# Potential Theory

- Most of the mass of stellar systems is in point-like objects.
- Calculating the potential by summing individual contributions is impracticle.

  But we usually consider systems containing many stars.

$\Rightarrow$ stellar systems can be represented by smooth density fields generating smooth potentials.

## 1 Poisson's equation

To calculate the force on a unit mass at at a point $\mathbf{x}$, we sum over all the contributions from each element $\delta^3\mathbf{x}'$:

$$\delta\mathbf{F}(\mathbf{x}) = G\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\delta m(\mathbf{x}') = G\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\rho(\mathbf{x}')\delta^3\mathbf{x}'.$$

This gives

$$\mathbf{F}(\mathbf{x}) = G\int\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\rho(\mathbf{x}')\mathrm{d}^3\mathbf{x}'.$$

If we introduce the gravitational potental which we define as:

$$\Phi(\mathbf{x}) = -G\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}\mathrm{d}^3\mathbf{x}',$$

and note that

$$\nabla_{\mathrm{x}}\left(\frac{1}{|\mathbf{x}'-\mathbf{x}|}\right) = \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3},$$

we have

$$\mathbf{F}(\mathbf{x}) = \nabla\int\frac{G\rho(\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}\mathrm{d}^3\mathbf{x}' = -\nabla\Phi.$$

Deriving the force from a potental field has advantanges:

- It constrains the force field to be conservative.
- The scalar field, $\Phi$, is easier to visualize than a vector field.
- A scalar field is often easier to calculate than a vector field.

The divergence of the force is:

$$\nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{x}) = G \int \nabla_{\mathbf{x}} \cdot \left( \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) \rho(\mathbf{x}') \mathrm{d}^3\mathbf{x}'.$$

But

$$\nabla_{\mathbf{x}} \cdot \left( \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) = -\frac{3}{|\mathbf{x}' - \mathbf{x}|^3} + \frac{3(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^5}.$$

which is 0 for $\mathbf{x}' \neq \mathbf{x}$. So we can restrict the volume of integration to an arbitrarily small sphere about $\mathbf{x}$.

$$\begin{aligned}
\nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{x}) &= G\rho(\mathbf{x}) \int_{|\mathbf{x}'-\mathbf{x}| \leq h} \nabla_{\mathbf{x}} \cdot \left( \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) \mathrm{d}^3\mathbf{x}' \\
&= -G\rho(\mathbf{x}) \int_{|\mathbf{x}'-\mathbf{x}| \leq h} \nabla_{\mathbf{x}'} \cdot \left( \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) \mathrm{d}^3\mathbf{x}' \\
&= -G\rho(\mathbf{x}) \int_{|\mathbf{x}'-\mathbf{x}| = h} \frac{(\mathbf{x}' - \mathbf{x}) \cdot \mathrm{d}^2\mathbf{S}'}{|\mathbf{x}' - \mathbf{x}|^3}.
\end{aligned}$$

Now on the surface, we have $|\mathbf{x}' - \mathbf{x}| = h$ and $\mathrm{d}^2\mathbf{S}' = (\mathbf{x}' - \mathbf{x})h\mathrm{d}^2\Omega$. So

$$\nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{x}) = -G\rho(\mathbf{x}) \int \mathrm{d}^2\Omega = -4\pi G\rho(\mathbf{x}).$$

Substituting $\Phi$ for $\nabla \cdot \mathbf{F}$ we have **Poisson's equation**:

$$\nabla^2\Phi = 4\pi G\rho.$$

Integrating Poisson's equation over an arbitrary volume gives

$$4\pi G \int \rho \mathrm{d}^3\mathbf{x} = 4\pi GM = \int \nabla^2\Phi \mathrm{d}^3\mathbf{x} = \int \nabla\Phi \cdot \mathrm{d}^2\mathbf{S}.$$

This is **Gauss's theorem**: *the integral of the normal component of $\nabla\Phi$ over any closed surface equals $4\pi G$ times the mass contained within that surface.*

## 2 The Potential Energy Tensor

The total work required to assemble a configuration of mass, $\rho(\mathbf{x})$, defined as the **potential energy**.
Then the work done to bring in $\delta m$ from infinity to $\mathbf{x}$ is $\delta m\Phi(\mathbf{x})$.
So the change in energy for adding a $\delta\rho(\mathbf{x})$ is

$$\delta W = \int \delta\rho(\mathbf{x})\Phi(\mathbf{x})\mathrm{d}^3\mathbf{x}.$$

But from Poisson's equation, we have

$$\delta W = \frac{1}{4\pi G}\int \Phi\nabla^2(\delta\Phi)\mathrm{d}^3\mathbf{x}.$$

Integrating by parts, we have

$$\begin{aligned}\delta W &= \frac{1}{4\pi G}\int \nabla\cdot(\Phi\nabla(\delta\Phi))\mathrm{d}^3\mathbf{x} - \frac{1}{4\pi G}\int \nabla\Phi\cdot\nabla(\delta\Phi)\mathrm{d}^3\mathbf{x}.\\ &= \frac{1}{4\pi G}\int \Phi\nabla(\delta\Phi)\mathrm{d}^2\mathbf{S} - \frac{1}{4\pi G}\int \nabla\Phi\cdot\nabla(\delta\Phi)\mathrm{d}^3\mathbf{x}.\end{aligned}$$

The surface integral vanishes at $r \to \infty$ and $\nabla\Phi\cdot\nabla(\delta\Phi) = \frac{1}{2}\delta|\nabla\Phi|^2$.

$$\delta W = -\frac{1}{8\pi G}\delta\left(\int |\nabla\Phi|^2\mathrm{d}^3\mathbf{x}\right).$$

Summing over all contributions, we have

$$W = -\frac{1}{8\pi G}\int |\nabla\Phi|^2\mathrm{d}^3\mathbf{x}.$$

Again integrating by parts,

$$W = -\frac{1}{8\pi G}\int \nabla\cdot(\Phi\nabla\Phi)\mathrm{d}^3\mathbf{x} + \frac{1}{8\pi G}\int \Phi\nabla^2\Phi\mathrm{d}^3\mathbf{x}.$$

$$
\begin{aligned}
&= -\frac{1}{8\pi G}\int \Phi\nabla\Phi \mathrm{d}^2\mathbf{S} + \frac{1}{8\pi G}\int \Phi\nabla^2\Phi \mathrm{d}^3\mathbf{x} \\
&= \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})\mathrm{d}^3\mathbf{x}.
\end{aligned}
$$

Note how this compares with calculating the total potential energy of a collection of particles:

$$W = \frac{1}{2}\sum_i m_i \sum_{j\neq i} \frac{-Gm_j}{|\mathbf{x}_i - \mathbf{x}_j|}.$$

The factor of 1/2 arises because the sum has double counted every particle-particle interaction.
Now examine the tensor:

$$W_{jk} = -\int \rho(\mathbf{x}) x_j \frac{\partial \Phi}{\partial x_k}\mathrm{d}^3\mathbf{x},$$

known as the **Chandrasekhar potential energy tensor**.
If we replace $\Phi$ by its definition in terms of $\rho$, we have

$$W_{jk} = G\int \rho(\mathbf{x}) x_j \frac{\partial}{\partial x_k}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\mathrm{d}^3\mathbf{x}'\mathrm{d}^3\mathbf{x}.$$

Taking the derivative inside the integral gives

$$W_{jk} = G\int\int \rho(\mathbf{x})\rho(\mathbf{x}')\frac{x_j(x'_k - x_k)}{|\mathbf{x}' - \mathbf{x}|^3}\mathrm{d}^3\mathbf{x}'\mathrm{d}^3\mathbf{x}.$$

Since $\mathbf{x}$ and $\mathbf{x}'$ are dummy variables, this can also be written

$$W_{jk} = G\int\int \rho(\mathbf{x})\rho(\mathbf{x}')\frac{x'_j(x_k - x'_k)}{|\mathbf{x} - \mathbf{x}'|^3}\mathrm{d}^3\mathbf{x}'\mathrm{d}^3\mathbf{x}.$$

Adding these two expressions gives

$$W_{jk} = -\frac{1}{2}G\int\int \rho(\mathbf{x})\rho(\mathbf{x}')\frac{(x'_j - x_j)(x'_k - x_k)}{|\mathbf{x} - \mathbf{x}'|^3}\mathrm{d}^3\mathbf{x}'\mathrm{d}^3\mathbf{x}.$$

This makes it obvious that $\mathbf{W}$ is symmetric.
Taking the trace of both sides,

$$\begin{aligned}\mathrm{trace}(\mathbf{W}) &= -\frac{1}{2}G\int\rho(\mathbf{x})\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3\mathbf{x}'\mathrm{d}^3\mathbf{x} \\ &= \frac{1}{2}\int\rho(\mathbf{x})\Phi(x)\mathrm{d}^3\mathbf{x}.\end{aligned}$$

So we see that trace($\mathbf{W}$) is $W$, the gravitational potential energy.
For spherical systems, a more useful formula can be found. Consider the trace of $\mathbf{W}$:

$$W = -\int\rho\mathbf{x}\cdot\nabla\Phi\mathrm{d}^3\mathbf{x}.$$

If the body is spherical,

$$\nabla\Phi = \frac{\mathrm{d}\Phi}{\mathrm{d}r}\hat{\mathbf{e}}_r = \frac{GM(r)}{r^2}\hat{\mathbf{e}}_r,$$

where $M(r)$ is the total mass interior to $r$. Substituting, and integrating over all angles, we have

$$W = -4\pi G\int\rho M(r)r\mathrm{d}r.$$

A quantity related to the potential energy is the **gravitational radius**:

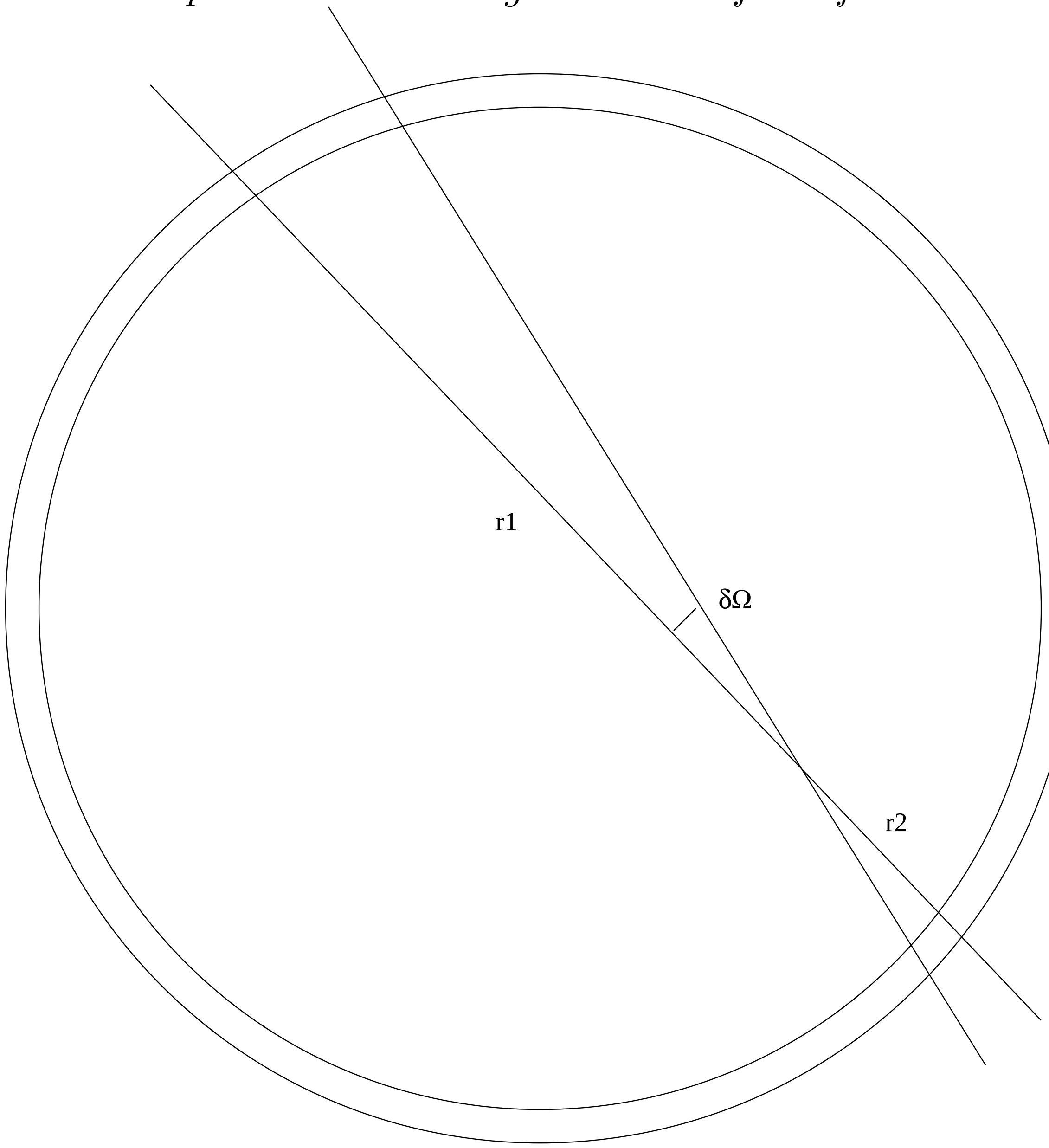
$$r_g \equiv \frac{GM^2}{|W|}.$$

This allows us to give a characteristic size to a system that does not have a sharp boundary.
Also note that for a spherical system $W_{jk}$ is diagonal ( $W_{jk}=0$ for $j\neq k$), and isotropic:

$$W_{jk} = \frac{1}{3}W\delta_{jk}.$$

# 3 Spherical Systems

Newton's first theorem: *A body that is inside a spherical shell of matter experiences no net graviatianal force from that shell.*




Note that for the portion of the shell enclosed by the cone

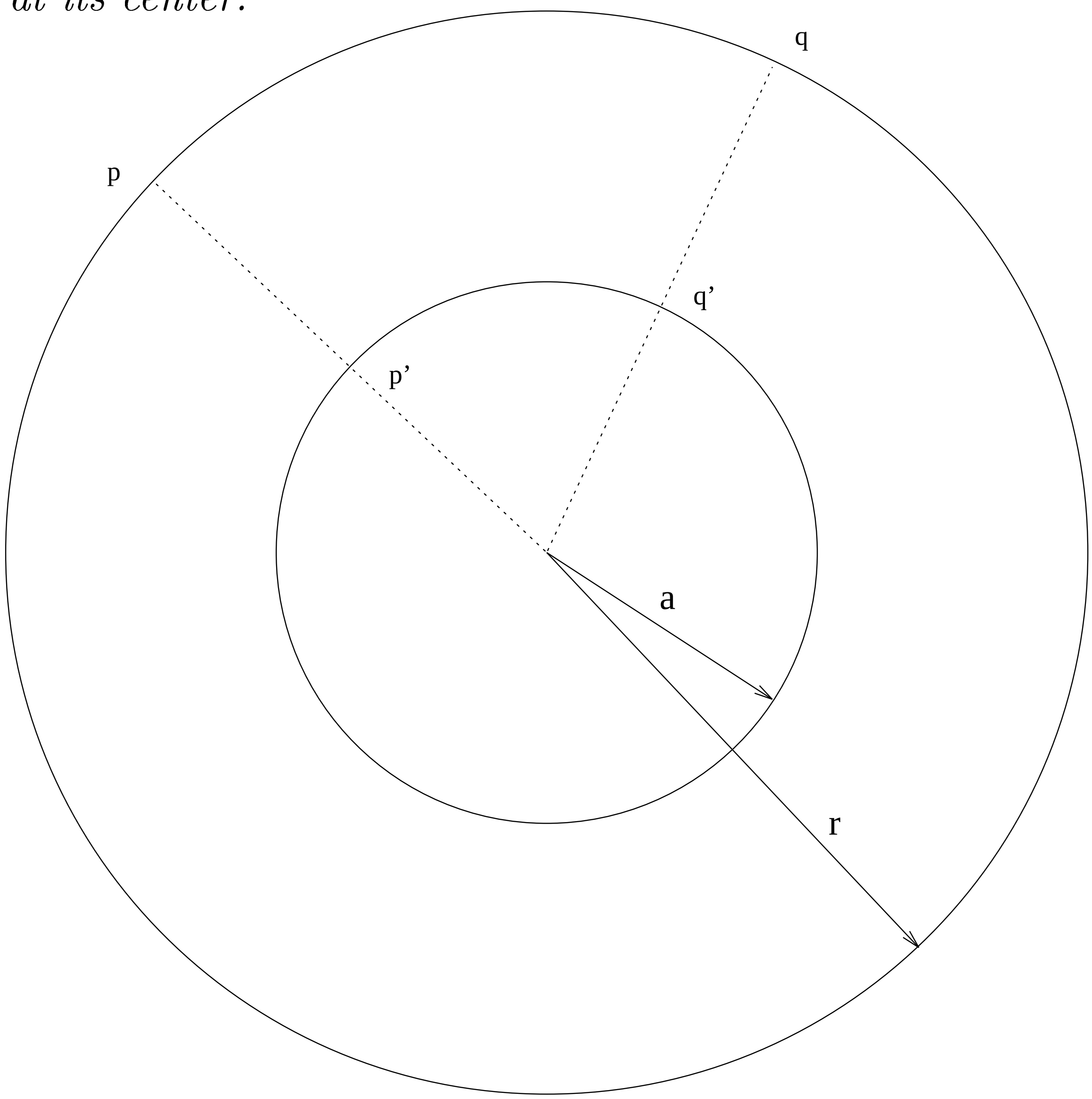
$$\frac{\delta m_1}{r_1^2} = \frac{\delta m_2}{r_2^2}.$$

Summing over all solid angles, there is no net force on an interior point due to the spherical shell.

Also $\nabla\Phi = -\mathbf{F} = 0$, $\Phi$ is a constant within the shell. Using the definition of $\Phi$ at the center of the sphere we have within the sphere:

$$\Phi = \frac{-GM}{R}.$$

Newton's second theorem: *The gravitational force on a body that lies outside a closed spherical shell of matter is tha same as it would be if all the shell's matter were concentrated into a point at its center.*



Compare the potential at point $p$ due to a patch at point $q'$ on a shell of radius $a$ of mass $M$:

$$\delta\Phi = -\frac{GM}{|\mathbf{p} - \mathbf{q}'|}\frac{\delta\mathbf{\Omega}}{4\pi}$$

with the potential at point $p'$ due to a patch at point $q$ on a shell of radius $r$ of mass $M$:

$$\delta\Phi' = -\frac{GM}{|\mathbf{p}' - \mathbf{q}|}\frac{\delta\mathbf{\Omega}}{4\pi}.$$

By symmetry $\delta\Phi = \delta\Phi'$, and by summing over $q$ and $q'$, $\Phi = \Phi'$. But we just calculated that $\Phi' = -GM/r$, so $\Phi = -GM/r$.
Since the potentials add linearly, the potential at any point in a spherical density distribution is the addition of the interior and exterior parts:

$$\Phi(r) = -4\pi G\left[\frac{1}{r}\int_0^r \rho(r')r'^2\mathrm{d}r' + \int_r^\infty \rho(r')r'\mathrm{d}r'\right].$$

The force at a point $r$ is entirely due to the mass inside $r$:

$$\mathbf{F}(r) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}\hat{\mathbf{e}}_r = -\frac{GM(r)}{r^2},$$

where $M(r)$ is the mass interior to $r$.
The velocity of a test particle on a circular orbit is the **circular speed**, $v_c$. Setting centripetal acceleration equal to the force:

$$v_c^2 = r\frac{\mathrm{d}\Phi}{\mathrm{d}r} = r|\mathbf{F}| = \frac{GM(r)}{r}.$$

So the circular speed is a measure of the mass interior to $r$.
The **escape speed**, $v_e$ is defined by

$$v_e(r) = \sqrt{2|\Phi(r)|}.$$

Where we set the kinetic energy equal to the absolute value of $\Phi$.

**Some Simple Spherical Systems**

Point mass:

$$\Phi(r) = -\frac{GM}{r} \quad ; \quad v_c(r) = \sqrt{\frac{GM}{r}} \quad ; \quad v_e(r) = \sqrt{\frac{2GM}{r}}.$$

The circular speed declining as $r^{1/2}$ it is referred to as **Keplerian**.

Homogeneous sphere:

$$M = \frac{4}{3}\pi r^3 \rho \quad ; \quad v_c = \sqrt{\frac{4\pi G\rho}{3}r}.$$

The equation of motion for a particle in such a body is

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM(r)}{r^2} = -\frac{4\pi G\rho}{3}r,$$

which is the equation of motion for a harmonic oscillator with period

$$T = \sqrt{\frac{3\pi}{G\rho}}.$$

Independent of $r$, if a particle is started at $r$, it will reach the center in a time

$$t_{dyn} = \frac{T}{4} = \sqrt{\frac{3\pi}{16G\rho}},$$

known as the **dynamical time**.

By integrating the density, we can get the potental:

$$\Phi = \begin{cases} -2\pi G\rho(a^2 - \frac{1}{3}r^2), & r < a \\ -\frac{4\pi G\rho a^3}{3r}, & r > a. \end{cases}$$

One would expect the center of a galaxy to have a potential of this type if there is no cusp in the central density.

Isochrone potential:

$$\Phi(r) = -\frac{GM}{b + \sqrt{b^2 + r^2}}.$$

The circular speed is

$$v_c^2 = \frac{GMr^2}{(b+a)^2 a},$$

where $a \equiv \sqrt{b^2 + r^2}$.
Using Poisson's equation, we can find the density:

$$\rho(r) = \frac{1}{4\pi G r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = M\left[\frac{3(b+a)a^2 - r^2(b+3a)}{4\pi(b+a)^3 a^3}\right].$$

So the central density is

$$\rho(0) = \frac{3M}{16\pi G b^3},$$

and the asymptotic density is

$$\rho(r) \approx \frac{bM}{2\pi r^4}.$$

## 4 Flattened Systems

Plummer-Kuzmin Models:
The Plummer model has the potential:

$$\Phi = -\frac{GM}{\sqrt{r^2 + b^2}}.$$

Poisson's equation in spherical coordinates gives us the density:

$$\rho = \left(\frac{3M}{4\pi b^3}\right)\left(1 + \frac{r^2}{b^2}\right)^{-\frac{5}{2}}.$$

Now consider a Kuzmin model with the axisymetric potential

$$\Phi(R, z) = -\frac{GM}{\sqrt{R^2 + (a + |z|)^2}}.$$

Note that $\nabla^2\Phi$ is everywhere 0 (except at $z = 0$). The surface density in the plane can be found using Gauss's theorem on a flat volume containing $z = 0$:

$$\sigma(R) = \frac{aM}{2\pi(R^2 + a^2)^{3/2}}.$$

Now consider

$$\Phi(R, z) = -\frac{GM}{\sqrt{R^2 + (a + \sqrt{z^2 + b^2})^2}}.$$

For $a = 0$, this reduces to a Plummer model. For $b = 0$, it reduces to the Kuzmin disk. From Poisson's equation:

$$\rho(R, z) = \left(\frac{b^2 M}{4\pi}\right) \frac{aR^2 + (a + 3\sqrt{z^2 + b^2})(a + \sqrt{z^2 + b^2})^2}{[R^2 + (a + \sqrt{z^2 + b^2})^2]^{5/2}(z^2 + b^2)^{3/2}}.$$

Logarithmic potentials:
If $v_c = v_0$, a constant, then $\mathrm{d}\Phi/\mathrm{d}R \propto 1/R$, $\Rightarrow \Phi \propto v_0^2 \ln R + C$. So consider

$$\Phi = \frac{1}{2} v_0^2 \ln\left(R_c^2 + R^2 + \frac{z^2}{q_\Phi^2}\right) + \text{constant}.$$

where $q_\Phi \leq 1$ for oblate potentials. Poisson's equation gives:

$$\rho = \left(\frac{v_0^2}{4\pi G q_\Phi^2}\right) \frac{(2q_\Phi^2 + 1)R_c^2 + R^2 + 2(1 - \frac{1}{2}q_\Phi^{-2})z^2}{(r_c^2 + R^2 + z^2 q_\Phi^{-2})^2}.$$

Poisson's equation for thin disks:
In an axisymetric system Poisson's equation is

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho.$$

For a thin disk, this simplifies to

$$\frac{\partial^2\Phi(R, z)}{\partial z^2} = 4\pi G\rho(R, z).$$

## 5 Multipole Expansions

Poisson's equation is linear, so if we find a complete set of orthogonal functions $\psi_n$ such that

$$\nabla^2\psi_n = \lambda_n\psi_n,$$

where orthogonal means

$$\int \psi_n^*\psi_m w(\mathbf{x})\mathrm{d}^3\mathbf{x} = \delta_{nm},$$

then the $\psi_n$ are eigenfunctions and the $\lambda_n$ are eigenvalues.
The density is decomposed into a sum over the eigenfunctions:

$$\rho(\mathbf{x}) = \sum_n C_n\psi_n(\mathbf{x}),$$

where

$$C_n = \int \psi_n^*\rho w(\mathbf{x})\mathrm{d}^3\mathbf{x}.$$

The potential is now just

$$\Phi(\mathbf{x}) = 4\pi G\sum_n \frac{C_n}{\lambda_n}\psi_n(\mathbf{x}).$$

and we are done.
For example, in cartesian coordinates,

$$\psi_{\mathbf{k}}(\mathbf{x}) = \rho_{\mathbf{k}}e^{\mathrm{i}(k_x x+k_y y+k_z z)}$$

are eigenfunctions of the Laplacian, with eigenvalues $-|\mathbf{k}|^2$. So the potential is just

$$\Phi = \sum_{\mathbf{k}} -\frac{4\pi G\rho(\mathbf{k})e^{\mathrm{i}(k_x x+k_y y+k_z z)}}{k^2}.$$

Poisson's equation in spherical coordinates is:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}=\lambda\psi.$$

The method of separation of variables begins by assuming $\psi$ will be a product of eigenfunctions of separarate equations in each of the independent variables.

$$\psi(r,\theta,\phi)=R(r)P(\theta)Q(\phi).$$

Substituting this in, we have

$$\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)-\lambda r^2\sin^2\theta+\frac{\sin\theta}{P}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right)=-\frac{1}{Q}\frac{\mathrm{d}^2Q}{\mathrm{d}\phi^2}.$$

Both sides must be equal to a constant, $m^2$. So

$$\frac{\mathrm{d}^2Q}{\mathrm{d}\phi^2}=-m^2Q,$$

whose eigenfunctions are easily seen as

$$Q_m(\phi)=e^{\mathrm{i}m\phi}.$$

Where $m$ must be an integer for single valued functions.
Rewriting the other side of the equation, we have

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)-\lambda r^2=\frac{m^2}{\sin^2\theta}-\frac{1}{P\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right).$$

Again, we can set both sides equal to a constant, $l(l+1)$. In terms of $x\equiv\cos\theta$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P}{\mathrm{d}x}\right]-\frac{m^2}{1-x^2}P=-l(l+1)P,$$

the eigenfunctions of which are associated Legendre functions, $P_l^{|m|}$, with eigenvalues, $l(l+1)$. The combination of $P_l^{|m|}$ and $e^{im\phi}$ are spherical harmonics, $Y_l^m(\theta,\phi)$, where

$$\int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi Y_l^{m*}(\theta,\phi) Y_{l'}^{m'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}.$$

The radial equation is

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \lambda r^2 R - l(l+1)R = 0.$$

The eigenfunctions of which are spherical Bessel functions, $j_l(kr)$ and $y_l(kr)$ with eigenvalues $-k^2$.

- It is impractical to use the radial eigenfunctions.
- Since many galaxies are nearly spherical, the $Y_l^m$ can be of use.

Consider the potential of a spherical shell. Everywhere except on the shell, we have Laplace's equation: $\nabla^2\Phi = 0$. The radial equation is now:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - l(l+1)R = 0.$$

The simplest solutions of which are

$$R(r) = Ar^l \quad \text{and} \quad Br^{-(l+1)}.$$

Note that spherical Bessel functions reduce to these in the limit of small $k$.

Now we expect the potential for a thin shell to be finite in its interior so it should be comprised of terms that look like:

$$\Phi_{\text{int}}(r,\theta,\phi) = Ar^l Y_l^m(\theta,\phi).$$

On the other hand externally, the field should fall to zero. So

$$\Phi_{\text{ext}}(r,\theta,\phi) = Dr^{-(l+1)} Y_l^m(\theta,\phi).$$

We now match these solutions at the shell by requiring $\Phi_{\rm int}(r') = \Phi_{\rm ext}(r')$ and

$$\left(\frac{\partial \Phi_{\rm ext}}{\partial r}\right)_{r=r'} - \left(\frac{\partial \Phi_{\rm int}}{\partial r}\right)_{r=r'} = 4\pi G\sigma,$$

where $\sigma$ is the surface density on the shell.
Summing over all shells, we have

$$\begin{aligned}\Phi(r,\theta,\phi) &= -4\pi G \sum_{l,m} \frac{Y_l^m(\theta,\phi)}{2l+1}\left[\frac{1}{r^{(l+1)}}\int_0^r \rho_{lm}(r') r'^{(l+2)} \mathrm{d}r' \right.\\ &\quad \left. + r^l \int_r^\infty \rho_{lm}(r') \frac{\mathrm{d}r'}{r'^{(l-1)}}\right].\end{aligned}$$

## 6 Designer basis functions

We are not forced to use orthogonal functions. If we choose any linearly independent $\psi_n$, then we can decompose the potential as

$$\Phi(\mathbf{x}) = \sum_n C_n \psi_n(\mathbf{x}),$$

where the $C_n$ satisfy

$$\sum_n M_{mn} C_n = 4\pi \int \psi_m^*(\mathbf{x}) \rho \mathrm{d}^3\mathbf{x},$$

where

$$M_{mn} = \int \psi_m^* \nabla^2 \psi_n \mathrm{d}^3\mathbf{x}.$$

The $M_{mn}$ need only be calculated once for any set of $\psi_n$.
A reasonable scheme is choose

$$\psi_{klm} = F_{kl}(r) Y_{lm}(\theta, \phi).$$

Now let

$$F_{kl} = W_l(r) U_{lp}(r)$$

Where $W_l(r)$ is a reasonable approximation to the radial dependence, and $U_{lp}(r) = (u(r))^p$.
One example would be:

$$W_l = \frac{r^l}{(1+r)^{2l+1}}, \quad u = \frac{r-1}{r+1}.$$

## 7 Disks

Consider a thin disk such that $\rho = 0$ for $z \neq 0$.
Remember that in an axisymetric system Laplace's equation is

$$\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{R\partial\Phi}{\partial R}\right) + \frac{\partial^2\Phi}{\partial z^2} = 0.$$

Just as in the spherical case, we can use separation of variables to find solutions of this equation:

$$\Phi(R, z) = J(R)Z(z).$$

This gives us

$$\frac{1}{J(R)R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}J}{\mathrm{d}R}\right) = \frac{-1}{Z(z)}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = -k^2,$$

with $k$ an arbitrary constant.
So we have

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} - k^2 Z = 0,$$

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}J}{\mathrm{d}R}\right) + k^2 J(R) = 0.$$

The potential is therefore

$$\Phi_k(R, z) = e^{-k|z|}J_0(kR).$$

where $J_0(kR)$ is a Bessel function.
If we match the solutions at $z = 0$, from Gauss's theorem we have

$$\Sigma_k(R) = -\frac{k}{2\pi G}J_0(kR).$$

Thus we can find the potential of a disk with surface density $\Sigma(R)$ if we can find a function $S(k)$ such that

$$\Sigma(R) = \int_0^\infty S(k)\Sigma_k(R)\mathrm{d}k = -\frac{1}{2\pi G}\int_0^\infty S(k)J_0(kR)k\mathrm{d}k.$$

This is a Hankel transform, and the inverse gives

$$S(k) = -2\pi G \int_0^\infty J_0(kR)\Sigma(R)R\mathrm{d}R.$$

So the potential is

$$\Phi(R, z) = -2\pi G \int_0^\infty \mathrm{d}k e^{-k|z|} J_0(kR) \int_0^\infty \Sigma(R')J_0(kR')R'\mathrm{d}R'.$$

The circular speed of a disk is easily found as

$$v_c^2 = R\left(\frac{\partial\Phi}{\partial R}\right)_{z=0} = -R\int_0^\infty S(k)J_1(kR)k\mathrm{d}k.$$

An example is the case of Mestel's disk where

$$\Sigma(R) = \frac{\Sigma_0 R_0}{R}.$$

In this case

$$S(k) = -\frac{2\pi G\Sigma_0 R_0}{k}$$

and

$$v_c^2(R) = 2\pi G\Sigma_0 R_0 R\int_0^\infty J_1(kR)\mathrm{d}k = 2\pi G\Sigma_0 R_0,$$

which is constant.