Reconceptualizing Mathematics:
Courses for Prospective and Practicing Teachers

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> *Recommendation 1. Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach.* The Mathematical Education of Teachers, CBMS, 2001, p. 1.

Over the past few years faculty at San Diego State University have been developing courseware for prospective and practicing elementary school mathematics teachers. Our goal is to provide instructional materials that will promote a *reconceptualization* of the mathematics they often think they already know. By this we mean that they revisit elementary school mathematics in ways that allow them to make sense of the mathematics they teach, to learn to verse page orderrepresent mathematical ideas in different ways and appreciate how and why some representations work better than others, to see the connections among mathematical topics and understand the underlying ideas that link them together, and to identify and focus on the "big ideas" of the curriculum that begin in the early grades and become more sophisticated as children move up through the K-12 mathematics curriculum. We believe that teachers should know school mathematics at a much deeper level than most people do. Teachers need to know the mathematics they teach in a way that allows them to hold conversations about mathematical ideas and mathematical thinking with their students. To do so, teachers must be able to understand their students' reasoning and to build on this reasoning in a manner that allows their students' mathematical understanding to develop.

"Sense-making" is the dominant theme of these course materials, with regular expectations that students provide explanations and justifications. Students should not expect to be shown "the right way," as though imitation is the only way one can learn mathematics and as though there is only one way in which certain tasks can be accomplished. Many of the exercises and discussion questions are rich enough that different solution methods are possible and even likely.

These instructional materials are also appropriate for prospective middle school teachers in that they provide a foundation for the additional coursework that middle school teachers need.

To illustrate how we design our materials to help teachers reconceptualize the mathematics of the elementary school, I focus here primarily on the first module, with briefer descriptions of other modules. (The modules are not equal in length and can be combined in various ways for two or three semester-long courses.)

Reasoning About Numbers and Quantities

We begin with a module we call *Reasoning About Numbers and Quantities (RNQ)*. One major goal of this course is that preservice or inservice elementary school teachers develop a deep understanding of the numbers and number operations. A second major goal is to help students develop habits of reasoning quantitatively when working with word problems. The key ideas presented in this course are as follows, although not necessarily in this order:

Numbers

N1: Our numeration system can only be understood if the significant role of place value is recognized.

N2: There are different generic contexts that can be modeled by arithmetic operations. These contexts lead to different ways of thinking about the operations.

N3: Standard algorithms are not the only ways to carry out arithmetic operations. Alternate algorithms can be valuable in developing understanding of the steps of a standard algorithm.

N4: Different types of calculating skills are appropriate for different problematic situations. Mental computation, estimation, paper-and-pencil computation, and calculators all have their places in our daily lives and in our classrooms.

N5: Understanding the meanings of different representations of fractions and decimal numbers, and facility in changing easily from one representation to another, are fundamental to having good number sense.

N6: The underlying mathematics in algorithms for operating on rational numbers can and should be understood.

N7: Understanding the role of referent units is fundamental to understanding both place value and arithmetic operations.

Quantities

Q1: Situations can be better understood when they are analyzed by recognizing the quantities involved and the relationships among these quantities.

Q2: Understanding quantities includes understanding the values that the quantity can take on. Very large values and fractional values are the most difficult to understand and need special attention.

Q3: Quantities can be compared additively or multiplicatively, depending on the nature of the situation. Students should recognize whether a situation is additive or multiplicative in nature and know how to respond.

Q4: A multiplicative comparison of two quantities results in a ratio, which is itself a quantity.

Q5: A ratio can be thought of as a measure of some attribute.

The *Reasoning About Numbers and Quantities* module begins by requiring that students distinguish between quantities as properties of objects that can be measured, and numbers as the values we attach to quantities. Students then begin to analyze quantitative situations by listing all the quantities associated with the problem and identifying the relationships that exist among those quantities (Q1 above). Many times these relationships become clear through a diagram or drawing. Only then are the values of the quantities considered. This type of analysis provides students with an approach to a problem that replaces the usual habit of beginning the solution by attending to the numbers and working with the numbers before fully understanding the problem.

Here is an example of a quantitative analysis of a situation:

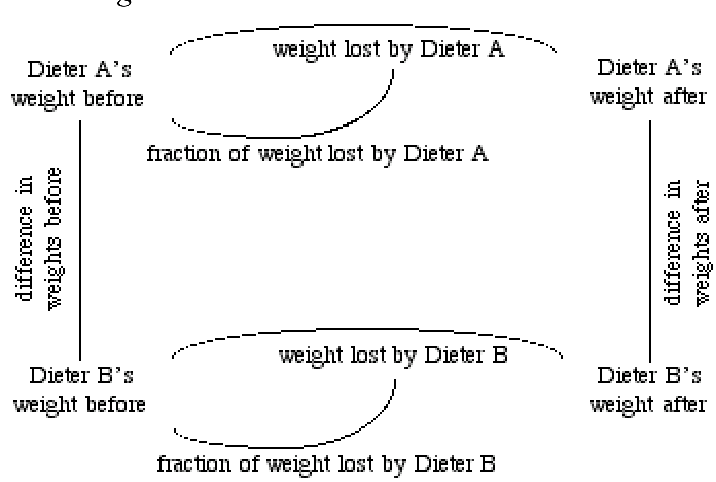
Dieter A: "I lost $\frac{1}{8}$ of my weight. I lost 19 pounds."

Dieter B: "I lost $\frac{1}{6}$ of my weight, and now you weigh 2 pounds less than I do." How much weight did Dieter B lose?

The following are some of the possibly relevant quantities:

- *Dieter A's weight before the diet*
- *Dieter A's weight after the diet*
- *Dieter B's weight before the diet*
- *Dieter B's weight after the diet*
- *The amount of weight lost by Dieter A*
- *The amount of weight lost by Dieter B*
- *The difference in their weights before the diets*
- *The difference in their weights after the diets*

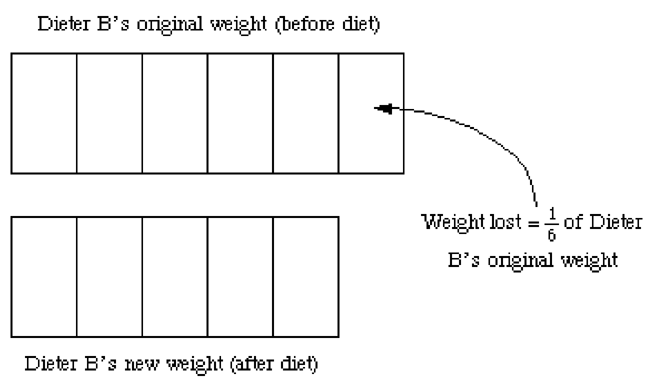
A diagram can help students see how these quantities are related to each. Here is one example of such a diagram:



The following key points need to be emphasized:

- This diagram involves only quantities—no actual values.
- Understanding the relationships in this diagram *IS* understanding the problem.
- It doesn't really matter what values are given in the problem (within reason, of course), because once we understand the diagram we can use the relationships between quantities to find the values that we need. Understanding the problem is independent of the values that are given in the problem.

This diagram shows that there are two different ways to compare the weights before and after the diet. One can either look at the difference (the weight lost) or the fraction of the weight lost for each dieter. Given the fraction of weight lost and the original weight, many students can find the new weight. However, students have difficulty trying to find the original weight of Dieter B, given the fraction lost and the new weight. To help students see the relationships between these quantities, another diagram is helpful:



As students move through the RNQ module, they encounter other types of problem situations for which a quantitative analysis is required, particularly in later sections that focus on proportional reasoning.

Of course, reasoning about numbers and number operations is also a focus. Teachers need to understand that different situations give rise to different ways of thinking about the same number operations (See N2 above). For example, division can be thought about as "sharing" when asked how many cards each person gets when 52 cards are passed out to 4 people. But if, from a deck of 52 cards, each person gets 13 cards, one can ask how many people get cards, a situation can be thought about as repeated subtraction; 13 can be subtracted from 52 four times. Even in the primary grades students can solve such problems through sharing and repeated subtraction, and later, algorithms that use repeated subtraction to find a quotient can lead to an understanding of the standard algorithm for division. When I have demonstrated how one can progress from repeated subtraction to the standard algorithm, some prospective teachers have actually become angry because, they claimed, they had suffered so long trying to learn to long division, and after the demonstration they could see that it was actually easy to understand.

Repeated subtraction can also lead to an understanding of the infamous “invert and multiply” rule for dividing fractions (See N6 above). For example, one could reason:

(a) $1 \div \frac{1}{3}$ *asks how many* $\frac{1}{3}$*s are in 1? (How many times can* $\frac{1}{3}$ *be subtracted from 1?)*

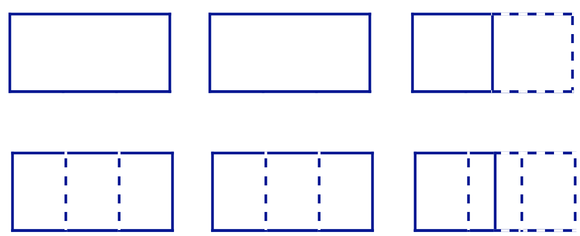
There are 3 $\frac{1}{3}$*s in 1, so* $1 \div \frac{1}{3} = 3$*, which is the same as* $\frac{3}{1}$.
So there are 3 one-thirds, or $\frac{3}{1}$ *one-thirds in 1. Notice that* $\frac{3}{1}$ *is the reciprocal of* $\frac{1}{3}$.

(b) $2 \div \frac{1}{3} = ?$ *There should be twice as many* $\frac{1}{3}$*s as for* $1 \div \frac{1}{3}$,
That is, $2 \div \frac{1}{3} = 2 \times \frac{3}{1} = 2 \times 3 = 6$.

(c) $5 \div \frac{1}{3} = ?$ *There should be five times as many* $\frac{1}{3}$*s as for* $1 \div \frac{1}{3}$,
That is, $5 \div \frac{1}{3} = 5 \times \frac{3}{1} = 5 \times 3 = 15$.

(d) $2\frac{1}{2} \div \frac{1}{3} = ?$ *There should be two and a half times as many* $\frac{1}{3}$*s as for* $1 \div \frac{1}{3}$,
That is, $2\frac{1}{2} \div \frac{1}{3} = 2\frac{1}{2} \times \frac{3}{1} = 2\frac{1}{2} \times 3 = 7\frac{1}{2}$.

$2\frac{1}{2} \div \frac{1}{3} = ?$ *can also be demonstrated with a diagram:*



$2\frac{1}{2}$ wholes, so for $2\frac{1}{2} \div \frac{1}{3}$, there are $7\frac{1}{2}$ thirds.
OR
$2\frac{1}{2} \times \frac{3}{1} = 7\frac{1}{2}$
Note: the half of a third in the diagram may need discussion.

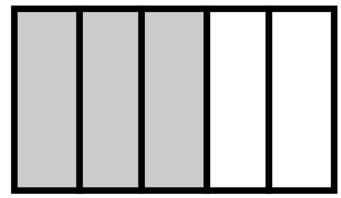
Once again, our experience with this explanation has elicited surprise from teachers, for whom it did not occur that there was a rationale for the rule of invert and multiply.

One reason that students, whether adults or children, have difficulty with multiplication and division of fractions is because the unit changes (see N7 above), and they do not attend to the unit described by each fraction.

In the problem above, the result is $7\frac{1}{2}$ ***thirds of a rectangle***. *The* $2\frac{1}{2}$ *and the* $\frac{1}{3}$ ***refer to the rectangle*** *but the* $7\frac{1}{2}$ *refers to the* ***thirds*** *of the rectangle.* So the problem $2\frac{1}{2} \div \frac{1}{3}$ can be interpreted as asking how many thirds of a rectangle are in 2 and a half of these rectangles.

Not attending to the referent units, which can change within a calculation, leads to many errors in arithmetic. Some explicit attention needs to be paid to this issue. Here is one example of how this might be done:

*a. Can you see $\frac{3}{5}$ of something in this picture? Where? Be explicit. ($\frac{3}{5}$ of **what**?)*



*b. Can you see $\frac{5}{3}$ of something in this picture? $\frac{5}{3}$ of **what?***

c. Can you see $\frac{2}{3}$ of something in this picture? .....is $\frac{2}{3}$ of .....

d. Can you see $\frac{2}{3}$ of $\frac{3}{5}$? What is the whole (the unit)? What part is $\frac{3}{5}$ of the whole? What part is $\frac{2}{3}$ of $\frac{3}{5}$ of the whole?

e. Can you see $1 \div \frac{3}{5}$? Think: How many $\frac{3}{5}$ are in 1?

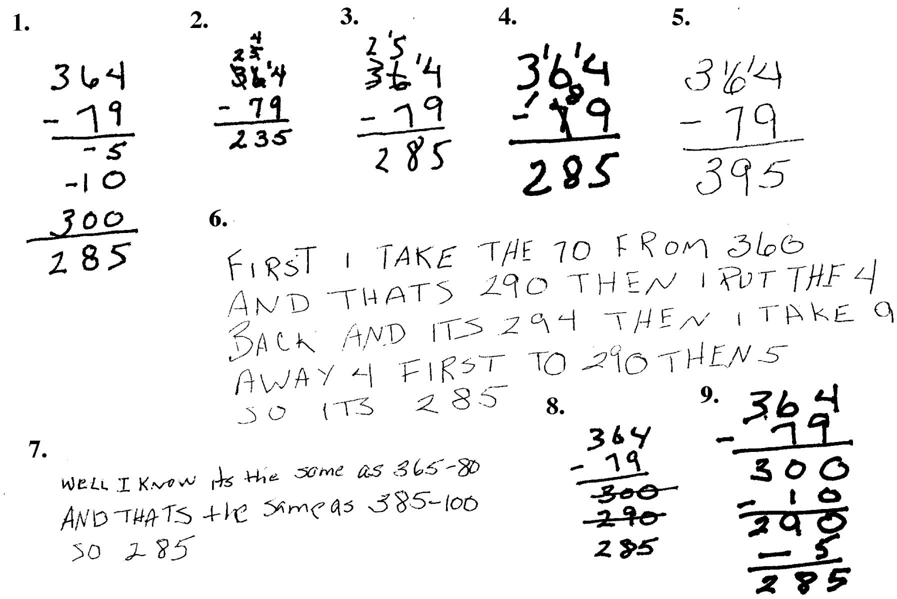
Children who have a good understanding of our base ten system can quite easily find ways of operating on numbers that are nonstandard (see N3 above). Teachers should be able to recognize when a student's reasoning is correct. In the following example, a subtraction calculation is solved in a variety of ways. When prospective teachers encounter this example, they often recognize that their own limited understanding of mathematics is not sufficient to comprehend the reasoning processes used by these children, and they become more aware of why they need to know mathematics at a much deeper level than they currently do.

Consider the work of six children, all solving 364 – 79

Identify: a. which students clearly understand what they are doing;

b. which students might understand what they are doing;

c. which students do not understand what they are doing.



Ratio and proportion often occupy only a few pages of an intermediate-grade textbook. Teachers themselves think of proportion as two equal ratios, and when one of the four numbers is unknown, cross-multiplication can be used to solve for the unknown. While this is not incorrect, it is a very limited understanding of both ratio and proportion. Ratio can in fact be used as a unit of measure. Consider this problem (see Q5 above) :

A new housing subdivision offers rectangular lots of three different sizes:

a. 75 feet by 114 feet

b. 455 feet by 508 feet

c. 185 feet by 245 feet

If you were to view these lots from above, which would appear "most square"? Which would appear "least square"? Explain your answers.

In this problem, what attribute or characteristic of the lots are we interested in? In what ways can this attribute be quantified?

This problem might seem trivial, but it is not. A large percentage of prospective teachers reason additively rather than multiplicatively for this problem; that is, they compare the differences rather than the ratios of the lengths and widths. Only by working with a variety of other sized rectangles do many of them understand that this is a multiplicative problem.

We include a CD from the Integrating Mathematics and Pedagogy Project (IMAP) with some videoclips of children doing mathematics that can help prospective teachers become more aware of the need to have a deep understanding of mathematics and the different ways of solving problems in order to teach mathematics well. The interviews are not only interesting, they are informative and highly motivational.

There are additional modules that together with the RNQ module offer flexibility to choose topics that fit into required and elective college and university courses at different institutions. At San Diego State we also use them for professional development leading to a mathematics specialist certificate. Together, the materials are adequate for three semester-long courses. (Using them for two courses will require that some sections will need to be deleted.)

The other modules are briefly described here.

Algebra and Change

In the Algebra and Change module students use algebra to solve problems numerically, graphically, and algebraically. Graphing and algebraic symbols are used to represent quantitative relationships, thus tying back to the module on quantitative reasoning.

An understanding of slope is approached through the practice of describing relationships between time, distance, and rate, in some instances using motion detectors. Average speed and weighted averages are explored. The ideas are all used to develop and interpret qualitative graphs of situations, and creating explanations involving quantitative relationships.

Shapes and Measurement.

Our experience suggests that beginning with the study of two-dimensional space leads students to think this course is "review." Thus beginning with the study of three dimensional space is a deliberate choice. We do this through a focus on developing visualization skills.

This module is quite long, and if all sections are included in a course it can serve as a geometry course for middle school teachers and is even worthwhile for secondary teachers. A secondary teacher we know who teaches this course on Saturdays for prospective teachers claims that the content of this course is far richer than the content of the geometry course she teaches for secondary students, even though formal proof is not an official part of this module.

Again, the focus is on reasoning and sense-making that permeates all modules. Some of the differences between this course and other geometry courses might be the following:

- Size changes are introduced in terms of scale factors rather than through proportions. This work with scale factors can lay a foundation for other work in mathematics.
- Algebraic reasoning is pursued in this module, primarily through pattern study and generalizations.
- A long measurement section builds on ideas of quantities and values of quantities from the RNQ module.

Collecting, Representing, and Interpreting Data

The primary goal for this module is to provide teachers with the background they need to teach the many statistical ideas now found in elementary and middle school mathematics curricula. Additionally, teachers as educated citizens need to be able to interpret common, everyday statistical statements, that is, to be statistically literate. Teachers need to understand statistical ideas they need as professional educators. Finally, teachers need to understand how children reason about statistical topics (such as measures of central tendency) throughout the elementary mathematics curriculum.

Uncertainty and Its Quantifications

What IS a probabilistic situation? Traditional ways of assigning probabilities to situations are explored. Simulations are studied via software demonstrations. Some traditional probability topics, such as independence, are addressed. This content, combined with the Data module, could be sufficient for a three unit semester course, but we recognize that most institutions do not offer a "statistics and probability" course for elementary teachers and will spend a limited amount of time on these topics.

Additional Features of Reconceptualizing Mathematics

Prospective teachers begin these courses with at least 12 years of apprenticeship in how mathematics is taught and what it means to learn mathematics. Many have never experienced mathematics as anything but the traditional "Review homework, watch the teacher do new problems, do some of the new problems as modeled by the teacher." We have tried to design these modules to allow instructors to model good pedagogy by introducing mathematical discussions in the classroom between the instructor and the students, and among students. A lesson typically has class activities and questions for discussion. Lecturing is not discouraged; rather, we view it as one of many modes of instruction to which prospective teachers should be exposed.

Prospective teachers need to understand the relevance of what they are learning. Examples from elementary school textbooks can be incorporated. Most chapters ends with a section called "Issues for Learning," which provides some information on research on learning the topics in the chapter.

We recognize that teaching from these modules will be quite different for instructors more used to traditional textbooks. We offer a great deal of assistance through sidenotes and endnotes with suggestions for instruction, questions to ask, explanations of why we chose to introduce topics the way we have, and other types of information that can help an instructor make full use of the text. A few videoclips taken from classrooms where these modules are being taught, together with guidance and discussion questions, can be used to prepare instructors to use these text materials. The clips will be available in 2006.

Development and Publication

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The Development Team for the original project materials included Judith Sowder (Project Director), Larry Sowder, Alba Thompson (now deceased), Patrick Thompson, Janet Bowers, Joanne Lobato, Nicholas Branca, and Randolph Philipp; and doctoral students Jamal Bernhard, Lisa Clement, Melissa Lernhardt, Susan Nickerson, and Daniel Siebert.

The revised *Reconceptualizing Mathematics* modules, by Judith Sowder, Larry Sowder, and Susan Nickerson will be published by Freeman Press beginning in the autumn of 2007. The are available now in a preprint form and can be obtained by contacting jsowder@sciences.sdsu.edu. They are currently being used in this form at many sites around the country.