Simulations of Giant Planet Migration in Gas Disks

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ABSTRACT

We present the results of a suite of giant planet migration simulations using Smoothed Particle Hydrodynamics (SPH). These simulations take a disk model with an embedded planet and evolve the system for several dynamical times, measuring the rates of migration and accretion. The simulations are global, three-dimensional, include the self-gravify of the disk, have free boundary conditions, and do not prescribe a model for accretion onto the planet. These are the first simulations to include these features simultaneously. The suite covers variation in the initial planet and disk masses and numerical parameters. The disk mass varied from 0.005 to 0.09 M⊙, and initial planet mass from 0.25 to 2 M_Jup. During the first few hundred years, most of the systems exhibited a constant rate of inward migration. A 1 M_Jup planet initially on a circular orbit at 5.2 AU in a 0.01 M_⊙ disk moved inward at 10^{-3} AU yr^{-1}. The dependence of migration rate on disk mass was linear, as predicted by the theory of type I migration. We found no dependence on planet mass, contrary to linear theory. This is because the disk response to the planet is non-linear. The process of accretion is a more complicated function of disk and planet mass, and can depend sensitively on numerical parameters.

Subject headings: accretion, accretion disks — methods: N-body simulations — planetary systems — planets and satellites: general
1. Introduction

Almost half of the known extrasolar planets orbit within 1 AU of their parent star (Schneider 2005). This has come as a surprise since both popular theories of giant planet formation, core accretion (Pollack et al. 1996) and disk instability (Boss 1997), cannot work at these distances, due to Keplerian shear and, less so, high temperature (Wuchterl et al. 2000). Therefore the standard explanation for these planets is that they form farther out and undergo a secular migration to smaller orbits (Lin et al. 1996). Since the amount of angular momentum change needed to cause this migration is large, and the circumstellar disk has significant angular momentum, interactions between the planet and the circumstellar disk has become the favored mechanism for this migration Ward & Hahn (2000).

Despite decades of focused theoretical and numerical research, this complex astrophysical problem remains largely unsolved. The properties of circumstellar disks are only loosely constrained Boss (1998), and the physical processes leading to planet formation occur over enormous ranges of mass, time, and distance. Furthermore, once planets have formed, there are still a few to hundreds of millions of years of evolution until a planetary system reaches the kind of stable configuration we see today in our own Solar System and elsewhere. Hence, there is a large disconnect between the processes of migration and observational constraints.

Given these limitations, rather than attempting to model the migration scenario in general, we will attempt to gain theoretical understanding of a particular slice of the problem: the initial migration and accretion rates after a gas giant has formed. We will do this numerically in as self-consistent way as possible, including the gravitational and gas dynamics, but making some assumptions about the gas equation of state and the properties of the planet. By performing a parameter study in planet mass and disk mass we hope to identify trends that can be broadly applicable.

1.0.1. Previous Work

The linear theory of migration via disk–planet interaction is well developed. It depends critically on two assumptions, that gas density fluctuations in the disk are linearly dependent on the perturbing potential of the planet, and that these fluctuations are tightly wound spirals that can be Fourier decomposed (Goldreich & Tremaine 1980). For two-dimensional, non-self-gravitating disks, the rate of migration can be derived; in the absence of a gap, this is known as type I migration (Ward 1986). More recently, the theory has been augmented in an attempt to account for three-dimensional disks (Tanaka et al. 2002) and a more sophisticated equation of state (Jang-Condell & Sasselov 2004).
Many numerical simulations have been conducted to test the assumptions of the linear theory, most with two-dimensional Eulerian grid codes. In this technique, the smallest resolvable distance scale, the grid size, is limited by the available computational resources, and requires an explicit treatment of boundary conditions. Recently, D’Angelo et al. (2002, 2003) used a system of nested grids to achieve excellent spatial resolution close to the planet. Their findings indicate that the circumplanetary material can have an important effect on the orbit of the planet. Two-dimensional self-gravitating grid simulations have been performed by Nelson & Benz (2003a,b), finding mixed agreement with the linear theory. We provide a detailed comparison of these results to our own simulations in Sections 4.1.2 and 4.1.3.

Using SPH for solar system-scale problems is still rare. Nelson et al. (1998, 2000) compared a grid code with SPH while studying the formation of spiral waves in unstable gas disks. Our previous paper (Lufkin et al. 2004) used three-dimensional, self-gravitating SPH to catalog the set of possible disk–planet interactions. Also, Schäfer et al. (2004) used SPH to simulate the migration of one and two planets in a gas disk, in the two-dimensional approximation and ignoring the self-gravity of the disk.

1.0.2. Outline

The organization of the paper is as follows. In Section 2 we discuss the numerical tool used to perform the simulations. Section 3 details the generation of initial conditions. Our results are presented and discussed at length in Section 4, and we conclude with Section 5.

2. Numerical Technique and Setup

The simulations described herein were performed with Gasoline, a sophisticated N-body code for gravity and hydrodynamics (Wadsley et al. 2004). Gravity is calculated between all particles in the simulation using a tree with hexadecapole expansions of the enclosed gravitational force. The tree is also used in the SPH algorithm to speed up the finding of nearest neighbors. The force of gravity is softened for each particle using a cubic-spline kernel (see Section 4.2.1 for details). The standard SPH formulation of artificial viscosity is used to stabilize the numerical integration and model shocks (Section 4.2.2).

Running the Simulations All the simulations described in this paper were run using Gasoline on computers in the Astronomy Department of the University of Washington. The jobs were distributed over a group of ~70 PCs running Linux by the Condor system (Thain
System of Units The simulations were carried out in heliocentric coordinates. The largest timestep, $\delta_0$, was about 2.5 years, or two percent of the orbital period of the outer edge of the disk. *Gasoline* uses a multi-stepping scheme to speed up the force calculations for particles with varying dynamical times. Smaller timesteps lie on “rungs”, binary subdivisions of the maximum timestep. The timestep for particles on the $i$th rung is $\delta_i = \delta_0/2^i$. For each particle, the timestep is chosen by the dynamical time of the largest gravitational interaction in which it participates (Richardson et al. 2000). In addition, the timestep is required to obey the Courant condition (Wadsley et al. 2004). In our simulations, the majority of the particles were within the first 4–5 rungs. The largest number of rungs used was usually 12, by the innermost particles.

Equation of State The equation of state of the gas used was locally isothermal with an imposed radial temperature profile

$$T(r, \theta, z) = T(r) = T_0 \left( \frac{r}{r_0} \right)^{-1}$$

with $T_0 = 538$ K and $r_0 = 1$ AU, which gives a plausible temperature at the current orbit of Jupiter. Given the vertical density distribution derived from pressure support, this is equivalent to a constant disk aspect ratio of $H/r = 0.05$. As a particle moved radially inward or outward, its temperature was changed to always match this profile. This enforces our assumption that the distance to the central star is the dominant factor in determining local temperature. This choice of power-law index differs from fits to infrared spectra by Beckwith et al. (1990), which suggested $T \propto r^{-1/2}$. However, it is a popular choice in simulations (Lubow et al. 1999; Kley et al. 2001; Bate et al. 2003; Papaloizou & Nelson 2003; Nelson & Papaloizou 2003; Papaloizou et al. 2004; Nelson & Papaloizou 2004), allowing us to make meaningful comparisons. The mean molecular weight of the gas was $\mu = 2$ m$_H$, again to allow us to make meaningful comparisons with previous work.

3. Initial Conditions

Our simulations begin after a central star has formed and an azimuthally symmetric, differentially rotating equilibrium disk has been established around it. Equilibrium pressure forces keep individual parcels of gas on vertically supported circular orbits. The disk extends from $r = R_{\text{in}}$ to $R_{\text{out}}$. We choose the complete temperature profile (Eq. 1) and the radial
density profile. Balancing the gravity of the central star, the disk self-gravity, and gas pressure forces with the centripetal force of circular orbits completes the system. The radial density profile is taken as $\Sigma(r) \propto r^{-3/2}$, motivated by the Minimum Mass Solar Nebula model (Hayashi et al. 1985) and observations (Beckwith et al. 1990). The full expression for density everywhere in the disk, between $R_{\text{in}}$ and $R_{\text{out}}$, is

$$\rho(r,z) = \rho_0 \left( \frac{r}{r_0} \right)^{-5/2} e^{-\frac{z^2}{2[H(r)]^2}}$$

with $r_0 = 1$ AU, $[H(r)]^2 = \frac{k_B T(r)}{\mu G M_\star} r^3$ the scale-height of the disk, and $\rho_0$ determined by the total disk mass. The velocity of gas on a circular orbit in such a disk is not easily expressed in closed form, but is not difficult to calculate numerically.

Given the parameters of the disk, we can discretize to create an ensemble of particles that embodies those parameters. This is done by randomly picking particle positions, weighted by the density profile. Once the position of a particle is chosen, the appropriate temperature and circular velocity are assigned to it. The generation of initial conditions is discussed in more detail in Lufkin (2004).

With these choices for the density and temperature profile and our standard disk mass $M_D = 0.01 M_\odot$, the minimum value of the Toomre stability criterion (Binney & Tremaine 1987) is at the outer edge of the disk, $Q = 16$. No boundary conditions were imposed, allowing the disk to expand and contract. Without an embedded planet, the disk exhibited little structure formation. To examine migration, a planet particle interacting via gravity only was placed on a circular orbit at 5.2 AU. The motion of the planet particle was not constrained by any external forces. Its motion was determined entirely by the gravitational forces due to the central star and the mass of the gas disk. Each gas particle feels the gravity of the central star, planet, and all other gas particles, and the gas pressure force due to its neighbors calculated via SPH. Its temperature is governed by its location in the disk, per the imposed temperature profile of the global isothermal equation of state. The central star was always 1 $M_\odot$.

4. Results and Discussion

In Lufkin et al. (2004) we considered the formation of giant planets via the disk instability. In this paper we are assuming that a planet has already formed on a circular orbit in an unperturbed disk. This is more consistent with the core accretion model of planet formation. Specifically, we are trying to start near the point where gas accretion becomes a runaway process. This is the time when the planet first starts to significantly perturb the
gas disk.

To extract migration and accretion rates from our simulations, we calculate the semi-major axis and mass of the planet for every output from the simulation. The mass of the planet is calculated using an iterative procedure that finds all the particles within the Hill sphere with density greater than the Hill density. The Hill sphere is the region where the gravity of the planet dominates that of the central star, and its radius is given by $r_H = a_p (M_p/3M_\star)^{1/3}$. The Hill density is the mass of the planet divided by the volume of the Hill sphere.

The simulation outputs are regularly spaced in time, typically 2.55 yr apart. When the semi-major axis and mass of the planet appear to vary linearly with time, the values are fit to a straight line using a least-squares fitting routine. The slopes of the best-fit lines are then quoted as the rates for migration and accretion for that simulation. Because we do not have uncertainties for each data point, the concept of goodness-of-fit does not apply. We examine each straight-line fit to ensure that the data does in fact appear to be linear.

4.1. Dependence on Planet and Disk Mass

The theory of type I migration, mentioned in Section 1.0.1, gives an expression for the migration rate of a planet embedded in a gas disk. We will compare equation 70 of Tanaka et al. (2002) to the results of our simulations. Substituting the size, density profile, and temperature profile of our disks, and changing to a rate, this equation becomes

$$\dot{a}_p = -217.5 \left( \frac{M_D}{M_\odot} \right) \left( \frac{M_p}{M_\odot} \right) \text{AU yr}^{-1}. \quad (3)$$

This rate is linearly dependent on the masses of both the planet and disk. We performed a suite of 231 simulations to test the predicted dependence.

In this suite, the total mass of the disk and initial mass of the planet were varied from 0.0025–0.09 M_\odot and 0.25–2 M_{\text{Jup}}, respectively. The softening lengths were determined with the default parameters described in Section 4.2.1. The artificial viscosity parameters were fixed at their default values (Section 4.2.2). The number of particles used ranged from 20000–1000000.
4.1.1. Overall Results

For the standard case, \( M_D = 0.01 \, M_\odot \) and \( M_p = 1 \, M_{\text{Jup}} \), the planet migrates inward at a constant rate of \( 10^{-3} \, \text{AU} \, \text{yr}^{-1} \). It also accretes gas from the surrounding disk at a constant rate of \( 2.5 \times 10^{-6} \, M_\odot \, \text{yr}^{-1} \). The quoted rates are the mean values from 114 different simulations with the specified disk and planet masses and a range of particle number. The standard deviation from the mean of the migration rate is \( 1.2 \times 10^{-4} \), about 11 percent. The standard deviation from the mean of the accretion rate is \( 1.4 \times 10^{-7} \), about 6 percent. Reasons why simulations with identical parameters yield different results are discussed in Section 4.2.

The migration and accretion continue at these rates for several hundred years (tens of orbits). The results presented in this section are all derived from this initial period. Later, migration slows as the planet creates a gap in the disk. Section 4.4 discusses the long-term behavior. For some of the simulations with more massive disks the timescale for gap formation is short. In those cases, the quoted migration and accretion rates were derived from only the first few orbits of the planet.

4.1.2. Dependence on Planet Mass

We found that migration rate was independent of planet mass. Figure 1 shows the migration rate as a function of the initial planet mass for 148 simulations with \( M_D = 0.01 \, M_\odot \). No trend is evident. The rates shown are a superset of the standard case result, yet the mean and standard deviation of the larger set are nearly identical.

To quantify the lack of trend, we need to estimate the errors of a linear fit to this data. Since we have no \textit{a priori} knowledge of the magnitude of the errors, we use the bootstrap method, assuming our data is independent and identically distributed (Press et al. 1992). This method derives error estimates from the distribution of datasets created from the known data by random drawing with replacement. For the data shown in Figure 1, the linear fit with errors is \( \dot{a}_p = (-1.02 \pm 0.03) - (0.063 \pm 0.037)M_p/M_{\text{Jup}} \times 10^{-3} \, \text{AU} \, \text{yr}^{-1} \). The intercept is well constrained and definitely non-zero, while the slope is poorly known and much smaller. Furthermore, the slope is within two standard deviations of zero. This shows quantitatively the absence of a linear trend, and disagrees with the prediction of linear theory. In our simulations, within the range \( 0.25 \, M_{\text{Jup}} < M_p < 2 \, M_{\text{Jup}} \), the rate of migration is unrelated to the initial mass of the planet.

Nelson & Benz (2003a) (hereafter, NB) performed similar simulations using a two-dimensional grid code. After performing a suite of simulations with \( M_D = 0.05 \, M_\odot \) and
Fig. 1.— The migration rate as a function of the planet mass in a 0.01 M_☉ disk. Varying the initial mass of the planet does not affect the rate of migration.
planet mass ranging from 0.1–2 M$_{\text{Jup}}$, they also concluded that migration rate was independent of planet mass. Figure 2 shows the results of a similar set of simulations that we performed. We made bootstrap estimates of the errors of a linear fit to this data, as above. Again we find a definitely non-zero intercept and small slope, supporting our claim of independence. Our simulations show a smaller spread in migration rate than those of NB, but a magnitude several times larger. This difference is likely due to our differing numerical approaches. They used a two-dimensional Eulerian grid; we used a three-dimensional SPH simulation.

The theory of type I migration assumes that the disk responds linearly to the presence of the planet. We found this not to be true, and suggest that it explains why the migration rate does not depend on the mass of the planet. We make this claim based on three measurements of our simulations.

First, the magnitude of the perturbed density in the spiral waves formed by the planet is not small. The linear theory deals with the perturbed surface density of the disk, assumed to be small with respect to the unperturbed, average density (Goldreich & Tremaine 1978). This is not true in our simulations. We vertically integrated the disk density to obtain a surface density. We then measured the maximum deviation in the surface density relative to the azimuthal average at each radius, $\delta \Sigma / \Sigma$. Close to the planet, the relative deviation in surface density was typically 3 to 5. Further away, near the Lindblad resonances, the deviation was routinely above 1. These large density fluctuations clearly do not satisfy the small perturbation required by the theory.

Second, the maximum density as measured above did not depend on the perturbing planet mass. In the linear regime, we would expect the amplitude of the spiral waves to be dependent on the mass of the planet exciting them.

Finally, the relative surface density was decomposed into azimuthal Fourier modes. According to theory, the magnitude of the torque exerted at the $m$th Lindblad resonance increases with $m$ up to about $m = 10$ (Ward 1997). Therefore, the high $m$ modes are most responsible for migrating the planet. We measured the power in the $m$th mode at the $m$th Lindblad resonance. It was independent of planet mass. NB performed a similar calculation with the same result.

Based on this evidence, we conclude that, for planet masses of 0.25–2 M$_{\text{Jup}}$, the response of the disk is nonlinear in proportion to the perturbing potential of the planet. Therefore, we should not expect the linear theory to correctly predict the rate of migration. Beyond a critical planet mass, the response of the disk saturates. That is, the spiral waves cannot be made sharper or higher in amplitude. Therefore the torque they exert does not increase
Fig. 2.— The migration rate as a function of the planet mass in a 0.05 M\odot disk. Compare this to Figure 9 of NB. We agree on the lack of trend, but find a magnitude several times greater. These simulations used \( N = 80000 \) particles.
as the planet mass increases. Presumably, probing ever-lighter planets would reveal the the minimum planet mass that saturates the disk response. Simulations of smaller-mass planets require more resolution in the disk, so we have been unable to look for this transition.

Unlike migration, we found that the initial planet mass does affect the rate of accretion. Figure 3, the counterpart to Figure 1, shows the accretion rate as a function of the initial planet mass for the simulations with $M_D = 0.01 \, M_\odot$. Figure 4 shows the similar result for the suite similar to NB. The accretion rate is linearly dependent on the initial planet mass. This agrees with our explanation of accretion via gas capture within the Hill sphere (Section 4.2.1). Because the volume of the Hill sphere is proportional to the planet mass, so is the gas available for accretion.

4.1.3. Dependence on Disk Mass

We found that the migration rate was linearly dependent on the total disk mass (at fixed size and profile, equivalently the surface density). This agrees with the theory of type I migration. Figure 5 shows the results of 129 simulations with $M_p = 1 \, M_{\text{Jup}}$ and disk masses ranging from 0.01–0.09 $M_\odot$.

NB also performed simulations varying the disk mass from 0.0025–0.05 $M_\odot$, using a 0.3 $M_{\text{Jup}}$ planet. We performed simulations for those conditions, using a disk of 80000 particles, obtaining the results shown in Figure 6. As in the simulations where the planet mass was varied, our rates are several times greater than NB. The migration rates are linearly dependent on the disk mass. This trend is slightly stronger than that found by NB. Our result agrees qualitatively with the predictions of the theory of type I migration (Section 1.0.1). Quantitatively, we are within a factor of a few of Equation 3. However, since we do not see the linear dependence on planet mass, this is likely a coincidence.

The linear fits shown in Figures 5 and 6 were made assuming identical errors. By creating bootstrap datasets, we estimate the error in the intercept to be 11% and 20%, and the error in the slope to be 4% and 2%, for Figures 5 and 6, respectively. Within this error, the intercept is definitely non-zero. This implies that, for zero disk mass, the planet still migrates. This could be due to numerical effects inherent in the simulation. Alternatively, the interaction with the disk could deviate from linearity at disk masses lower than we simulated. Note that at the lowest disk masses, the disk is only a few times more massive than the planet.

Our quoted migration rates imply that giant planets should migrate rapidly and be accreted by their parent star within only a few thousand years. However, as discussed in
Fig. 3.— The accretion rate as a function of the planet mass in a $0.01 \, M_\odot$ disk. The data are well approximated by a linear function of the planet mass. A least-squares fit of $\dot{M} = (0.2 + 2.3 M_p/M_{\text{Jup}}) \times 10^{-6} \, M_\odot \, \text{yr}^{-1}$ is shown.
Fig. 4.— The accretion rate as a function of the planet mass in a 0.05 $M_\odot$ disk. The data are well approximated by a linear function of the planet mass. A least-squares fit of $\dot{M} = (0.7 + 1.4 M_p / M_{\text{Jup}}) \times 10^{-5} \; M_\odot \; \text{yr}^{-1}$ is shown.
Section 4.4, larger disk mass results in more rapid gap formation, halting migration sooner. Therefore, we believe that some planets can survive this period of rapid migration, provided they open a gap quickly.

We expect the accretion rate to be linearly dependent on the disk mass. Figures 7 and 8 show the accretion rate of 1 and 0.3 $M_{\text{Jup}}$ planets, respectively. We have not superimposed linear fits, as the data suggest a slightly stronger dependence. We have not investigated this further.

4.2. Dependence of Migration Rate on Simulation Resolution

In order to test that our simulations have accurately captured the relevant physics of the disk–planet system, we varied the simulation resolution. To measure the convergence with resolution, we conducted simulations of the standard migration system with particle numbers ranging from $2 \times 10^4$ to $10^6$. At each resolution we ran several simulations, varying the seed used by the random number generator in constructing the initial conditions. The discretization of the continuous fluid introduces some small amount of Poisson noise onto the particle distribution. By varying the random seed, we quantify the effect this discretization noise has on the aggregate results we are interested in. Figures 9 and 10 show the dependence of the migration and accretion rates on the resolution for our standard disk and planet masses. In this instance the error bars are calculated from the standard deviation of the rates for each population of simulations at a particular resolution. By measuring the variance in this way, we have a baseline with which to judge the effects of other parameters. For example, when we say that the gravitational softening does not affect the migration rate, we mean that the migration rates measured in runs with different softening lengths were within the standard deviation for the resolution of the simulations. Based on the convergence of migration rate, we are confident that we have accurately resolved the length scales important for the interaction of the planet and disk. In contrast, the non-convergence of accretion rates suggests caution when interpreting our accretion results. However, because we observed that the migration rate, which occurs over much longer length scales than accretion, was independent of planet mass, the migration results should not be affected.

4.2.1. Varying Gravitational Softening

Particle simulations with gravity use a softening prescription to approximate the extended physical size of the modeled objects. We use a technique known as cubic-spline
Fig. 5.— The migration rate of a $1 \text{ M}_{\text{Jup}}$ planet as a function of the disk mass. The data are well approximated by a linear function of the disk mass: $\dot{a}_p = (0.0006 - 0.165 \frac{M_D}{M_\odot}) \text{ AU yr}^{-1}$. 
Fig. 6.— The migration rate of a 0.3 M$_{\text{Jup}}$ planet as a function of the disk mass. The data are well approximated by a linear function of the disk mass: $\dot{a}_p = (0.00023 - 0.130 M_D / M_\odot)$ AU yr$^{-1}$. Compare this to Figure 10 of NB.
Fig. 7.— The accretion rate of a 1 M_{Jup} planet as a function of the disk mass.
Fig. 8.— The accretion rate of a 0.3 M$_{\text{Jup}}$ planet as a function of the disk mass.
Fig. 9.— Migration rate of a 1 M$_{\text{Jup}}$ planet in a 0.01 M$_{\odot}$ disk as a function of the simulation resolution. Each point is the result from a separate simulation of the same physical system. Mean values for a particular number of particles are shown as crosses, and the error bars represent one standard deviation of the population. Increasing resolution decreases the spread in migration rates, and the average value appears to be relatively fixed. The independence of average migration rate on particle number indicates that we have sufficient resolution to capture the formation and dissipation of the spiral waves excited by the planet that lead to migration.
Fig. 10.— Accretion rate of a 1 M$_{\text{Jup}}$ planet in a 0.01 M$_{\odot}$ disk as a function of the simulation resolution. Each point is the result from a separate simulation of the same physical system. Mean values for a particular number of particles are shown as crosses, and the error bar represents one standard deviation of the population. The deviation tends to decrease with increasing resolution. In addition, the mean value fluctuates, possibly trending lower. We suggest that this dependence on resolution is due to a finer sampling of the region within the Hill radius of the planet, where gas particles can be captured. In addition, the extent of dissipation via artificial viscosity decreases with resolution.
softening. In this method the gravitational force is equal to the Newtonian value for interaction distances greater than twice the softening length $\epsilon$. Inside the softening length the force is that due to a spherically symmetric extended object with mass distribution equal to the standard smoothing kernel. The softening length for a pairwise interaction is the average of the softening lengths of the two interacting particles.

In our simulations each gas particle has the same softening length. The planet particle is more massive than the gas particles but, we imagine, has a more compact physical size, so is given a different softening length. In addition, the force due to the central star is softened on a third distinct length scale.

How do we pick appropriate values for these softening lengths? In a study of Jeans collapse, Bate & Burkert (1997) found that softening and smoothing should be commensurate in SPH simulations. Dehnen (2001) suggests picking a softening length by minimizing the error in the gravitational force calculation. For unstable disk simulations, Mayer et al. (2003) empirically found that the softening should be smaller than the most unstable Toomre wavelength (Binney & Tremaine 1987). We used an empirical study to choose softening lengths by looking at accretion onto the planet.

Section 4.3 details the physical picture of accretion. For this mechanism to work, the softening lengths must be a fraction of the Hill radius of the planet. To find a precise value, we performed simulations varying the softening length of the gas particles (Figure 11), and planet particle (Figure 12). In both cases, too large a softening leads to a sharp cutoff in accretion. The accretion plateaus with smaller softening length, and becomes more computationally expensive. Thus we pick values close to the cutoff, allowing accretion.

To scale as we varied the planet mass, the softening length of the planet particle was chosen as a fixed fraction of the initial Hill radius. We used $\epsilon_{\text{planet}} = r_H/5$. To scale with the disk mass $M_D$ and number of particles $N$, the softening length of the gas particles was determined from a fixed density $\rho_{\text{gas}}$:

$$\epsilon_{\text{gas}} = \left(\frac{M_D}{N} \frac{4\pi}{3}\rho_{\text{gas}}\right)^{1/3}. \quad (4)$$

We used $\rho_{\text{gas}} = 3.73 \times 10^{-5} \, M_\odot \, \text{AU}^{-3}$.

In the experiments varying the softening, the migration rate of the planet did not change significantly. From this we conclude that the distance scales relevant to migration are much greater than the softening length.

The potential of the central star is softened with a length scale $\epsilon_\odot = 0.4 \, \text{AU}$. This is within the initial internal edge of the disk in all our simulations. In addition, it is well within
Fig. 11.— The accretion rate onto the planet as a function of the gravitational softening of the gas particles. These data are taken from simulations with $M_D = 0.01 \, M_\odot$, $M_p = 1 \, M_{\text{Jup}}$, and $N = 80000$. At this resolution the choice $\rho_{\text{gas}} = 3.73 \times 10^{-5} \, M_\odot \, \text{AU}^{-3}$ corresponds to $\epsilon_{\text{gas}} = 0.093 \, \text{AU}$, close to the transition value.
Fig. 12.— The accretion rate onto the planet as a function of the gravitational softening length of the planet. These data are taken from simulations with $M_D = 0.01 \, M_\odot$ and $M_p = 1 \, M_{\text{Jup}}$. Simulations with $N = 40000$ are points, $N = 80000$ are crosses. At this resolution the choice $r_H/5$ corresponds to $\epsilon_{\text{planet}} = 0.072 \, \text{AU}$. 
the inner Lindblad resonance of a planet at 5.2 AU, so should not affect the formation and dissipation of spiral density waves.

4.2.2. The Effect of Artificial Viscosity

We use the standard form of artificial viscosity for SPH from Monaghan (1992), which is parametrized with two constants, $\alpha$ and $\beta$, designed to handle shocks in gas and reduce particle inter-penetration. Conventional wisdom in the SPH community suggests that $\alpha = 1$, $\beta = 2$ is appropriate for a broad class of simulations. Recently, Mayer et al. (2003) has found that SPH simulations of planet formation via gravitational instability are sensitive to the particular choices for these parameters. We found that varying the artificial viscosity parameters by a factor ten smaller or larger affected the magnitude of the migration rate by only a few percent. When both parameters are set to identically zero, no shocks occur and the planet does not capture surrounding gas particles (there is no accretion). This is consistent with our assertion that shocks are the source of dissipation that leads to the planet capturing gas particles. Given the weak dependence, we assert that artificial viscosity was not a dominant force in our simulations. Accordingly, the default values $\alpha = 1$, $\beta = 2$ were used in all other simulations.

4.3. Capturing Gas Particles

In our simulations, the planet captures gas particles from the disk, growing in mass. These particles are gravitationally bound to the planet, forming a flattened sphere of gas rotating in the same sense as the orbital motion. A gas particle must come within the Hill sphere of the planet and lose some of its relative velocity to be captured by the planet. This is possible because, in addition to gravity, the gas particles are subject to the dissipative forces of SPH. Figure 13 shows a series of zero-velocity curves for the restricted three-body problem appropriate to the Sun-Jupiter combination. The shape of each zero-velocity curve depends on the Jacobi constant (equivalently, the energy in the rotating frame) of a test particle. As the energy decreases, the zero-velocity curve encloses the planet, trapping particles around it. Because SPH is collisional and dissipative, gas particles that lose energy while within the Hill sphere of the planet will become bound to the planet. This occurs when a particle on an orbit slightly external (internal) to the planet is caught by (catches) the trailing (leading) density wave of the planet. The particle collides with the density wave, accelerating and thus, in the rotating frame, losing energy. If the particle is within the Hill radius of the planet, it is trapped, becoming part of the envelope. This scenario for the capture of gas
particles requires that the gravitational softening between the planet and gas particles be appreciably smaller than the Hill radius, so that the three-body problem approximation is valid. Figure 14 highlights the particles that will soon become part of the planet.

4.4. Long-term Behavior

The results quoted above for migration and accretion rates are all derived from the first few hundred years of the evolution of the system. What happens if we continue the simulation, letting the planet move farther inward? The planet forms a gap in the disk. This changes the dynamics of the system, halting migration and changing the accretion rate.

A gap forms as the planet accretes matter along its orbit and exerts tidal torques that push material away from corotation. A gap physically disconnects the planet and its envelope from the inner and outer portions of the disk. This disconnect results in slower accretion and halts the type I migration of the planet. At this point we expect type II migration, which depends on the overall viscosity of the disk and has a much longer timescale, to occur. Because we minimized the viscosity in the disk, our simulations did not exhibit type II migration. Figure 15 shows the semi-major axis and mass of a planet forming a gap in a 0.05 $M_\odot$ disk. The vertically and azimuthally averaged surface density in the disk is shown for this simulation in Figure 16. As the planet accretes and exerts torques on the disk, it depletes a ring of gas around its orbit. The depth of the gap grows, resulting in a change in the migration of the planet. The disconnect from the disk is not complete, as shown in Figure 17. Gas from the disk can still accrete onto the planet through the spiral waves that reach through the gap.

We found the speed of gap formation to be sensitive primarily to the disk mass. Figures 15, 16 and 17 show a gap forming within about 250 years in a 0.05 $M_\odot$ disk. In contrast, a planet in a 0.1 $M_\odot$ disk carved a gap in about 100 years, and a planet in a 0.01 $M_\odot$ disk carved a gap in about 1500 years.

5. Conclusions

We have simulated the migration of giant planets in gaseous circumstellar disks using the smoothed particle hydrodynamics technique. These are the first fully three-dimensional simulations of this problem that treat the self-gravity of the disk, allow for the planet to dynamically migrate and accrete, and have free boundary conditions. We used a simple isothermal equation of state with an imposed radial temperature profile of $T(r) \propto r^{-1}$. The
Fig. 13.— Zero-velocity curves in the restricted three-body problem with mass ratio appropriate for the Sun–Jupiter system. The right-hand figures are closeups of the area around the planet. A particle with the specified value of the Jacobi constant \( C_J \) is prohibited from entering the shaded region. In this system of units \( C_J = (x^2 + y^2) + 2 \left( \frac{1-\mu_2}{\sqrt{(x+\mu_2)^2+y^2}} + \frac{\mu_2}{\sqrt{(x-(1-\mu_2))^2+y^2}} \right) - (\dot{x}^2 + \dot{y}^2) \) where \( \mu_2 = 0.001 \) is the Sun–Jupiter mass ratio. As the Jacobi constant increases, the prohibited region encloses the planet. If dissipative processes decrease the energy (equivalently, increase the Jacobi constant) of a particle while it is near the planet, the zero-velocity boundary closes in, trapping the particle. It is then effectively a part of the planet.
Fig. 14.— A planet in a gas disk, showing the trailing and leading edge of the spiral density waves. Within two time steps, all the particles highlighted in green will be part of the planet. The exterior particles will be caught by the super-Keplerian trailing wave, and the interior particles will catch the sub-Keplerian leading wave. The shock will slow them in the rotating frame, trapping them as part of the gas envelope of the planet. For non-planet particles, the color scale represents the logarithm of the gas density.
Fig. 15.— Semi-major axis and mass of a planet forming a gap in a 0.05 $M_\odot$ disk. Notice that migration and accretion slow between 100 and 200 years after the start of the simulation. The initial mass of the planet was 1.9 $M_{\text{Jup}}$ and the disk was composed of 80000 particles. Compare with the density profiles of this disk shown in Figure 16.
Fig. 16.— Surface density profiles of a planet forming a gap in a 0.05 M⊙ disk. The straight line, topmost, is the initial condition, $\Sigma(r) \propto r^{-3/2}$. The four curves below are at intervals of 81 years, showing the density depression around the orbit of the planet deepen as the simulation proceeds. Relating to the semi-major axis and mass shown in Figure 15, we see the gap move inward along with the planet, deepening as the planet grows in mass. When the simulation stops, after 321 years, the density in the gap is almost 100 times less than the initial conditions.
Fig. 17.— A snapshot of a simulation where the planet has formed a gap in the disk. The region along the orbit of the planet has been almost entirely depleted of material. However, the planet still excites spiral density waves, which reach in to touch the planet, allowing for continued accretion across the gap. The color scale represents the logarithm of the gas density.
initial surface density of the disk was $\Sigma(r) \propto r^{-3/2}$, as suggested by the minimum mass solar nebula model. The initial vertical distribution was determined by the equilibrium pressure support of the tide from the central star. We used disks of mass 0.005–0.09 M$_\odot$ around 1 M$_\odot$ stars, with embedded planets of mass 0.25–2 M$_{\text{Jup}}$ on initially circular orbits at 5.2 AU.

Our most robust result concerns the type I inward migration rate of the planet. We find that the migration rate is independent of the planet mass and linearly dependent on the disk mass. This result is consistent with previous 2D simulations. However, theoretical treatments of this problem suggest that migration rate should be proportional to both the planet and disk mass. We provide evidence that this inconsistency is due to the disk response to the planet becoming saturated, in contrast to the assumptions of the linear theory. Furthermore our measured migration rate is independent of the numerical parameters of our simulations such as particle number and softening length.

Our results on accretion onto the planet are less robust. We obtained convergence only at our largest particle numbers and with gravitational softening lengths less than 1/5 the Hill radius of the planet. With these caveats, we found the accretion rate onto the planet to be a linear function of the planet mass and a more complicated function of the disk mass. Given that the accretion process seems to involve the shocking of captured material approaching the Roche limit, a more detailed study of this region, including an improved treatment of radiative transfer, will be needed to understand this process.

The tidal torques exerted by the planet will push disk material on nearby orbits away from corotation. Combined with accretion, this process will form a gap in the disk, where gas density is several orders of magnitude below the initial value. The gap slows both the migration of the planet and its mass accretion. A giant planet of 1 M$_{\text{Jup}}$ at 5.2 AU in a 0.01 M$_\odot$ disk will form a gap in about 1500 years, with a final orbit of 4.6 AU. At this point, viscosity in the disk may lead to further, type II, migration (Ward 1997).

Although we provide evidence that non-linear dynamics plays a significant role in the migration of giant planets, we confirm the rapid planet migration puzzle that comes out of linear theory. This suggests that planets would quickly be swallowed by their parent star after forming, yet observational evidence shows that planet formation is robust. Areas for further research include more sophisticated treatment of radiative transfer, the gravitational interaction of material close to the planet, the process of accretion onto the planet and the behavior of multiple-planet systems. In addition, new ideas for halting migration should be pursued.
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