Orbits

The Continuum Approximation

How well does a smooth density and potential model a distribution of point masses?
Consider an infinite homogeneous system of identical stars with mass m, and number density n. A star with velocity $\mathbf{v}$ and impact parameter b will receive an impulse

$$|\delta \mathbf{v}_\perp| \simeq \frac{2Gm}{bv}$$

as it passes a background star. Since the background stars are distributed randomly, the sum of the impulses after a time t is:

$$\begin{aligned} |\Delta \mathbf{v}_\perp|_t^2 &= \sum |\delta \mathbf{v}_\perp|^2 = \frac{4G^2m^2t}{v^2} \int \frac{nv2\pi b \mathrm{d}b}{b^2} \\ &= \frac{8\pi G^2 m^2 n t}{v} \int \frac{\mathrm{d}b}{b} \end{aligned}$$

The test star's trajectory deviates significantly from a straight line when the impact parameter becomes smaller than $b_{\mathrm{min}} \equiv Gm/v^2$.
Also b cannot be greater than the charactaristic size of the system, R.
We therefore define the **Coulomb logarithm** by

$$\ln \Lambda \equiv \ln(b_{\mathrm{max}}/b_{\mathrm{min}}) \simeq \ln\left(\frac{Rv^2}{Gm}\right),$$

we conclude that $|\Delta \mathbf{v}_\perp|_t^2 \simeq v^2$ after a time

$$t_{\mathrm{relax}} = \frac{v^3}{8\pi (Gm)^2 n \ln \Lambda}.$$

According to the virial theorem

$$v^2 \simeq \frac{GmN}{R}, \quad \text{where} \quad N \equiv \frac{4}{3}\pi n R^3.$$

So the ratio of the relaxation time to the crossing time, $t_{\rm cross} \equiv R/v$ is

$$\frac{t_{\rm relax}}{t_{\rm cross}} = \frac{v^3 N \frac{4}{3}\pi R^3}{8\pi (v^2 R)^2 \ln\Lambda}\frac{v}{R} = \frac{N}{6\ln\Lambda}.$$

Crossing times and relaxation times for astronomical objects.

Object	N	$t_{\rm cross}$	$t_{\rm relax}$	Age
Open Cluster	10^2	10^6 yr	10^7 yr	10^8 yr
Globular Cluster	10^5	10^5 yr	10^9 yr	10^{10} yr
Galactic Nucleus	10^8	10^4 yr	10^{11} yr	10^{10} yr
Galaxy	10^{11}	10^8 yr	10^{17} yr	10^{10} yr
Galaxy Cluster	10^3	10^9 yr	10^{11} yr	10^{10} yr

Orbits in spherical potentials

The equation of motion in a spherical potential is:

$$\ddot{\mathbf{r}} = F(r)\hat{\mathbf{e}}_r.$$

Note that the angular momentum vector, $\mathbf{L} \equiv \mathbf{r} \times \mathbf{v}$ is conserved.

The equations of motion in the plane are

$$\begin{aligned} \ddot{r} - r\dot{\psi}^2 &= F(r) \\ 2\dot{r}\dot{\psi} + r\ddot{\psi} &= 0 \Rightarrow r^2\dot{\psi} = L. \end{aligned}$$

Eliminating ψ, we have a one dimensional equation of motion:

$$\ddot{r} - L^2/r^3 = F(r).$$

From this we define an effective potential

$$\Phi_{\rm eff} \equiv \Phi(r) + \frac{L^2}{2r^2}.$$

The energy of the oscillator is:

$$\frac{1}{2}\dot{r}^2 + \Phi_{\rm eff} = \frac{1}{2}\dot{r}^2 + \Phi + \frac{L^2}{2r^2} \equiv E.$$

If we eliminate t rather than ψ, then we have an equation for the orbit's shape. In terms of the variable $u \equiv 1/r$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + u = -\frac{F(1/u)}{L^2 u^2}.$$

Now let us look at motion in a Kepler potential, $\Phi = GM/r$:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + u = \frac{GM}{L^2} \quad \Rightarrow \quad u = \frac{GM}{L^2}[1 + e\cos(\psi - \psi_0)].$$

The motion is periodic in ψ with period 2π. This gives a closed orbit. In terms of frequences, $2\pi/T_r \equiv \omega_r = \omega_\psi$.
The harmonic potential is

$$\Phi = \Phi_0 + \frac{1}{2}\Omega^2 r^2.$$

The motion decouples in cartesian co-ordinates and the solution is:

$$x = X\cos(\Omega t + \phi_x), \quad y = Y\sin(\Omega t + \phi_y),$$

where X, Y, ϕ_x and ϕ_y are arbitrary constants. In this case, $\omega_r = 2\omega_\psi$.

Integrals

We define an integral to be a function $I(\mathbf{x}, \mathbf{v})$ of the phase-space coordinates such that

$$\left.\frac{\mathrm{d}I}{\mathrm{d}t}\right|_{\rm orbit} = 0.$$

For the Kepler problem we know of four integrals, $\mathbf{L}$, and E. The dimensionality of a Kepler orbit is one, so we expect 5 integrals.
To find the fifth, consider

$$r = \frac{a(1-e^2)}{1+e\cos(\psi-\psi_0)}$$

where

$$a \equiv \frac{L^2}{GM(1-e^2)} = -\frac{GM}{2E}$$

is the semi-major axis. Solving for ψ_0, we find that it is also an integral:

$$\psi_0(\mathbf{x},\mathbf{v}) = \psi - \arccos\left\{\frac{1}{e}\left[\frac{a}{r}(1-e^2)-1\right]\right\}.$$

So we've found that the number of integrals is (6 − the dimensionality of the orbit). However the number of integrals can exceed this. Consider

$$\Phi = -GM\left(\frac{1}{r}+\frac{r_0}{r^2}\right).$$

An orbit in this potential creates a rosette, yet the potential still has a fifth integral, ψ_0. The equation of motion is now

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + \left(1-\frac{2GMr_0}{L^2}\right)u = \frac{GM}{L^2}.$$

The solution of which is

$$u = \frac{GM}{L^2}\left[K^2 + e\cos\left(\frac{\psi-\psi_0}{K}\right)\right],$$

where

$$K \equiv 1\Big/\sqrt{1-\frac{2GMr_0}{L^2}}.$$

So ψ_0 is an integral. To see why it is not isolating, look at the solution for ψ:

$$\psi = \psi_0 + K \arccos\left\{\frac{1}{e}\left[\frac{a}{r}(1-e^2) - K^2\right]\right\}.$$

But we can always add $2m\pi$ to the value of the arccos, and K will almost always be irrational. Therefore, ψ can approach any given value and ψ_0 imposes no useful constraint on the particle's motion.

Axisymmetric potentials

The equations of motion in an axisymmetric potential are

$$\frac{\mathrm{d}}{\mathrm{d}t}(R^2\dot{\phi}) = 0 \quad \Rightarrow \quad R^2\dot{\phi} = L_z.$$

And

$$\begin{aligned} \ddot{R} &= -\frac{\partial \Phi_{\mathrm{eff}}}{\partial R} \\ \ddot{z} &= -\frac{\partial \Phi_{\mathrm{eff}}}{\partial z} \end{aligned} \qquad \text{where} \qquad \Phi_{\mathrm{eff}} \equiv \Phi + \frac{L_z^2}{2R^2}$$

Now consider the potential

$$\Phi = \frac{1}{2}v_0^2 \ln\left(R^2 + \frac{z^2}{q^2}\right)$$

This reduces the orbit to motion in the meridional plane (R, z). The only analytical integral is

$$E = \frac{1}{2}(\dot{R}^2 + \dot{z}^2) + \Phi_{\mathrm{eff}}.$$

But the orbit seems to be constrained by an additional integral.

Surface of Section

In the spherical or nearly spherical case, the **third integral** can be found analytically.
In the general 2-D case, we can use a graphical device: Poincaré's surface of section.

1. Choose an energy condition.
2. Choose a coordinate condition.
3. Integrate the orbit.
4. Plot the other coordinate vs. its conjugate momentum (the consequent) whenever the coordinate condition is satisfied.

- If the orbit is not restricted by another integral, the consequents will fill an area.
- If the orbit is restricted by another integral, the consequents will lie on a curve.

The epicycle approximation

Expand $\Phi_{\rm eff}$ in a Taylor series about its minimum:

$$\Phi_{\rm eff} = {\rm const} + \frac{1}{2}\kappa^2 x^2 + \frac{1}{2}\nu^2 z^2 + \cdots,$$

where

$$x \equiv R - R_g, \quad \kappa^2 \equiv \left.\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right|_{(R_g,0)}, \quad \nu^2 \equiv \left.\frac{\partial^2 \Phi_{\rm eff}}{\partial z^2}\right|_{(R_g,0)}.$$

The equations of motion decouple and we have two integrals:

$$x = X\cos(\kappa t + \phi_0) \qquad z = Z\cos(\nu t + \zeta)$$
$$E_R \equiv \frac{1}{2}[v_R^2 + \kappa^2 (R - R_g)^2] \qquad E_z \equiv \frac{1}{2}[v_z^2 + \nu^2 z^2].$$

Now compare the **epicycle frequency**, κ, with the angular frequency, Ω.

$$\Omega^2 \equiv \frac{v_c^2}{R^2} = \frac{1}{R}\frac{\partial\Phi}{\partial R} = \frac{1}{R}\frac{\partial\Phi_{\rm eff}}{\partial R} + \frac{L_z^2}{R^4},$$

$$\kappa^2 = \frac{\partial(R^2\Omega^2)}{\partial R} + \frac{3L_z^2}{R^4} = R\frac{\partial\Omega^2}{\partial R} + 4\Omega^2.$$

Since Ω always decreases, but never faster than Keplerian,

$$\Omega \leq \kappa \leq 2\Omega.$$

The epicycle approximation also makes a prediction for the ϕ-motion since $L_z = R^2\dot{\phi}$ is conserved. Let

$$y \equiv R_g[\phi - (\phi_0 + \Omega t)]$$

be the displacement in the ϕ direction from the "guiding center". If we expand L_z to first order in displacements from the guiding center, we obtain

$$\phi = \phi_0 + \Omega t - \frac{2\Omega X}{\kappa R_g}\sin(\kappa t + \phi_0).$$

Therefore

$$y = -Y\sin(\kappa t + \phi_0) \quad \text{where} \quad \frac{Y}{X} = \frac{2\Omega}{\kappa} \equiv \gamma \geq 1.$$

$\Rightarrow$ The epicycles are elongated tangentially.
Now eliminate $(R - R_g)$ from E_R. Let $\mathbf{v} \equiv \mathbf{v}' - \mathbf{v}_{LSR}$ be the velocity w.r.t. the Local Standard of Rest. Then

$$\begin{aligned} v_\phi(R_0) &\equiv v'_\phi(R_0) - v_c(R_0) = R_0(\dot{\phi} - \Omega_0) \\ &= R_0(\dot{\phi} - \Omega_g + \Omega_g - \Omega_0) \\ &\simeq R_0\left[(\dot{\phi} - \Omega_g) - \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)_{R_0} x\right]. \end{aligned}$$

Replacing $\dot{\phi}$ by $\Omega_g(1 - 2x/R_g)$ this becomes

$$\begin{aligned} v_\phi &\simeq -R_0 x \left[\frac{2\Omega_g}{R_g} + \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)_{R_0}\right] \simeq -x \left[2\Omega_0 + R_0 \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)_{R_0}\right] \\ &\equiv 2Bx. \end{aligned}$$

So $E_R \simeq \frac{1}{2}[v_R^2 + \gamma^2 v_\phi^2]$ where $\gamma = \frac{1}{2}\kappa/B = 2\Omega_g/\kappa$. So E_R looks like a kinetic energy of peculiar motion.

Non-axisymmetric potentials

Consider the 2-D potential

$$\Phi(x, y) = \frac{1}{2} v_0^2 \ln(R_c^2 + x^2 + y^2/q^2).$$

Two different families of orbits are present: **loop orbits** and **box orbits**.

Orbits in both families are constrained by another integral.

Stäckel potentials

Finding the right coordinate system helps to find the integrals. We need a coordinate system (u, v) such that:

1. $u =$ constant and $v =$ constant are boundaries of orbits.
2. $H(u, v, p_u, p_v) \times f(u, v)$ breaks into a sum $F_u(u, p_u) + F_v(v, p_v)$.

Consider the coordinates defined by

$$x = \Delta \sinh u \cos v, \quad y = \Delta \cosh u \sin v.$$

The Lagrangian in these coordinates is

$$\mathcal{L} = \frac{1}{2}\Delta^2(\sinh^2 u + \cos^2 v)(\dot{u}^2 + \dot{v}^2) - \Phi.$$

The momenta are therefore

$$\begin{aligned} p_u &\equiv \frac{\partial \mathcal{L}}{\partial \dot{u}} = \Delta^2(\sinh^2 u + \cos^2 v)\dot{u}, \\ p_v &\equiv \frac{\partial \mathcal{L}}{\partial \dot{v}} = \Delta^2(\sinh^2 u + \cos^2 v)\dot{v}, \end{aligned}$$

and the Hamiltonian is

$$\begin{aligned} H(u, v, p_u, p_v) &= p_u \dot{u} + p_v \dot{v} - \mathcal{L} \\ &= \frac{1}{2}\Delta^2(\sinh^2 u + \cos^2 v)(\dot{u}^2 + \dot{v}^2) + \Phi \\ &= \frac{p_u^2 + p_v^2}{2\Delta^2(\sinh^2 u + \cos^2 v)} + \Phi. \end{aligned}$$

It is clear that the Hamiltonian will only separate if

$$\Phi(u, v) = \frac{U(u) - V(v)}{\sinh^2 u + \cos^2 v}.$$

Then $(\sinh^2 u + \cos^2 v) \times (H = E)$ breaks into

$$E \sinh^2 u - \frac{p_u^2}{2\Delta^2} - U(u) \equiv I_2 = \frac{p_v^2}{2\Delta^2} - V(v) - E\cos^2 v,$$

where I_2 is a constant.
The momenta are therefore

$$\begin{aligned} p_u &= \pm\sqrt{2}\Delta \sinh u \sqrt{E - U_{\text{eff}}}, \quad \text{where} \quad U_{\text{eff}} \equiv \frac{I_2 + U(u)}{\sinh^2 u}, \\ p_v &= \pm\sqrt{2}\Delta \cos v \sqrt{E - V_{\text{eff}}}, \quad \text{where} \quad V_{\text{eff}} \equiv \frac{I_2 + V(v)}{\cos^2 v}. \end{aligned}$$

Now look at the potential of a **perfect ellipsoid** which corresponds to a density of

$$\rho(\mathbf{x}) = \frac{\text{constant}}{1 + m^2}, \quad \text{where} \quad m^2(\mathbf{x}) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}.$$

For the case $a > b = c$ we have

$$\begin{aligned} U(u) &= -W \sinh u \arctan\left(\frac{\Delta \sinh u}{c}\right) \\ V(v) &= W \cos v \,\mathrm{arctanh}\left(\frac{\Delta \cos v}{c}\right) \end{aligned}$$

For $I_2 > 0$ we have loop orbits and for $I_2 < 0$ we have box orbits.

Review of Hamiltonian Mechanics

Lagrangian Mechanics

The principle of virtual work is that in equilibrium,

$$\sum_i \mathbf{F}_i \cdot \delta \mathbf{q}_i = 0,$$

where $\mathbf{F}_i$ is the applied force and $\delta \mathbf{q}_i$ is a displacement consistent with constraints.
But any dynamical state can be an equilibrium if we add a force, $-\dot{\mathbf{p}}$ so

$$\sum_i (\mathbf{F}_i \cdot \delta \mathbf{q}_i - \dot{\mathbf{p}}_i \cdot \delta \mathbf{q}_i) = 0.$$

Integrating over a finite time, we have

$$\int_{t1}^{t2} \left[-\delta\Phi + \sum \mathbf{p}_i \cdot \frac{\mathrm{d}(\delta \mathbf{q}_i)}{\mathrm{d}t} \right] \mathrm{d}t - \left[\sum \mathbf{p}_i \cdot \delta \mathbf{q}_i \right]_{t_1}^{t2}.$$

Now we need a function T_i such that

$$\delta T_i = \mathbf{p}_i \cdot \delta \mathbf{v}_i = \mathbf{p}_i \cdot \frac{\mathrm{d}(\delta \mathbf{q}_i)}{\mathrm{d}t}.$$

This is obviously $T_i = \frac{1}{2} m_i \mathbf{v}_i^2$, so

$$\delta \int_{t1}^{t2} (T - \Phi) \mathrm{d}t = \delta \int_{t1}^{t2} L \mathrm{d}t = 0,$$

where $L = T - \Phi$ is the **Lagrangian**.
L plays a rôle in dynamics similar to that of Φ in statics.
Note that L is a function of $\mathbf{q}$ and $\dot{\mathbf{q}}$.
$\Rightarrow$ we cannot have velocity dependent coordinates in Lagrangian mechanics.
Can we come up with new coordinates to use as dynamical variables?

Since the δq vanish at the end-points, we have

$$\int_{t1}^{t2}\left[\delta L-\frac{\mathrm{d}(\mathbf{p}\cdot\delta\mathbf{q})}{\mathrm{d}t}\right]\mathrm{d}t=0.$$

for any function $\mathbf{p}$.

$$\int_{t1}^{t2}\left[\left(\frac{\partial L}{\partial\dot{\mathbf{q}}}-\mathbf{p}\right)\cdot\delta\dot{\mathbf{q}}+\left(\frac{\partial L}{\partial\mathbf{q}}-\dot{\mathbf{p}}\right)\cdot\delta\mathbf{q}\right]\mathrm{d}t=0.$$

Choose $\mathbf{p}=\partial L/\partial\dot{\mathbf{q}}$ and note that this gives $\dot{\mathbf{p}}=\partial L/\partial\mathbf{q}$. Now define

$$H(\mathbf{p},\mathbf{q},t)=\mathbf{p}\cdot\dot{\mathbf{q}}-L$$

so that

$$\mathrm{d}H=\dot{\mathbf{q}}\cdot\mathrm{d}\mathbf{p}-\frac{\partial L}{\partial\mathbf{q}}\cdot\mathrm{d}\mathbf{q}-\frac{\partial L}{\partial t}\mathrm{d}t.$$

this implies

$$\dot{\mathbf{q}}=\frac{\partial H}{\partial\mathbf{p}},\quad\dot{\mathbf{p}}=\frac{\partial L}{\partial\mathbf{q}}=-\frac{\partial H}{\partial\mathbf{q}},$$

which are Hamilton's equations of motion.
The variational principle is now

$$\delta\int(\mathbf{p}\cdot\dot{\mathbf{q}}-H)\mathrm{d}t=0.$$

The dynamical variables are now $\mathbf{p}$ and $\mathbf{q}$.

Poisson brackets and symplectic structure

In terms of cartesian coordinates one defines the **Poisson bracket** $[A,B]$ of any two functions $A(\mathbf{x},\mathbf{v})$, $B(\mathbf{x},\mathbf{v})$ on phase space by

$$[A,B]\equiv\frac{\partial A}{\partial\mathbf{x}}\cdot\frac{\partial B}{\partial\mathbf{v}}-\frac{\partial A}{\partial\mathbf{v}}\cdot\frac{\partial B}{\partial\mathbf{x}}. \tag{1}$$

We can verify that the coordinates $(\mathbf{x},\mathbf{v})$ satisfy the **canonical commutation relations**

$$[v_i,v_j]=[x_i,x_j]=0 \quad\text{and}\quad [x_i,v_j]=\delta_{ij}.$$

If we write ($z_i\equiv x_i,\ z_{3+i}\equiv v_i\quad i=1,2,3$), and define the **symplectic matrix c** by

$$c_{\alpha\beta}\equiv[z_\alpha,z_\beta]=\begin{cases}\pm1 & \text{for } \alpha=\beta\mp3,\, 1\le\alpha,\beta\le6;\\ 0 & \text{otherwise,}\end{cases} \tag{2}$$

we have

$$[A,B]=\sum_{\alpha,\beta=1}^{6}c_{\alpha\beta}\frac{\partial A}{\partial z_\alpha}\frac{\partial B}{\partial z_\beta}. \tag{3}$$

Any set $\{Z_\alpha,\ \alpha=1,\dots,6\}$ is called a set of **canonical coordinates** if $[Z_\alpha,Z_\beta]=c_{\alpha\beta}$. Let $\{Z_\alpha\}$ be such a set; then with (3) and the chain rule we have

$$\begin{aligned}[A,B] &= \sum_{\alpha,\beta=1}^{6}c_{\alpha\beta}\frac{\partial A}{\partial z_\alpha}\frac{\partial B}{\partial z_\beta}=\sum_{\kappa\lambda}\left(\sum_{\alpha\beta}c_{\alpha\beta}\frac{\partial Z_\kappa}{\partial z_\alpha}\frac{\partial Z_\lambda}{\partial z_\beta}\right)\frac{\partial A}{\partial Z_\kappa}\frac{\partial B}{\partial Z_\lambda}\\ &= \sum_{\kappa\lambda}[Z_\kappa,Z_\lambda]\frac{\partial A}{\partial Z_\kappa}\frac{\partial B}{\partial Z_\lambda}=\sum_{\kappa\lambda}c_{\kappa\lambda}\frac{\partial A}{\partial Z_\kappa}\frac{\partial B}{\partial Z_\lambda}.\end{aligned} \tag{4}$$

Thus the value of (1) is *coordinate independent*.

The **Hamiltonian** is the particle's energy written as a function of the phase-space variables. For Kepler motion it is

$$H_K(\mathbf{x},\mathbf{v})=\frac{1}{2}v^2-\frac{GM}{r}.$$

Hamilton's equation express the rate of change of $(\mathbf{x}, \mathbf{v})$ in terms of gradients of H:

$$\dot{x}_i = \frac{\partial H}{\partial v_i} \quad ; \quad \dot{v}_i = -\frac{\partial H}{\partial x_i}. \tag{5}$$

Hamilton's eqns may be written

$$\dot{x}_i = [x_i, H] \quad ; \quad \dot{v}_i = [v_i, H]. \tag{6}$$

The rate of change of an arbitrary canonical coordinate Z_α along an orbit is

$$\dot{Z}_\alpha = \sum_{\beta=1}^{6} \frac{\partial Z_\alpha}{\partial z_\beta} \dot{z}_\beta,$$

where, as usual, $\mathbf{z} \equiv (\mathbf{x}, \mathbf{v})$. With Hamilton's equations (6) and equation (3) this becomes

$$\begin{aligned} \dot{Z}_\alpha &= \sum_{\beta=1}^{6} \frac{\partial Z_\alpha}{\partial z_\beta}[z_\beta, H] = \sum_{\beta\gamma\delta} \frac{\partial Z_\alpha}{\partial z_\beta} c_{\gamma\delta} \frac{\partial z_\beta}{\partial z_\gamma} \frac{\partial H}{\partial z_\delta} = \sum_{\gamma\delta} c_{\gamma\delta} \frac{\partial Z_\alpha}{\partial z_\gamma} \frac{\partial H}{\partial z_\delta} \\ &= [Z_\alpha, H]. \end{aligned} \tag{7}$$

Thus Hamilton's equations (6) are valid in *any* canonical coordinate system.
Given $H(\mathbf{q}, \mathbf{p})$ and $H'(\mathbf{Q}, \mathbf{P})$, then

$$\delta \int (\mathbf{p} \cdot \dot{\mathbf{q}} - H) dt = \delta \int (\mathbf{P} \cdot \dot{\mathbf{Q}} - H') dt.$$

A total differential can be added so:

$$\mathbf{p} \cdot \dot{\mathbf{q}} - H = \mathbf{P} \cdot \dot{\mathbf{Q}} - H' + \frac{\mathrm{d}}{\mathrm{d}t} F.$$

If $F = F(\mathbf{q}, \mathbf{Q})$ we see that in order for the above to hold, we must have

$$\mathbf{p} = \frac{\partial F}{\partial \mathbf{q}}, \quad \mathbf{P} = -\frac{\partial F}{\partial \mathbf{Q}}, \quad H' = H + \frac{\partial F}{\partial t}.$$

Likewise, if we consider $F = F(\mathbf{q}, \mathbf{P})$ then we can add $\mathbf{Q} \cdot \dot{\mathbf{P}}$ to both sides, set $S = F + \mathbf{Q} \cdot \mathbf{P}$ and get the requirement that

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}}$$

and

$$\mathbf{Q} = \frac{\partial S}{\partial \mathbf{P}}.$$

The transformation $(\mathbf{p}, \mathbf{q}) \rightarrow (\mathbf{P}, \mathbf{Q})$ is called a **canonical transformation** and S the **generating function** of the transformation.

Hamilton-Jacobi Equation

A **constant of motion** is a function $I(\mathbf{q}, \mathbf{p})$ on phase space which takes the same value at all points on an orbit:

$$0 = \frac{\mathrm{d}I}{\mathrm{d}t} = \sum_\alpha \frac{\partial I}{\partial z_\alpha} \dot{z}_\alpha.$$

Suppose there are 3 constants of motion I_1, I_2, I_3, and there was a system of canonical coordinates $(\mathbf{J}, \mathbf{w})$ such that $J_i = I_i$ etc. The e.o.m. for the J's would be trivial,

$$\begin{aligned} 0 &= \dot{J}_i = [J_i, H] \\ &= -\frac{\partial H}{\partial w_i}. \end{aligned}$$

This implies that $H(\mathbf{J})$ is independent of the w's. Therefore, the e.o.m. for the w's: are

$$\dot{w}_i = \frac{\partial H}{\partial J_i} \equiv \omega_i(\mathbf{J}), \quad \text{a constant} \quad \Rightarrow \quad w_i(t) = w_i(0) + \omega_i t.$$

and the problem is solved! $(\mathbf{J}, \mathbf{w})$ are called **action-angle** coordinates, the J's are actions and w's are angles. If the orbit is bound, we can expand $\mathbf{x}$ in a Fourier series

$$\mathbf{x}(\mathbf{w},\mathbf{J}) = \sum_{\mathbf{n}} \mathbf{X}_{\mathbf{n}}(\mathbf{J}) e^{\mathrm{i}\mathbf{n}\cdot\mathbf{w}},$$

where the sum is over all vectors $\mathbf{n}$ with integer components. The 3-surface of fixed $\mathbf{J}$ and varying $\mathbf{w}$ is a 3-torus. The best labels for this torus are the Poincaré invariants

$$\begin{aligned} J_i' &\equiv \frac{1}{2\pi}\oint_{\gamma_i} \mathbf{p}\cdot \mathrm{d}\mathbf{q} && (8)\\ &= \frac{1}{2\pi}\iint \sum_i \mathrm{d}p_i \mathrm{d}q_i, && (9) \end{aligned}$$

where the path γ_i is that followed when w_i is varied from 0 to 2π while holding everything else constant. Actually, $J_i' = J_i$. To see this, let $S(\mathbf{J},\mathbf{x})$ be the generating function of the transformation from Cartesian to action-angle coordinates. Then

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}} \quad ; \quad \mathbf{w} = \frac{\partial S}{\partial \mathbf{J}},$$

so

$$\begin{aligned} J_i' &\equiv \frac{1}{2\pi}\oint_{\gamma_i} \frac{\partial S}{\partial \mathbf{q}}\cdot \mathrm{d}\mathbf{q}\\ &= \frac{\Delta S}{2\pi} \end{aligned}$$

But we also have

$$\Delta S = \oint_{\gamma_i} \mathbf{J}\cdot \mathrm{d}\mathbf{w} = 2\pi J_i.$$

We can use $S(\mathbf{J},\mathbf{q})$ to eliminate $\mathbf{p} = \partial S/\partial \mathbf{q}$ from H, expressing H as a function of $(\mathbf{J},\mathbf{q})$:

$$H(\mathbf{J},\mathbf{q}) = H(\frac{\partial S}{\partial \mathbf{q}}, \mathbf{q}).$$

If we vary the q_i along an orbit H must remain constant at the energy E of the orbit in question. This suggests that we investigate the partial differential equation

$$H(\frac{\partial S}{\partial \mathbf{q}}, \mathbf{q}) = E, \qquad \textbf{(Hamilton-Jacobi equation)}.$$

If we can solve this equation, we identify the arbitrary constants on which the solution $S(\mathbf{q})$ depends with functions of the J_i.

Delaunay variables

The H-J equation for Kepler motion is

$$\begin{aligned} E &= \frac{1}{2}|\nabla S|^2 - \frac{GM}{r} \\ &= \frac{1}{2}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial S}{\partial \theta}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi}\right)^2\right] - \frac{GM}{r} \end{aligned}$$

Let $S(\mathbf{x}) = S_r(r) + S_\theta(\theta) + S_\phi(\phi)$:

$$\begin{aligned} \text{constant} \equiv L_z^2 &= \left(\frac{\partial S_\phi}{\partial \phi}\right)^2 = p_\phi^2 && (10) \\ L^2 - \frac{L_z^2}{\sin^2\theta} &= \left(\frac{\partial S_\theta}{\partial \theta}\right)^2 = p_\theta^2 && (11) \\ 2E + 2\frac{GM}{r} - \frac{L^2}{r^2} &= \left(\frac{\partial S_r}{\partial r}\right)^2 = p_r^2. && (12) \end{aligned}$$

These equations define a 3-surface: the orbit torus.
The inclination is $i = \frac{\pi}{2} - \theta_{\min}$ and from equation (11), $\sin\theta_{\min} = L_z/L$, so

$$i = \arccos(L_z/L).$$

The actions are

$$J_\phi = \frac{1}{2\pi}\oint p_\phi(\phi)\,\mathrm{d}\phi = L_z$$

$$
\begin{aligned}
J_\theta &= \frac{1}{2\pi}\oint p_\theta(\theta)\,\mathrm{d}\theta \\
&= \frac{2}{\pi}\int_{\theta_{\min}}^{\pi/2}\sqrt{L^2-\frac{L_z^2}{\sin^2\theta}}\,\mathrm{d}\theta = L - L_z \\
J_r &= \frac{1}{2\pi}\oint\sqrt{2E+\frac{2GM}{r}-\frac{L^2}{r^2}}\,\mathrm{d}r \\
&= \frac{GM}{\sqrt{2|E|}} - L.
\end{aligned}
$$

The last equation can be written

$$
E = H = -\frac{G^2M^2}{2(J_r+J_\theta+J_\phi)^2},
$$

therefore $\omega_r = \omega_\theta = \omega_\phi = -2H/(J_r+J_\theta+J_\phi)$: a double degeneracy.

Two differences in angles are constant, e.g. $w_1 \equiv w_\phi - w_\theta$ and $w_2 \equiv w_\theta - w_r$. Make a canonical transformation to a set which includes these with frequencies of

$$
\omega_1 = 0; \quad \omega_2 = 0.
$$

The generating function of this transformation is

$$
S' = (w_\phi - w_\theta)J_1 + (w_\theta - w_r)J_2 + w_r J_3.
$$

Differentiating S' w.r.t. the old angles:

$$
\begin{aligned}
J_\phi &= J_1 & & & \\
J_\theta &= J_2 - J_1 &\quad\Rightarrow\quad & J_2 &= J_\theta + J_\phi = L \\
J_r &= J_3 - J_2 &\quad\Rightarrow\quad & J_3 &= J_r + J_\theta + J_\phi.
\end{aligned}
$$

Replacing the old actions by the new:

$$
H = -\frac{G^2M^2}{2J_3^2} = -\frac{GM}{2a} \quad\Rightarrow\quad J_3 = \sqrt{GMa},
$$

$$
\begin{aligned}
e &= \sqrt{1-\frac{L^2}{GMa}}=\sqrt{1-J_2^2/J_3^2}, \\
i &= \arccos(J_1/J_2).
\end{aligned}
$$

It can be shown that w_1 is Ω, w_2 is ω and w_3 is N. e.g. w_1: We have $w_i=(\partial S/\partial J_i)$. Replacing E, L and L_z in our expressions for $\partial S_\phi/\partial\phi$ etc. we find

$$
S=\int J_1 \mathrm{d}\phi - \int\sqrt{J_2^2-\frac{J_1^2}{\sin^2\theta}}\,\mathrm{d}\theta + \int\sqrt{-\frac{G^2M^2}{J_3^2}+2\frac{GM}{r}-\frac{J_2^2}{r^2}}\,\mathrm{d}r. \tag{13}
$$

(Note the negative square root in S_θ.) Differentiating we get

$$
\begin{aligned}
w_1=\frac{\partial S}{\partial J_1} &= \phi + J_1\int\frac{\mathrm{d}\theta}{\sin^2\theta\sqrt{J_2^2-J_1^2\csc^2\theta}} \\
&= \phi+\int\frac{\mathrm{d}\theta}{\sin\theta\sqrt{\sin^2\theta\sec^2 i-1}} \\
&= \phi+\int\frac{\cot i\csc^2\theta\mathrm{d}\theta}{\sqrt{1-\cot^2 i\cot^2\theta}} \\
&= \phi-u,
\end{aligned}
$$

where

$$
\sin u\equiv\cot i\cot\theta.
$$

The new variable u is actually $\phi-\Omega$, therefore $w_1=\Omega$: (w_i, J_i) are called Delaunay variables. Let the Hamiltonian of a planet be $H=H_K+\epsilon H_1$, where H_K is the Kepler Hamiltonian and $\epsilon \ll 1$; the e.o.m. of the Delaunay variables are

$$
\begin{aligned}
\dot{J}_1=-\epsilon\frac{\partial H_1}{\partial w_1} \quad &; \quad \dot{J}_2 = -\epsilon\frac{\partial H_1}{\partial w_2} \quad ; \quad \dot{J}_3=-\epsilon\frac{\partial H_1}{\partial w_3} \\
\dot{w}_1=\epsilon\frac{\partial H_1}{\partial J_1} \quad &; \quad \dot{w}_2 = \epsilon\frac{\partial H_1}{\partial J_2} \quad ; \quad \dot{w}_3=-\frac{2H_K}{J_3}+\epsilon\frac{\partial H_1}{\partial J_3}
\end{aligned}
$$

Perturbation Theory

If Jupiter and the asteroid are on circular orbits in the ecliptic. Then

$$H_1 = -\frac{Gm_J}{a_J}((1+\alpha^2 - 2\alpha\cos\varphi)^{-1/2} - \alpha\cos\varphi), \tag{14}$$

where $\alpha \equiv a_a/a_J$ and φ is the angle between $\mathbf{r}_a$ and $\mathbf{r}_J$. Expand the radical in (14) in a Fourier series to give

$$H_1 = -\frac{Gm_J}{a_J}\sum_{\mathbf{n}} h_{\mathbf{n}}(\alpha)e^{\mathrm{i}(n_1\nu_J t + n_3 w_3)}, \tag{15}$$

where for circular orbits

$$\phi_J = \nu_J t \quad \text{and} \quad \phi_a = w_3 = \phi_0 + \nu_a t,$$

and ν_J and ν_a are the orbital frequencies of Jupiter and the asteriod, respectively.

The equations of motion of J_3 and w_3 are now

$$\begin{aligned}\dot{J}_3 &= \frac{Gm_J}{a_J}\sum_{\mathbf{n}} \mathrm{i}n_3 h_{\mathbf{n}}(\alpha)e^{\mathrm{i}(n_1\nu_J t + n_3 w_3)} \\ \dot{w}_3 &= -\frac{Gm_J}{a_J}\sum_{\mathbf{n}} \frac{\partial h_{\mathbf{n}}}{\partial\alpha}\frac{\partial\alpha}{\partial J_3}e^{\mathrm{i}(n_1\nu_J t + n_3 w_3)}.\end{aligned} \tag{16}$$

The n_i range over all positive and negative integers. For an asteroid with $\nu_a = 3\nu_J$, terms like $\mathbf{n} = (3, -1)$ cause disaster when we integrate up (16):

$$J_3 = \frac{Gm_J}{a_J}\sum_{\mathbf{n}} \frac{n_3 h_{\mathbf{n}}(\alpha)}{n_1\nu_J + n_3\nu_a}e^{\mathrm{i}t(n_1\nu_J + n_3\nu_a)}e^{\mathrm{i}(n_3\phi_0)} + \text{constant}.$$

The near vanishing of some of the denominators $n_1\nu_J + n_3\nu_a$ is called the 'small divisor problem'.

Resonances and Pendulum equations

The small denominators are the interesting terms.
For example, an asteroid along whose orbit $N_1\nu_J + N_3\nu_a \simeq 0$.
Let's set $J_2 = w_2 = 0$ and look at the evolution of (J_3, w_3).
Define a variable

$$\psi \equiv N_1\nu_J t + N_3 w_3$$

In H_1, only use zero-frequency term (the **secular** term) and the terms that comprise $\cos(\psi)$:

$$H_1 \simeq h_0 + h_{\rm res}\cos\psi.$$

This is called 'averaging the Hamiltonian'. The generating function for canonical transformation $(w_3, J_3) \leftrightarrow (\psi, J_\psi)$ is

$$S(w_3, J_\psi) = (N_1\nu_J t + N_3 w_3)J_\psi \quad \Rightarrow \quad \begin{aligned} \psi &= \frac{\partial S}{\partial J_\psi} = N_1\nu_J t + N_3 w_3, \\ J_3 &= \frac{\partial S}{\partial w_3} = N_3 J_\psi. \end{aligned}$$

Since the generating function contains explicit t-dependence in the new variables, the Hamiltonian is

$$\begin{aligned} H_1(J_\psi, \psi) &= H_1(J_3, w_3, t) + \frac{\partial S}{\partial t} \\ &= h_0 + N_1\nu_J J_\psi + h_{\rm res}\cos\psi. \end{aligned}$$

The motion is confined to surfaces of constant

$$\begin{aligned} H(J_\psi, \psi) &= H_0 + h_0 + N_1\nu_J J_\psi + h_{\rm res}\cos\psi \\ &= \alpha(J_\psi) - \beta(J_\psi)\cos\psi. \end{aligned}$$

Assume that $\beta > 0$. Note that $\beta/|\alpha| \simeq m_J a_a/m_s a_J$ is small.
There is an orbit where

$$0 = \dot\psi = \partial H_0/\partial J_\psi = {\rm d}\alpha/{\rm d}J_\psi.$$

So Taylor expanding α in powers of $\Delta \equiv J_\psi - J_0$ we have

$$H(\Delta, \psi) = A + \frac{1}{2}B\Delta^2 - \beta\cos\psi,$$

where A and B are constants. Since β is small, $\beta(J_\psi) \simeq \beta(J_0)$ and the equations of motion are approximately

$$\begin{array}{ll} \dot{\psi} & = B\Delta \\ \dot{\Delta} & = -\beta\sin\psi \end{array} \quad \Rightarrow \quad \ddot{\psi} = -B\beta\sin\psi.$$

This is the equation of motion of a pendulum. Quantitatively it circulates when

$$\text{K.E. at bottom} = \frac{1}{2}\dot{\psi}^2 > 2B\beta = \text{P.E. difference between top and bottom.}$$

When ψ is oscillating around $\psi = 0$, the asteroid is **librating** in w_3. There are asteroids trapped in this way for $N_1/N_2 = 1$ (**Trojan asteroids**) and 3/2 (**Hilda asteroids**). Other important resonances are marked by **Kirkwood gaps**.
Resonances overlap leads to chaos. Whether or not resonance overlap occurs depends on (i) the magnitude of the $h_{\mathbf{n}}$, and (ii) the nearness of the resonances.
Around Saturn the resonances are sufficiently well separated for order to be the rule. Around Uranus many important resonances overlap and chaos is widespread.

Numerical integration and Mappings

A numerical integration can be thought of as a transformation

$$(\mathbf{q}_0, \mathbf{p}_0) \to (\mathbf{q}_\tau, \mathbf{p}_\tau),$$

which is repeated.

We want a canonical or symplectic transformation : $(\mathbf{q}_0, \mathbf{p}_0)$ are related to $(\mathbf{q}_\tau, \mathbf{p}_\tau)$ by a generating function.
Now consider our Hamiltonian of

$$H = H_K + H_1.$$

Hamilton's equations are

$$\dot{\mathbf{z}} = \{\mathbf{z}, H_K + H_1\},$$

where $\mathbf{z}$ is $(\mathbf{q}, \mathbf{p})$. Introducing the operators

$$K \equiv \{, H_K\} \quad \text{and} \quad P \equiv \{, H_1\},$$

the formal solution of the above is

$$\mathbf{z}(\tau) = \exp[\tau(K + P)]\mathbf{z}(0).$$

Approximate $\exp[\tau(K+P)]$ as a product of terms like $\exp(\tau K)$ and $\exp(\tau P)$ via the Baker-Campbell-Hausdorff (BCH) identity:

$$\exp K \exp P = \exp\left(K + P + \frac{1}{2}[K, P] + \frac{1}{12}[K - P, [K, P]] + \langle\text{more terms}\rangle\right)$$

Consider

$$\exp(\tau K)\exp(\tau P) = \exp[\tau(K + P) + \langle\text{error}\rangle].$$

This operator is evolving the system under the under the Hamiltonian

$$H_K + H_1 + H_{\text{err}}.$$

where H_{err} is of order τ^2, so this is a first order integration scheme.
We can use this has a tool to calculate surfaces of section using an **Area preserving map**.

The standard map:

$$\begin{aligned} x_1 &= x_0 + a\sin(x_0 + y_0) && \mod(2\pi) \\ y_1 &= x_0 + y_0 && \mod(2\pi) \end{aligned}$$

demonstrates many charactaristics of a general system with 2 DOF.
Characteristics:

- Hamiltonian system with two degrees of freedom.
- The only integral is the Hamiltonian, (or the Jacobi integral).
- Orbits lie on a 3-d hypersurface.
- Second integral confines the orbit to a 2-d torus.
- Quick transition to chaos when last KAM surface breaks

With higher dimensions:

- Orbits do not divide phase space.
- Possibility of Arnold diffusion or other long term evolution.

Applications

- Resonance overlap around planets: ejection of bands of width $\Delta a \propto \mu^{2/7}$.
- Origin of gaps in asteroid belt.

Secular Perturbation Theory

For very long times, consider only the part of H_1 that is independent of the mean anomalies, that is $n_1 = 0$ and $n_3 = 0$:

$$H_{10} = -\frac{Gm_J}{a_J} \sum_{n_2} h_{0n_20}(e, e')e^{in_2\omega_2}.$$

Note that this is the same as averaging over the orbits of Jupiter and the asteroid.

$$H_{10} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} H_1 d(\nu_J t) d(\nu_a t).$$

Now consider several interacting planets, a canonical set of variables to use are:

$$x_p = 2a_p\sqrt{m_p\nu_p}\sin(\frac{e_p}{2})e^{-i\varpi_p}$$

and

$$y_p = 2a_p\sqrt{m_p\nu_p}\sin(\frac{e_p}{2})e^{i\varpi_p}$$

These are small, so expand H_{10} as a power series in x and y to second order: be:

$$\frac{dx_p}{dt} = \frac{\partial H_{10}}{\partial y_p} = \sum_{p'} a_{pp'}x_{p'}$$

and

$$\frac{dy_p}{dt} = \frac{\partial H_{10}}{\partial x_p} = \sum_{p'} a_{pp'}y_{p'}.$$

This has the solution:

$$x_p = \sum_{p'} c_{pp'}e^{s_{p'}t+\delta_{p'}}$$

and

$$y_p = \sum_{p'} c_{pp'}e^{-s_{p'}t+\delta_{p'}},$$

where s_p are the eigenvalues of the matrix $a_{pp'}$, c is the matrix of eigenvectors and δ is determined by the initial conditions.
The **mean motion resonances** cause trouble averaging the Hamiltonian.
Secular resonances occur when rational combinations of the secular normal modes are near zero.
Most research in the area tends to use the variables

$$\begin{aligned} h &= e \sin \varpi \\ k &= e \cos \varpi \\ p &= \sin \frac{i}{2} \sin \Omega \\ q &= \sin \frac{i}{2} \cos \Omega \end{aligned}$$

to describe phase space instead of a canonical set of variables.