Asymptotic Theory for Density Ridges

Yen-Chi Chen

Christopher R. Genovese   Larry Wasserman

Department of Statistics
Carnegie Mellon University

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Density Ridges: High Density Curves

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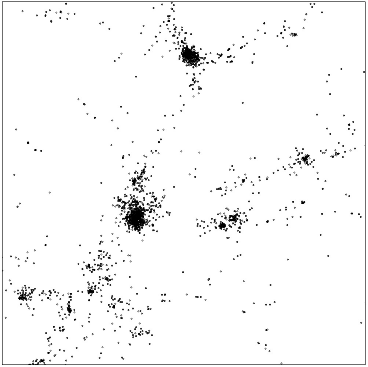

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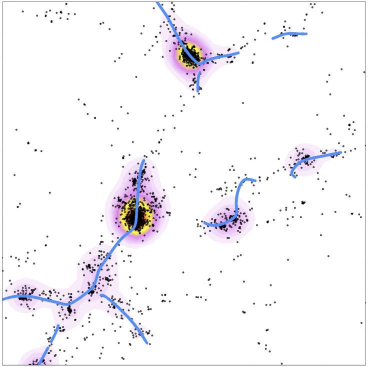
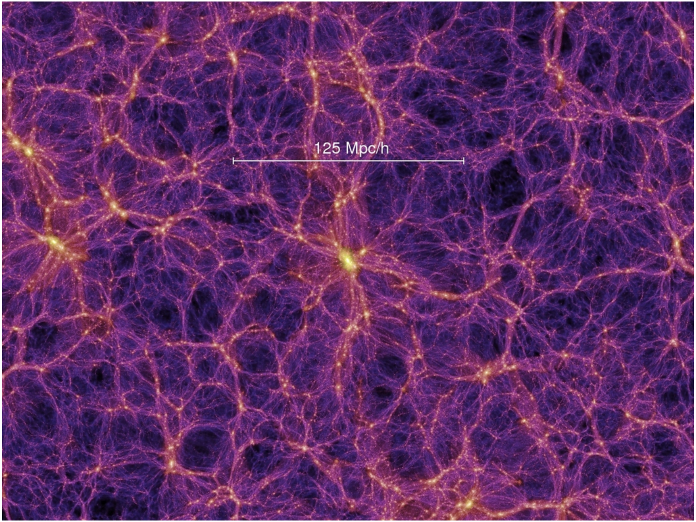


Application of Ridges: Cosmology



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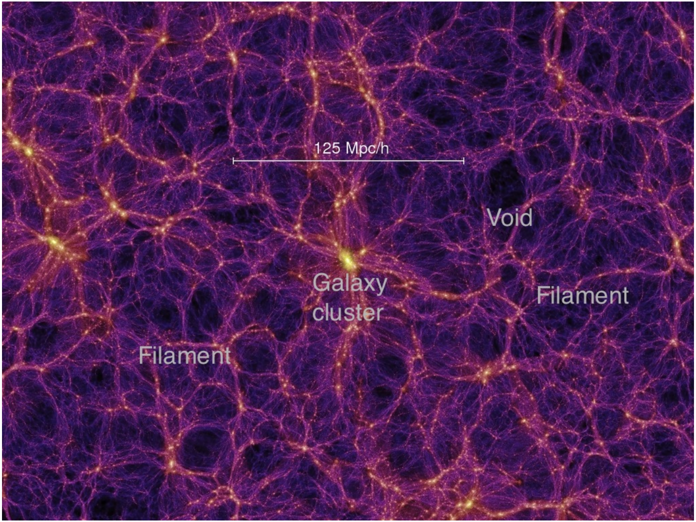

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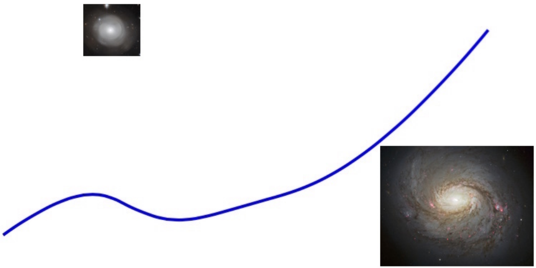

→ **Chen** et al. 'Detecting Effects of Filaments on Galaxy Properties in Sloan Digital Sky Survey III' (2015)

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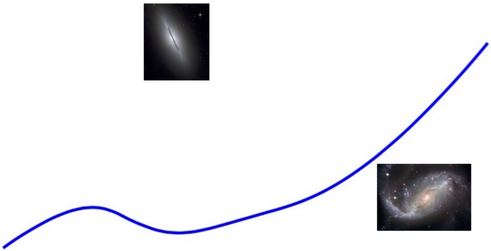
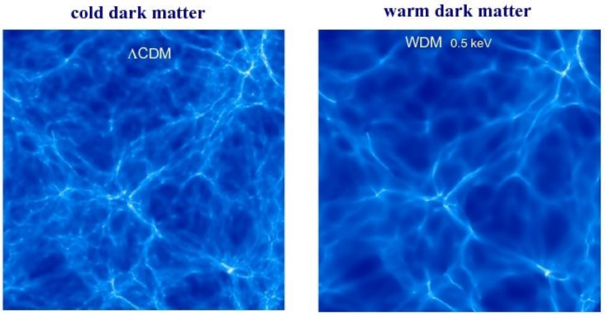


→ **Chen** et al. 'Investigating Galaxy-Filament Alignment in Hydrodynamic Simulations using Density Ridges' (Mon. Not. Roy. Astro. Soc. 2015)

The Importance of Filaments

Filaments play key roles in astronomy research.

- A galaxy's brightness, mass, and size are associated with filaments.
- A galaxy's alignment is associated with filaments.
- Filaments can be used to test cosmological models.



- Credit: Kavli Institute for Cosmology, Cambridge

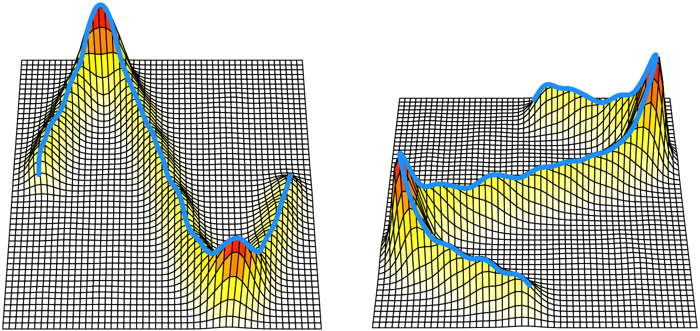
Density Ridges

A statistical model for filaments is the *density ridges*.

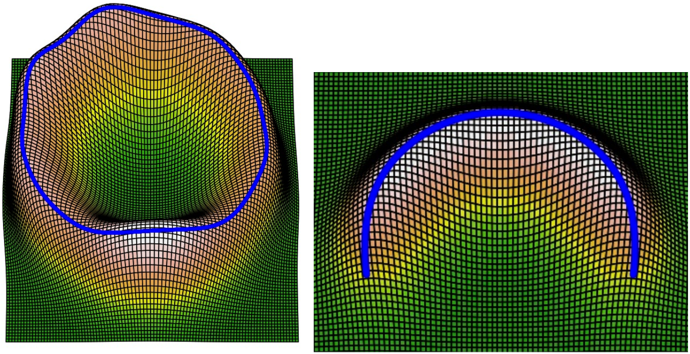
Example: Ridges in Mountains

Credit: Google

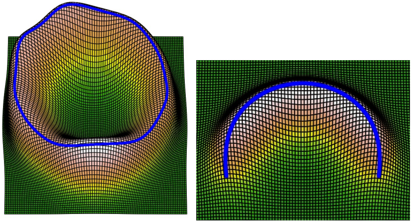
Example: Ridges in Smooth Functions



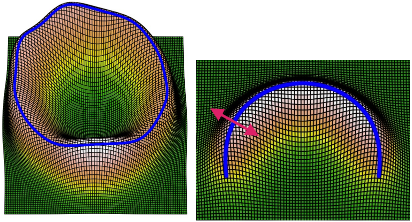
Example: Ridges in Smooth Functions



Ridges: Local Modes in Subspace



- A generalized local mode in a specific 'subspace'.



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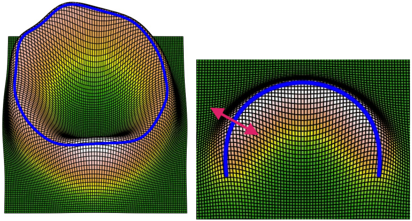

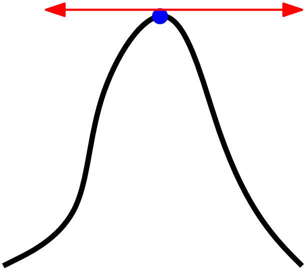

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Formal Definition of Density Ridges

- $p(x)$: the density function.

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$$R = \mathrm{Ridge}(p) = \{x : V(x)V(x)^T \nabla p(x) = 0, \lambda_2(x) < 0\},$$

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$$R = \mathrm{Ridge}(p) = \{x : V(x)V(x)^T \nabla p(x) = 0, \lambda_2(x) < 0\},$$

- Local modes:

$$\mathrm{Mode}(p) = \{x : \nabla p(x) = 0, \lambda_1(x) < 0\}.$$

Estimator and Algorithm

We use the plug-in estimate:

$$\widehat{R}_n = \mathrm{Ridge}(\widehat{p}_n),$$

where $\widehat{p}_n = \frac{1}{nh^d} \sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)$ is the KDE.

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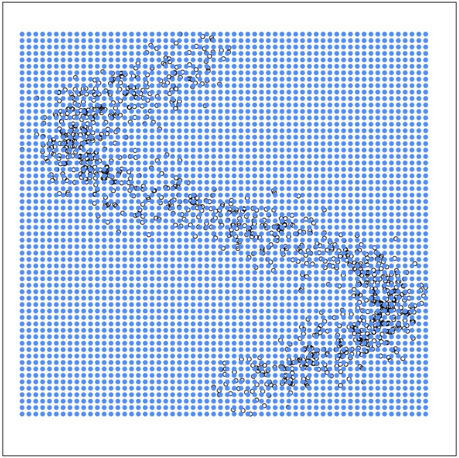
- In general, finding ridges from a given function is hard.
- The Subspace Constraint Mean Shift (SCMS; Ozertem2011) algorithm allows us to find $\widehat{R}_n$, ridges of the KDE.

SCMS: Ridge Recovery Algorithm



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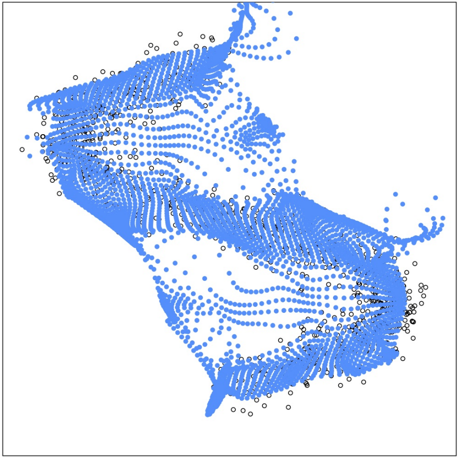

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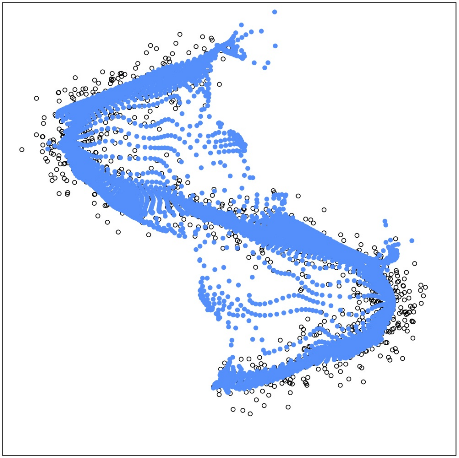

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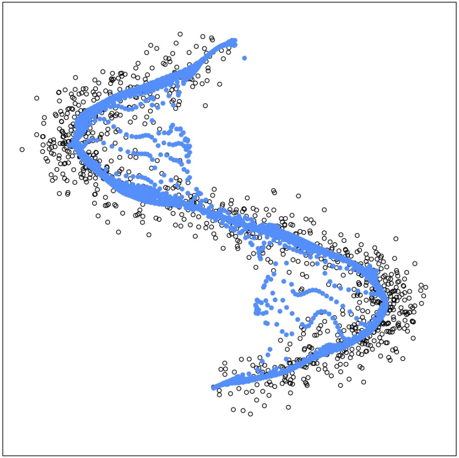

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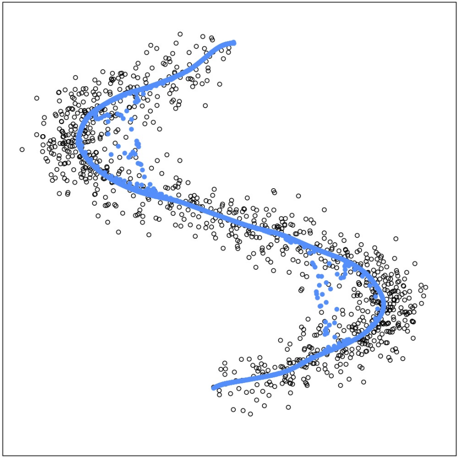

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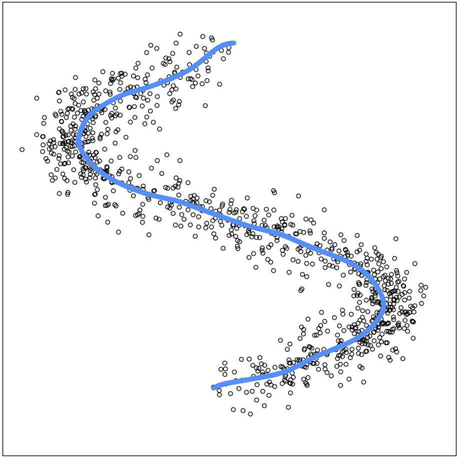
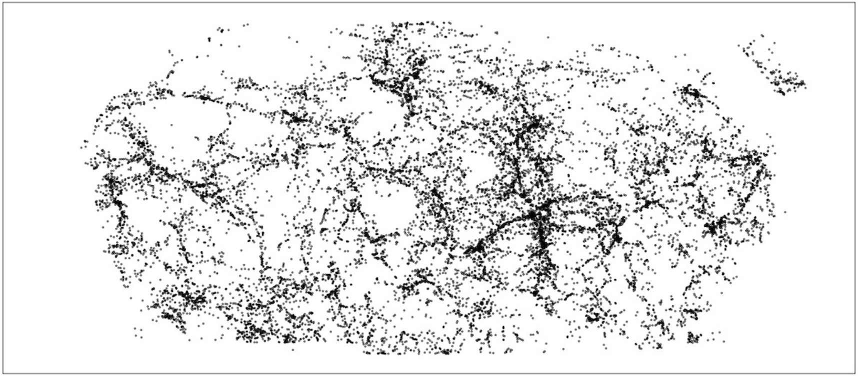


Example for Estimated Density Ridges



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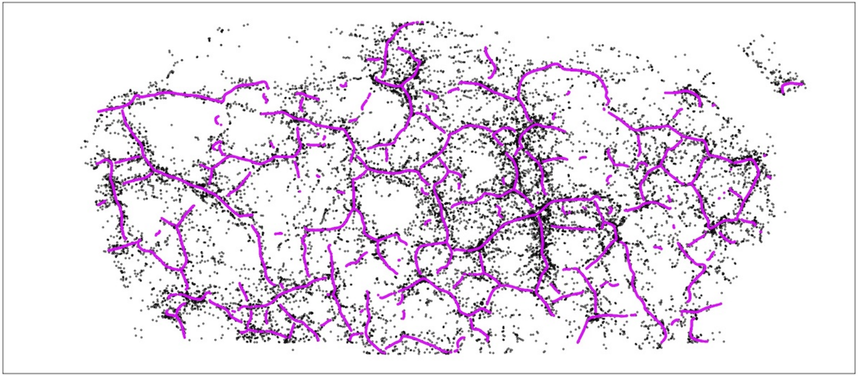

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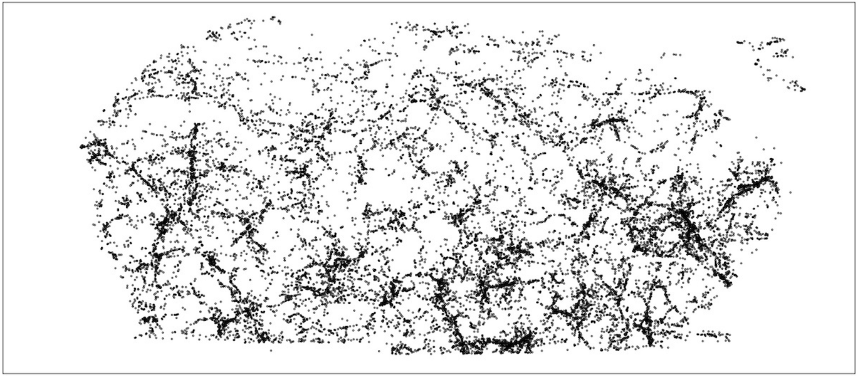

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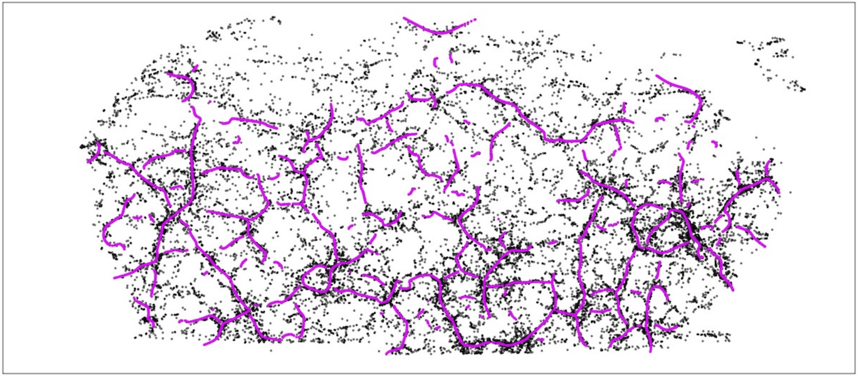
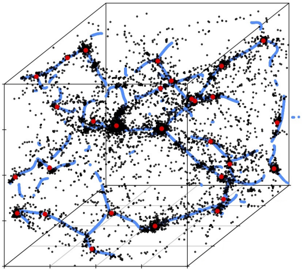


3D Example for Estimated Ridges



Blue curves: density ridges.
Red points: density local modes.

Statistical Inference: Confidence Sets

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In what follows, we ignore the bias for estimating R and focus only on the stochastic variation of $\widehat{R}_n$.

Useful Metric: Hausdorff Distance

We introduce a useful metric–*the Hausdorff distance* for sets:

$$\mathrm{Haus}(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\right\},$$

where $d(x, A) = \inf_{y\in A} \|x - y\|$ is the projection distance.

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- Haus is an $\mathcal{L}_\infty$ metric for sets.
- Consistency: $\mathrm{Haus}(\widehat{R}_n, R) = o_{\mathbb{P}}(1)$.

The $\oplus$ Operation

We define $A \oplus r = \{x : d(x, A) \leq r\}$.

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Then we have the following inclusion property:

$$A \subset B \oplus \text{Haus}(A, B), \quad B \subset A \oplus \text{Haus}(A, B).$$

Hausdorff Distance and Confidence Sets

We can use Hausdorff distance and $\oplus$ operation to construct confidence sets.

Let F_n be the CDF for $\mathrm{Haus}(\widehat{R}_n, R)$ and $t_{1-\alpha} = F_n^{-1}(1-\alpha)$ be the $1-\alpha$ quantile.

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- It can be shown that

$$\mathbb{P}\left(R \subset \widehat{R}_n \oplus t_{1-\alpha}\right) \geq 1-\alpha.$$

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- We need to find the distribution F_n.

Asymptotic Theory

Key observation:

$$\begin{aligned}\sqrt{nh^{d+2}}\mathrm{Haus}(\widehat{R}_n, R) &\approx \sqrt{nh^{d+2}} \sup_{x\in R} d(x, \widehat{R}_n)\\ &\approx \sup\left\{\text{Empirical process on } R\right\}\\ &\approx \sup\left\{\text{Gaussian process on } R\right\}.\end{aligned}$$

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$$

Theorem

Under regularity conditions and $\frac{\log n}{nh^{d+6}} \to 0$*, there exists a tight Gaussian process* $\mathbb{B}$ *defined on a certain function space* $\mathcal{F}$ *such that*

$$
\sup_t \left| \mathbb{P}\left(\sqrt{nh^{d+2}}\mathrm{Haus}(\widehat{R}_n, R) < t\right) - \mathbb{P}\left(\sup_{f\in\mathcal{F}} |\mathbb{B}(f)| < t\right)\right| = O\left(\left(\frac{\log^7 n}{nh^{d+2}}\right)^{1/8}\right).
$$

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$\longrightarrow$ A solution: the bootstrap.

The Bootstrap Consistency

- Bootstrap sample $\Longrightarrow$ bootstrap ridges $\widehat{R}_n^*$.
- Compute Haus$(\widehat{R}_n^*, \widehat{R}_n)$ to get a CDF estimator $\widehat{F}_n$.
- Choose $\widehat{t}_{1-\alpha}$ be the $1-\alpha$ quantile for $\widehat{F}_n$.

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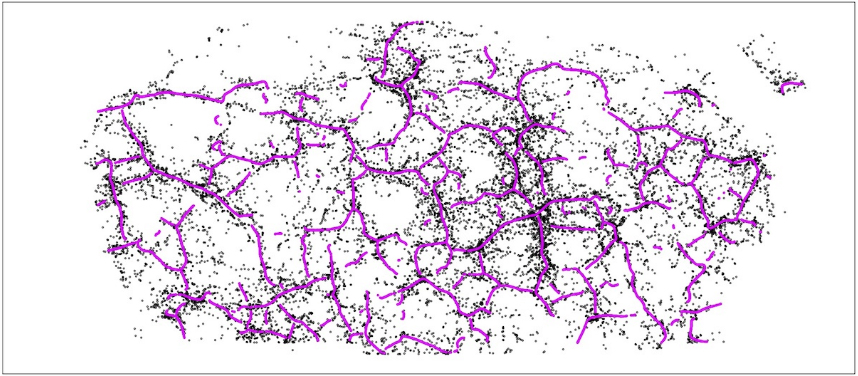
Theorem

Under regularity conditions and $\frac{\log n}{nh^{d+6}} \to 0$,

$$\mathbb{P}\left(R \subset \widehat{R}_n \oplus \widehat{t}_{1-\alpha}\right) = 1-\alpha + O\left(\left(\frac{\log^7 n}{nh^{d+2}}\right)^{1/8}\right).$$

Example for Confidence Sets



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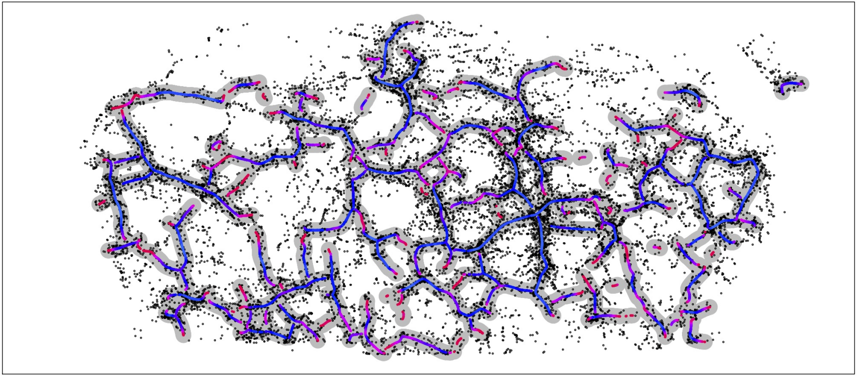
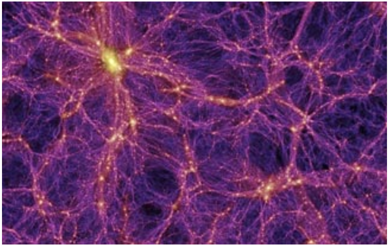


Concluding Remarks

Density ridges are very cool objects because

1. they have cosmological applications,



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Density ridges are very cool objects because

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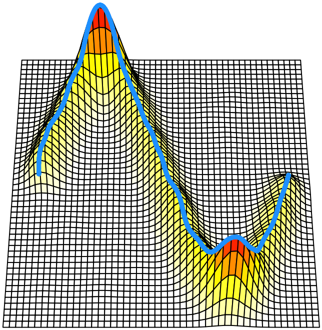

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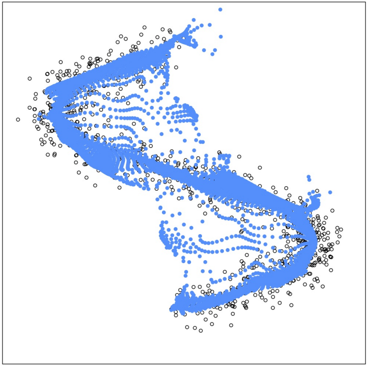

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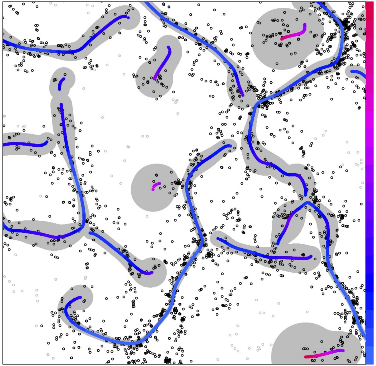
Density ridges are very cool objects because

1. they have cosmological applications,
2. they are well-defined objects,
3. there is a fast algorithm to compute them,
4. their statistical properties are well-studied.



Thank you!

More details can be found in: `http://www.stat.cmu.edu/~yenchic/`

References

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Regularity Conditions

(K1) The kernel function K is $\mathbf{BC}^4$ and integrable.

(K2) K satisfies the VC-type class condition.

(P1) The density p is in $\mathbf{BC}^4$.

(P2) The eigengap $\lambda_1(x) - \lambda_2(x) \geq \beta_0 > 0$ for points around ridges.

(P3) The orientation of each ridge point is close to the gradient.

Regularity Conditions on Kernel Functions

(K1) The kernel K is in $\mathbf{BC}^4$ and $\|K\|^*_{\infty,4} < \infty$.

(K2) Let

$$\mathcal{K}_r = \left\{ y \mapsto K^{(\alpha)}\left(\frac{x-y}{h}\right) : x \in \mathbb{R}^d, |\alpha| = r \right\},$$

where $K^{(\alpha)}$ is the α-th derivative and let $\mathcal{K}^*_l = \bigcup_{r=0}^{l} \mathcal{K}_r$. We assume that $\mathcal{K}^*_4$ is a VC-type class. i.e. there exists constants A, v and a constant envelope b_0 such that

$$\sup_Q N(\mathcal{K}^*_4, \mathcal{L}^2(Q), b_0\epsilon) \leq \left(\frac{A}{\epsilon}\right)^v, \tag{1}$$

where $N(T, d_T, \epsilon)$ is the ϵ-covering number for an semi-metric set T with metric d_T and $\mathcal{L}^2(Q)$ is the L_2 norm with respect to the probability measure Q.

Regularity Conditions on Distributions

(P1) The density p is in $\mathbf{BC}^4$.

(P2) There exists constants $\beta_0, \beta_1, \beta_2, \delta_0 > 0$ such that

$$\begin{aligned} \lambda_2(x) &\leq -\beta_1 \\ \lambda_1(x) &\geq \beta_0 - \beta_1 \\ \|g(x)\| \max_{|\alpha|=3} |p^{(\alpha)}(x)| &\leq \beta_0(\beta_1 - \beta_2) \end{aligned} \tag{2}$$

for all $x \in R \oplus \delta_0$.

(P3) For each $x \in R$, $|e(x)^T g(x)|^2 \geq \frac{\lambda_1(x)}{\lambda_1(x) - \lambda_2(x)}$ where $e(x)$ is the direction of R at point $x \in R$.

Smoothed Density Ridges

In particular, we focus on making inference for the smoothed version of the density, denoted as p_h:

$$p_h(x) = p \otimes K_h(x) = \mathbb{E}\left(\widehat{p}_n(x)\right), \quad K_h(x) = \frac{1}{h^d} K\left(\frac{x}{h}\right),$$

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  - Fast rate of convergence.

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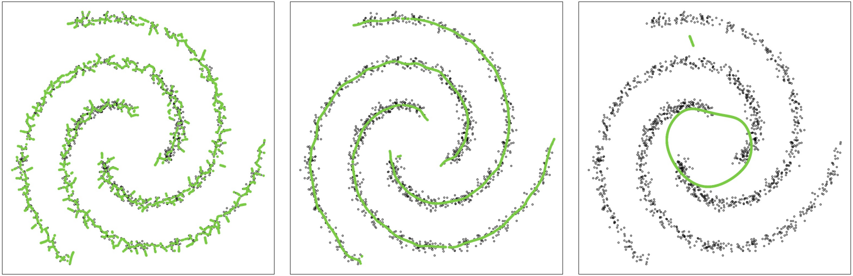
- One can always slightly undersmooth so that inference for R_h is asymptotically valid for R.

Bandwidth Selection for Density Ridges

Effect of Smoothing Bandwidth



Risk for Ridges

Let R and $\widehat{R}_n$ be the density ridges and their estimators.
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Define

$$W_n = d(U_R, \widehat{R}_n), \quad \widetilde{W}_n = d(U_{\widehat{R}_n}, R)$$

be the projected distance of U_R onto $\widehat{R}_n$ and $U_{\widehat{R}_n}$ onto R.
We define L_2 risk as

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- This is a generalized mean integrated square errors.
- Similarly, one can define $\mathrm{Risk}_{1,n}$ using L_1 loss.

Estimating Risks

We can use bootstrap or data splitting to estimate the risk $\mathrm{Risk}_{2,n}$. Let $\widehat{R}_n^*$ be the bootstrap version of $\widehat{R}_n$. Let

$$W_n^* = d(U_{\widehat{R}_n}, \widehat{R}_n^*), \quad \widetilde{W}_n^* = d(U_{\widehat{R}_n^*}, \widehat{R}_n)$$

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$$\widehat{\mathrm{Risk}}_{2,n} = \frac{1}{2}\mathbb{E}(W_n^{*2} + \widetilde{W}_n^{*2} | X_1, \cdots, X_n).$$

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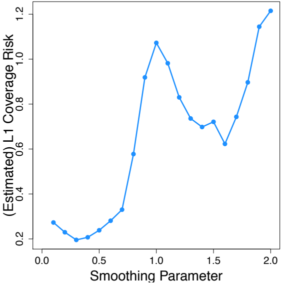
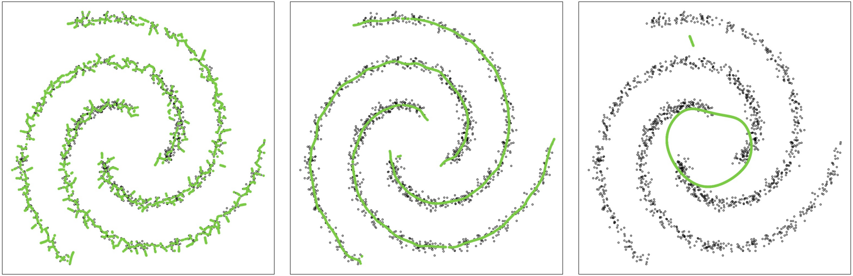
Theorem

Under regularity conditions,

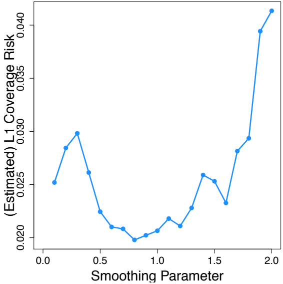
$$\frac{\widehat{\mathrm{Risk}}_{2,n}}{\mathrm{Risk}_{2,n}} \xrightarrow{P} 1, \quad \frac{\widehat{\mathrm{Risk}}_{1,n}}{\mathrm{Risk}_{1,n}} \xrightarrow{P} 1.$$

Bandwidth Selection via Risk Minimization





Application to Cosmology Dataset



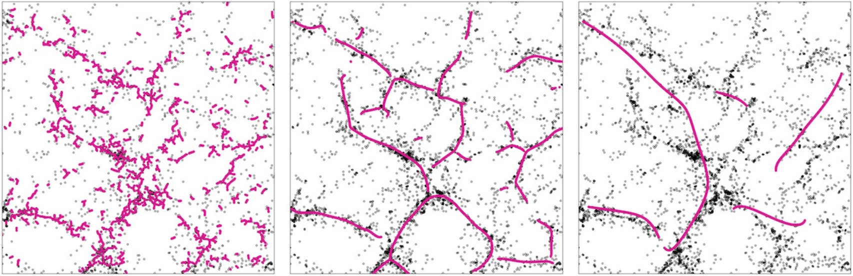
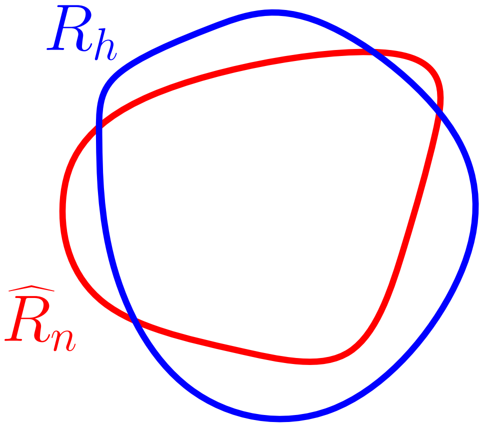
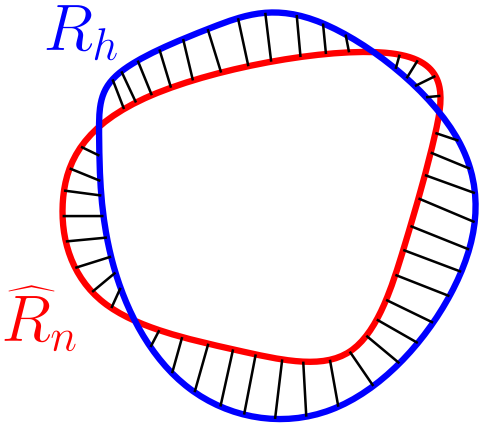


Illustration for Asymptotic Theory

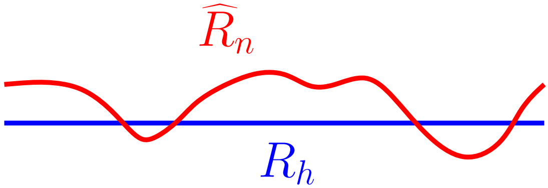
Asymptotic Theory



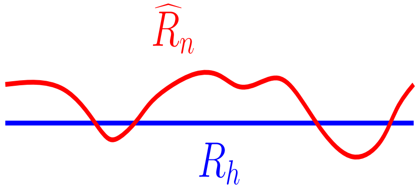
Asymptotic Theory



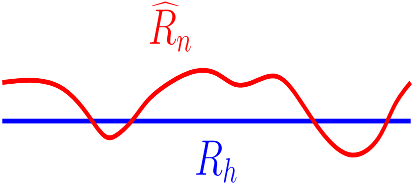
Asymptotic Theory



1. Thus, the projection distance $\approx$ a stochastic process.



1. Thus, the projection distance $\approx$ a stochastic process.
2. This stochastic process $\approx$ empirical process.



Asymptotic Theory

1. Thus, the projection distance $\approx$ a stochastic process.
2. This stochastic process $\approx$ empirical process.
3. $\mathrm{Haus}(\widehat{D}_n, D_h) =$ $\sup\{$projection distance$\} \approx$ $\sup\{$Empirical process$\}$.