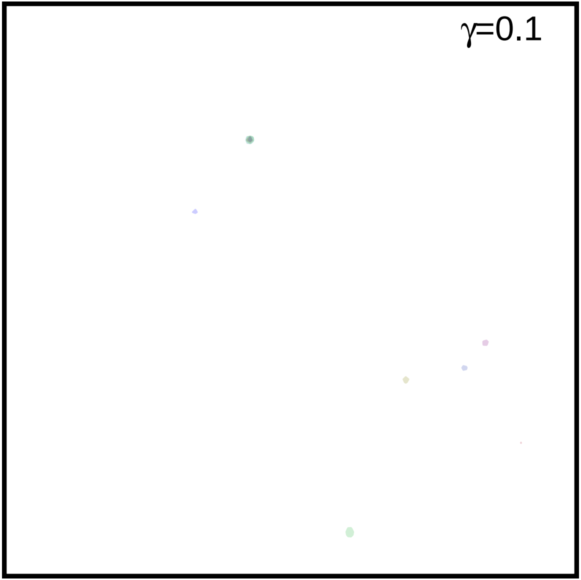
  be the (upper) level set.
- We compare the density ranking of each individual by overlapping their level sets/clusters at different levels.

Clusters of GPS Point Clouds



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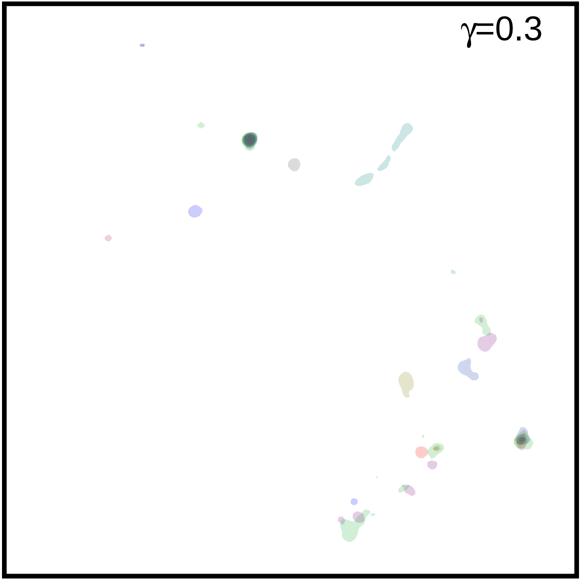

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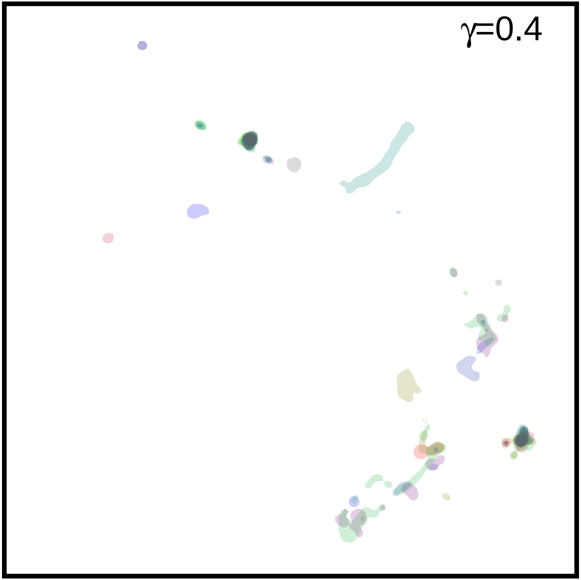

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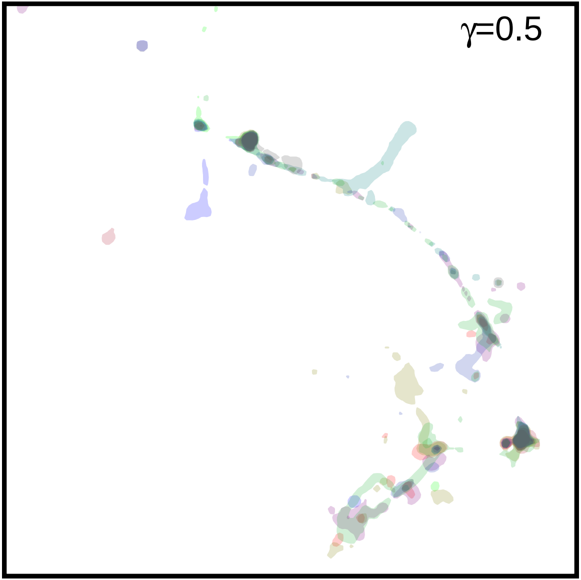

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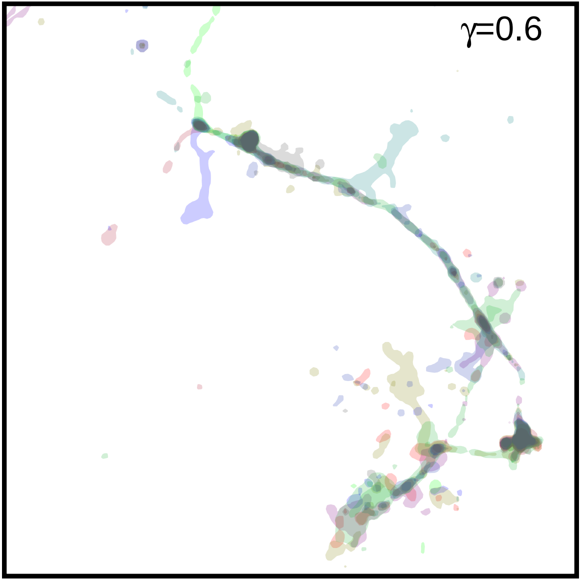

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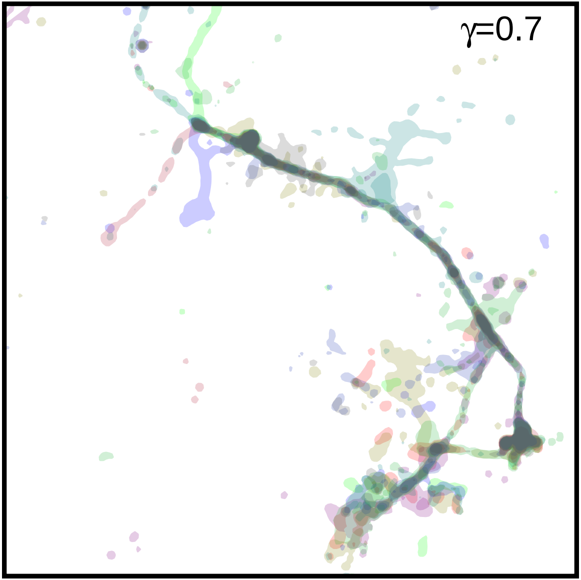

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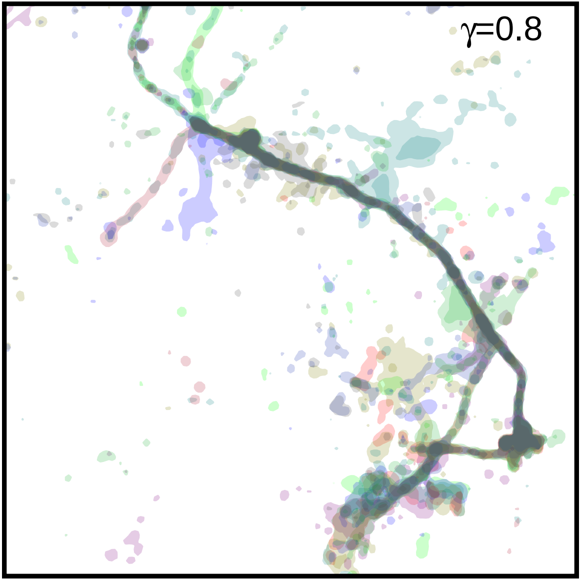

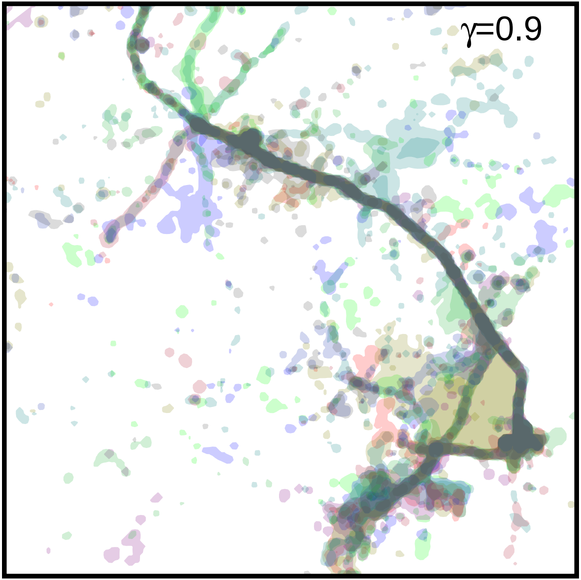
γ=0.9

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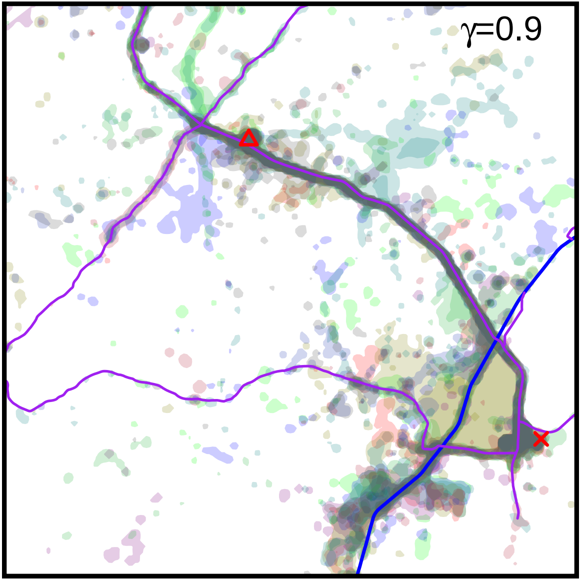
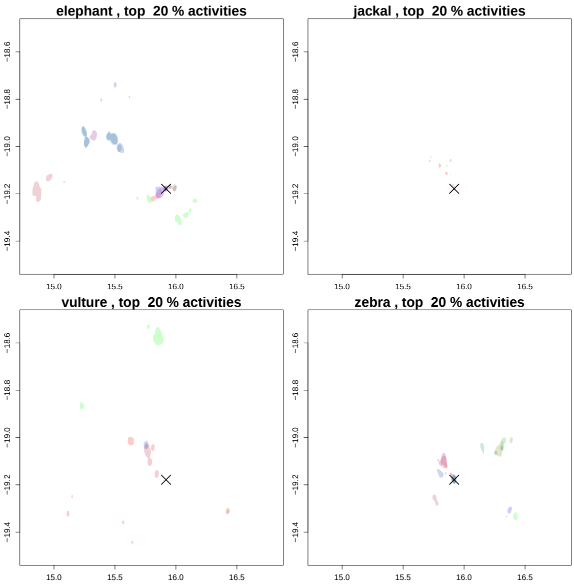





Applying to African Animal Datasets



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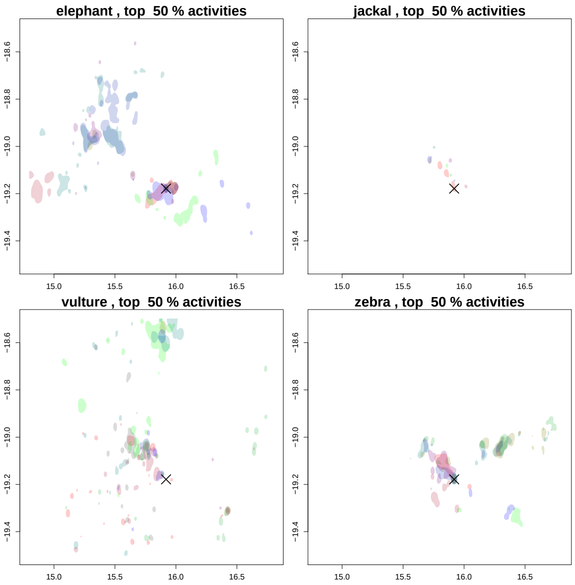

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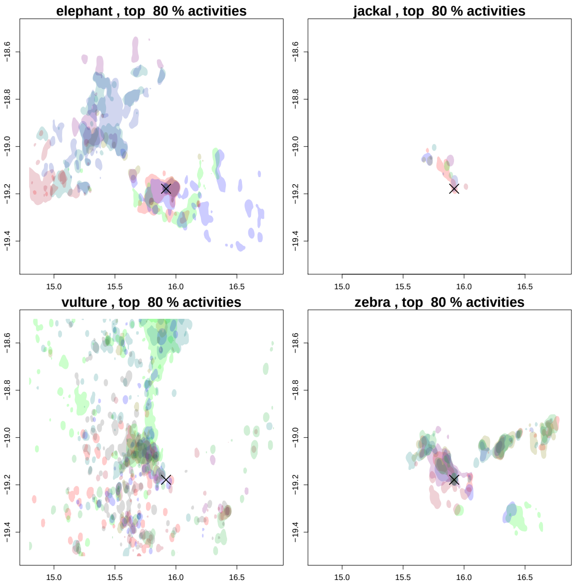

Conclusion

- When a point cloud is from a singular measure, the traditional density estimator will fail.
- However, the density ranking may still be a well-defined quantity and we can estimate it consistently.
- Using the idea of density ranking, we can analyze complex datasets such as the GPS data.
- Many open questions: generalizing to point processes, modeling the temporal trends, assessing the uncertainty.

Thank You!

An R script for density ranking:

https://github.com/yenchic/density_ranking

More details can be found in

http://faculty.washington.edu/yenchic/

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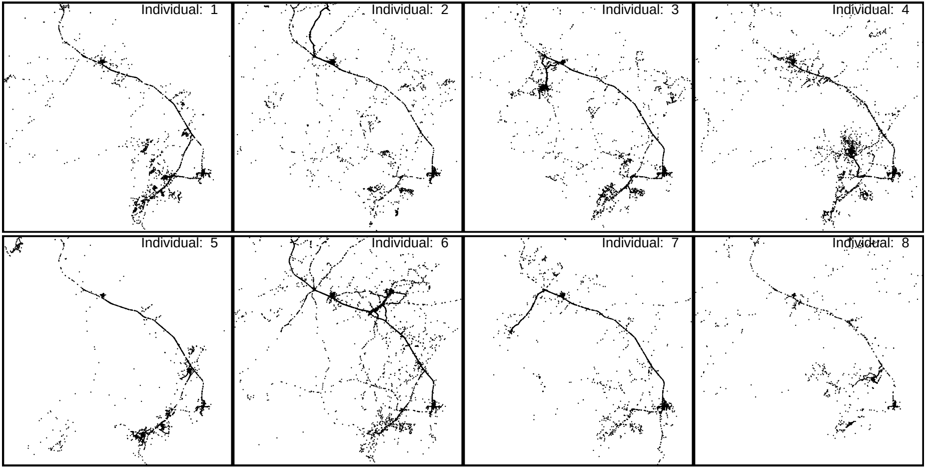
Real Person Datasets

- This data is about 10 real person's GPS records from Chen and Dobra (2017).
- All these participants share the same work place.
- The ages of the study participants were between 34 and 48 years.
- Each person has around $3,500$ to $8,500$ GPS records.

Real Persons Datasets: Raw Data



African Animal Datasets

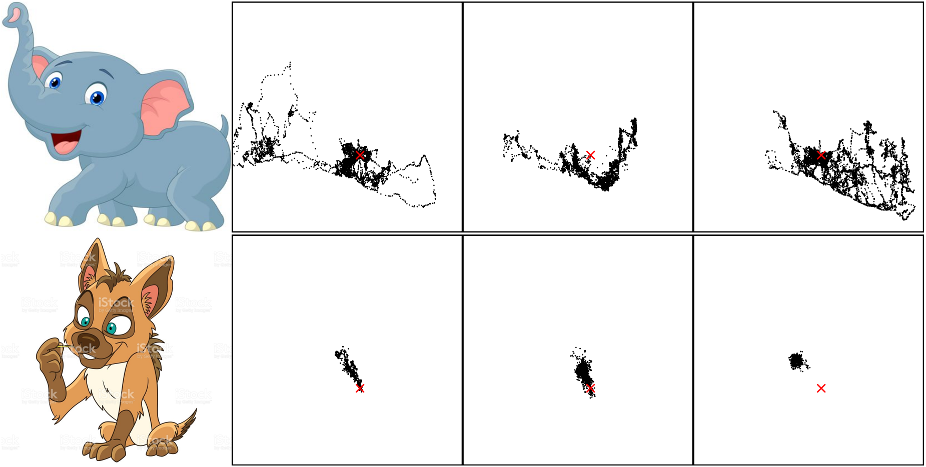
- This data is from the Movebank Data Repository[1] and was analyzed in Abrahms et al. (2017).
- Here we compare 4 different types of animals: elephants, jackals, vultures, and zebras.
- In this data, we have 8 elephants, 15 jackals, 10 vultures, and 9 zebras.
- Each animal has a set of GPS records with record size ranging from $1,000$ to $10,000$.

[1] https://www.datarepository.movebank.org/

African Animal Datasets: Raw Data




African Animal Datasets: Raw Data



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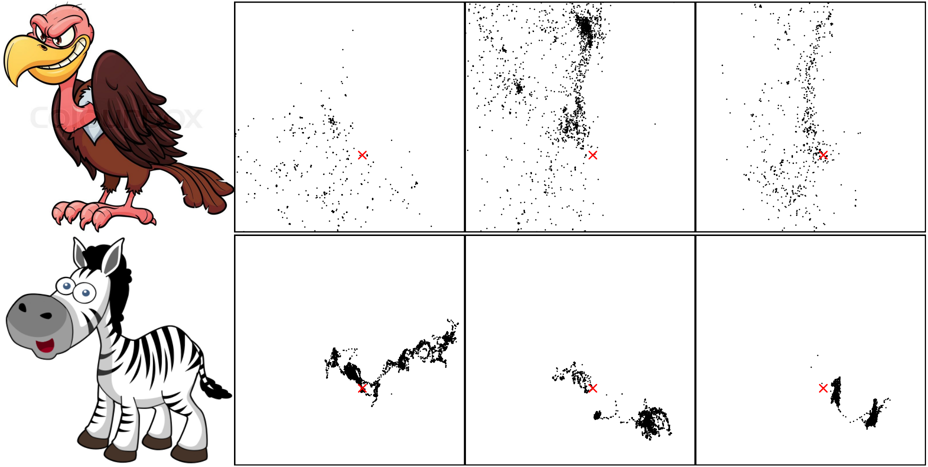

A statistical model for GPS dataset -1

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  where P_{GPS} is a probability distribution.
- Because of the natural of GPS records, we can decompose P_{GPS} as

$$P_{\mathsf{GPS}}(x) = \pi_0 P_0(x) + \pi_1 P_1(x) + \pi_2 P_2(x),$$

  where $P_0(x)$ is a distribution of point mass, and $P_1(x)$ is a distribution of a $1D$ density function, and $P_2(x)$ is a distribution of a $2D$ density function, and $\pi_0 + \pi_1 + \pi_2 = 1$ with $\pi_j \geq 0$ are proportions.

A statistical model for GPS dataset - 2

$$P_{\mathsf{GPS}}(x) = \pi_0 P_0(x) + \pi_1 P_1(x) + \pi_2 P_2(x).$$

- $P_0(x)$: a distribution that puts probability on the anchor/key locations.
- $P_1(x)$: a distribution describing the path/road that an agent takes.
- $P_2(x)$: a distribution describing the activity on an open space.

Assumptions for Regular Distributions

(R1) The density function p has a compact support $\mathbb{K}$.

(R2) The density function is a Morse function and is in $\mathbf{BC}^3$.

(K1) The kernel function K is in $\mathbf{BC}^2$ and integrable.

(K2) K satisfies the VC-type class condition.

(K2) Let

$$\mathcal{K}_r = \left\{ y \mapsto K^{(\alpha)}\left(\frac{x-y}{h}\right) : x \in \mathbb{R}^d, |\alpha| = r \right\},$$

where $K^{(\alpha)}$ is the α-th derivative and let $\mathcal{K}_l^* = \bigcup_{r=0}^{l} \mathcal{K}_r$. We assume that $\mathcal{K}_2^*$ is a VC-type class. i.e. there exists constants A, v and a constant envelope b_0 such that

$$\sup_Q N(\mathcal{K}_2^*, \mathcal{L}^2(Q), b_0\epsilon) \leq \left(\frac{A}{\epsilon}\right)^v, \tag{1}$$

where $N(T, d_T, \epsilon)$ is the ϵ-covering number for an semi-metric set T with metric d_T and $\mathcal{L}^2(Q)$ is the L_2 norm with respect to the probability measure Q.

Assumptions for Singular Distributions

(S1) The support can be partitioned into

$$K = K_0 \bigcup K_1 \bigcup \cdots \bigcup K_d,$$

where $K_\ell = \{x \in \mathbb{K} : \tau(x) = \ell\}$.

(S2) There exist $\rho_{\min}, \rho_{\max}$ such that $0 < \rho_{\min} \leq \rho(x) \leq \rho_{\max} < \infty$ for every $x \in \mathbb{K}$.

(S3) Restricted to each $\mathbb{K}_\ell$ where $\ell > 0$, $\rho(x)$ is a Morse function.

(K1') The kernel function K is in $\mathbf{BC}^2$, integrable, and supported in $[-1, 1]$.

(K2) K satisfies the VC-type class condition.

Estimating a Density Tree (Continue)

- To measure the estimation error, a simple metric is

$$d_\infty(\widehat{T_p}, T_p) = \sup_x \|\widehat{p}_n(x) - p(x)\|,$$

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- Under smoothness conditions and $n \to \infty, h \to 0$,

$$P_n \geq 1 - e^{-nh^{d+4} \cdot C_p},$$

for some constant C_p depending on the density function p.

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- Hartigan consistency (Chaudhuri and Dasgupta 2010; Balakrishnan et al. 2013) is another way to measure the consistency of a tree estimator.
- Note: density tree can also be recovered by a kNN approach; see Chaudhuri and Dasgupta (2010) and Chaudhuri et al. (2014) for more details.

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Convergence under Singular Measure: Density Ranking

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- Example of non-convergence of supreme norm: consider a sequence of points on a higher dimensional space but moving toward a lower dimensional space within distance $\frac{h}{2}$.

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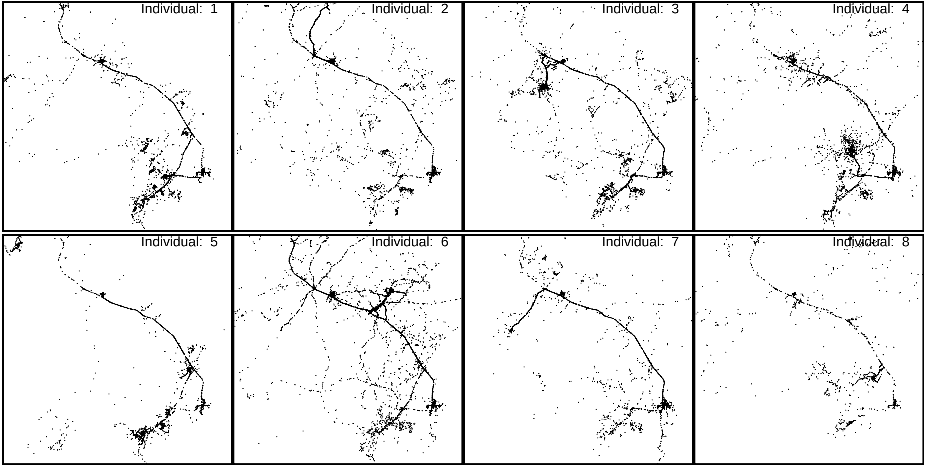
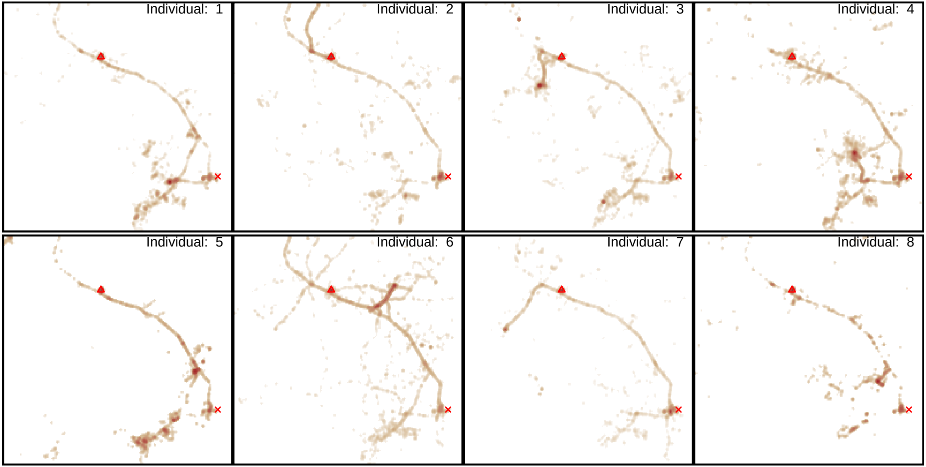
- Although we do not have uniform convergence, we can still recover the topology of the tree.
- In addition, the height of each branch of the tree will also converge.

Application of Density Ranking: GPS dataset - 1



Joint work with Adrian Dobra and Zhihang Dong.

Application of Density Ranking: GPS dataset - 2



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Summarizing Multiple Density Ranking: Level Plots - 1

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  be the (upper) level set.
- We can compare the density ranking of each individual by overlapping their level sets at different levels.

Summarizing Multiple Density Ranking: Level Plots - 2

- Note that we use $1-\gamma$ as the level in the set $\widehat{A}_{\gamma}$.
- This is because such a set has a natural interpretation in activity space.
- Activity space: the spatial regions where an individual undertakes his/her daily life.

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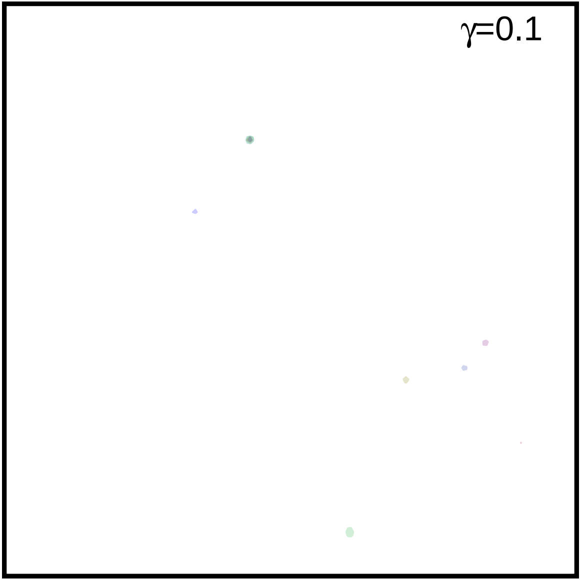
- Activity space: the spatial regions where an individual undertakes his/her daily life.
- We can interpret $\widehat{A}_\gamma$ as the (top) $\gamma \cdot 100\%$ activity space because they are regions containing at least $\gamma \cdot 100\%$ GPS records.
- Namely, $\widehat{A}_{\gamma=0.3}$ is the (top) 30% activity space.


γ=0.1

Level Plots: Example

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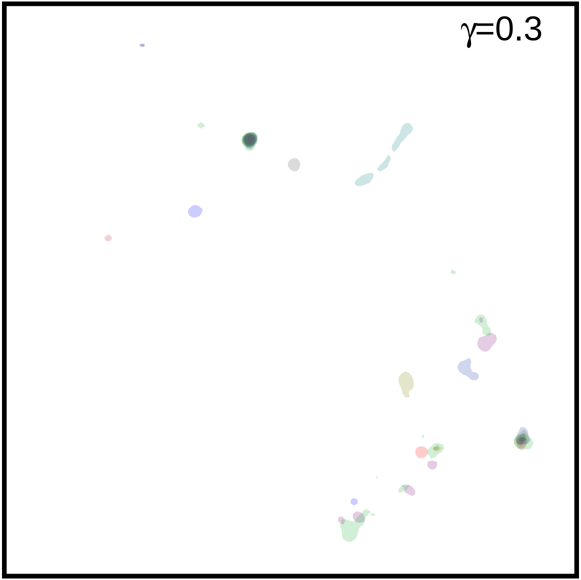

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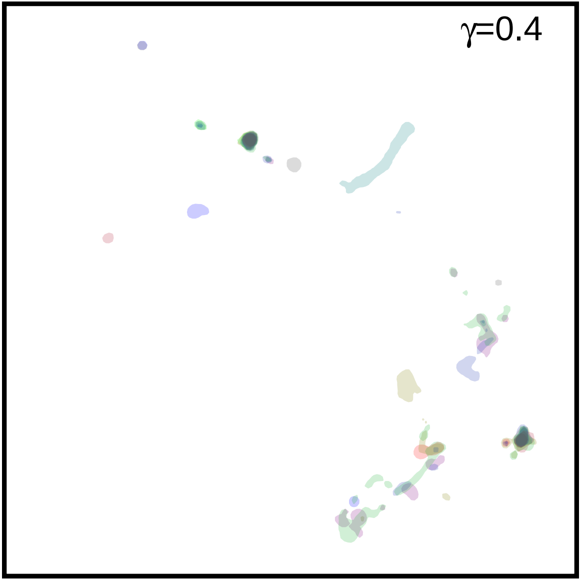

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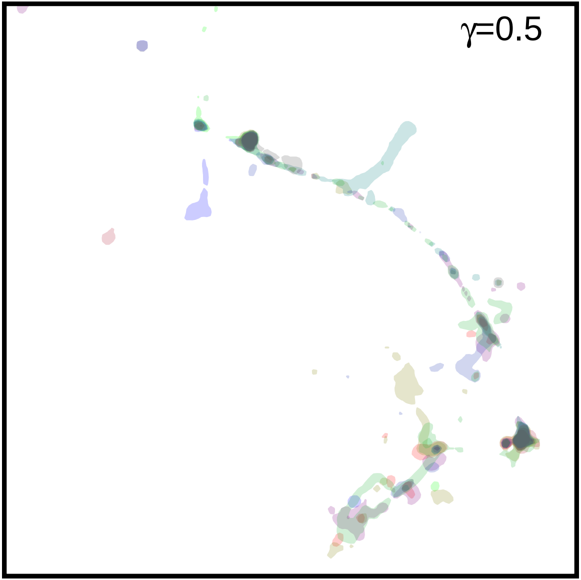
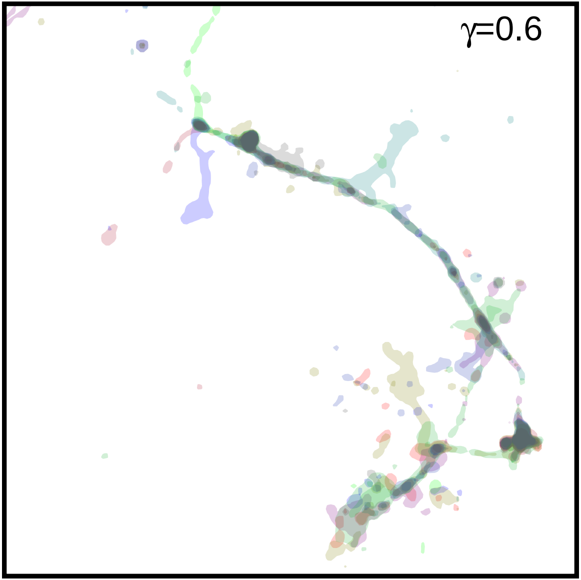
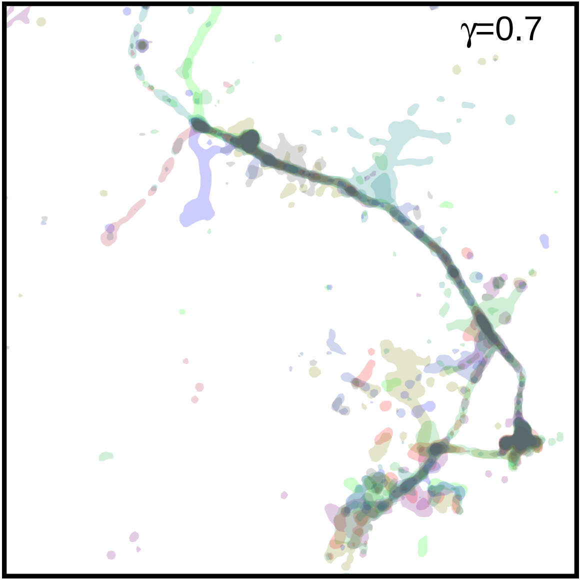
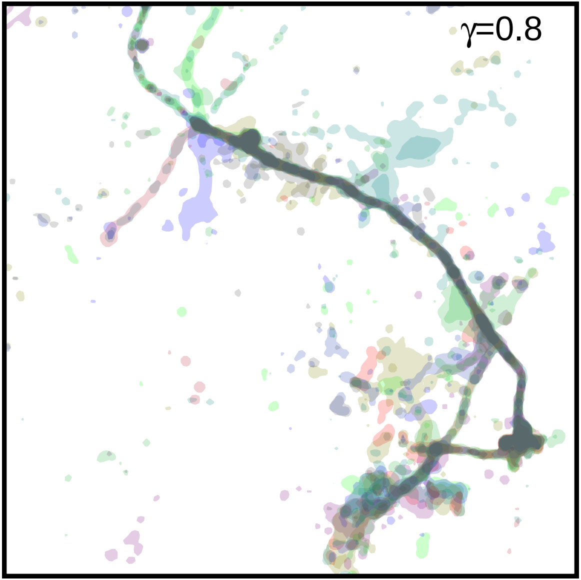


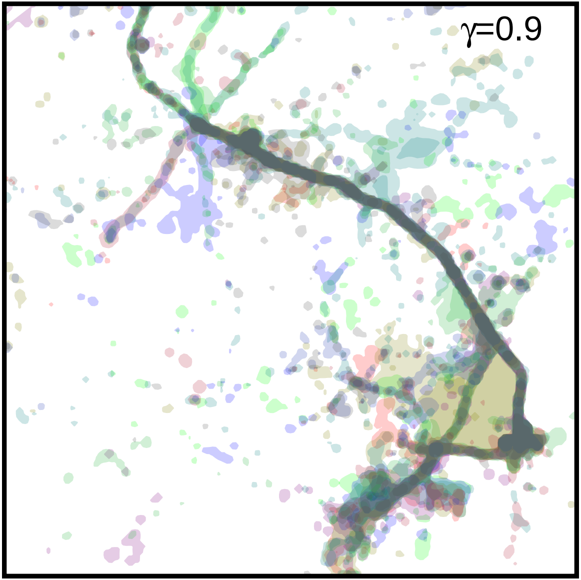

γ=0.6


γ=0.7

Level Plots: Example



Level Plots: Example



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