Chapter 2 : Vector Spaces and Bases

We begin with a definition:

A <u>vector space</u> is a set V equipped with two operations.

1) Addition: $\vec{v}, \vec{w} \in V \Rightarrow \vec{v} + \vec{w} \in V$
2) Scalar Multiplication: $\vec{v} \in V,\ c \in \mathbb{R} \Rightarrow c\vec{v} \in V$.

... these are subject to the following axioms:

a) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

b) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

c) $\vec{v} + 0 = \vec{v}$

d) For each $\vec{v} \in V$
$$\vec{v} + (-\vec{v}) = 0$$

e) $(c + d)\vec{v} = c\vec{v} + d\vec{v}$ & $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$

f) $c(d\vec{v}) = (cd)\vec{v}$

g) $1\vec{v} = \vec{v}$

<u>Remark</u>: This is an abstraction of $\mathbb{R}^n$, the set of all n-dimensional vectors.

Ex Let $V = \mathcal{M}_{n \times m}$, the set of all $n \times m$ matrices. We've already shown this is a vector space.

Ex

$$\mathbb{P}_n = \{ p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 x^0 \}.$$

The set of polynomials of degree $\leq n$.

Ex Let $I \subset \mathbb{R}$ be an interval, say $[a, b]$.

$$\mathcal{F} = \{ \text{All functions } f : I \to \mathbb{R} \}.$$

Let's verify the properties of a vector space for $\mathcal{F}$.

$$\begin{cases} f, g \in \mathcal{F} \\ c, d \in \mathbb{R} \end{cases}$$

Need addition.

$h(x) = f(x) + g(x)$ ← real-valued function defined on I ✓

Need scalar mult.

$h(x) = c f(x)$ ← real-valued function defined on I ✓

Properties:

a) It is clear

$$h(x) = f(x) + g(x) = g(x) + f(x) \checkmark$$

b) $f(x) + (g(x) + h(x)) = (f(x) + g(x)) + h(x) \checkmark$

c) The zero element is the "zero function"

$$f(x) + 0 = f(x)$$

d) $-f(x) \in \mathcal{F}$ so that

$$f(x) + (-f(x)) = 0.$$

e) $(c+d)f(x) = cf(x) + df(x)$

$c(f(x) + g(x)) = cf(x) + cg(x)$

f) $c(df(x)) = (cd)f(x)$

g) $1 f(x) = f(x)$.

Subspaces:

Definition

A subspace of a vector space V is a subset $W \subset V$ such that W is itself a vector space with operations (addition & scalar multiplication) inherited from V.

Proposition:

A subset $W \subset V$, V being a vector space is a subspace if and only if

a) For every $\vec{v}, \vec{w} \in W$, $\vec{v} + \vec{w} \in W$,

b) For every $\vec{v} \in W$, $c\vec{v} \in W$.

Remark: If we already know V is a vector space and $W \subset V$ we don't have to check all the axioms for W. We just verify that

a) W is closed under addition.

b) W is closed under scalar multiplication.

Remark: a, b can be checked in one step by showing if $c, d \in \mathbb{R}$ and $\vec{v}, \vec{w} \in W$ then

$$c\vec{v} + d\vec{w} \in W.$$

Ex Let $V = \mathbb{R}^3$ and we look for subspaces.

- Trivial sub space $W = \{0\}$

Let $c, d \in \mathbb{R}$, $\vec{v}, \vec{w} \in W$

$$c\vec{v} + d\vec{w} = c0 + d0 = (c+d)(0) = 0 \in W \checkmark$$

b) $W = \mathbb{R}^3$ (possibly the other trivial subspace)

c) $W = \left\{ v \in V : v = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \right\}$
$x \in \mathbb{R}$

Let $c, d \in \mathbb{R}$ $\quad \vec{v} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}$

then $c\vec{v} + d\vec{w} = c\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} cx + dy \\ 0 \\ 0 \end{pmatrix}.$$

$cx + dy \in \mathbb{R} \Rightarrow \begin{pmatrix} cx+dy \\ 0 \\ 0 \end{pmatrix} \in W.$ ✓

d) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 0 \right\}$

Let

$\vec{v} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad \begin{cases} x_1 + y_1 + z_1 = 0 \\ x_2 + y_2 + z_2 = 0 \end{cases}$

$c, d \in \mathbb{R}$.

$c\vec{v} + d\vec{w} = \begin{pmatrix} cx_1 + dx_2 \\ cy_1 + dy_2 \\ cz_1 + dz_2 \end{pmatrix}$. We need the sum condition:

$$cx_1 + dx_2 + cy_1 + dy_2 + cz_1 + dz_2$$
$$= c(\overbrace{x_1 + y_1 + z_1}^{0}) + d(\overbrace{x_2 + y_2 + z_2}^{0}) = 0$$ ✓

What isn't a subspace?:

a) $W = \left\{ \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} : x, y \in \mathbb{R} \right\}$.

Why?

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \notin W.$$

OR

$$0 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin W$$ (a subspace must contain the zero element).

b) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x \geq 0, y \geq 0, z \geq 0 \right\}$

contains zero element. What goes wrong?

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in W, \quad -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \notin W!$$

In $\mathbb{R}^3$ we have four different types of subspaces:

a) $W = \mathbb{R}^3$

b) A plane passing through the origin

c) A line passing through the origin

d) $\{0\}$

Span and Linear Independence

Let $v_1, \dots, v_K$ belong to a vector space V.
A sum of the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_K = \sum_{i=1}^{k} c_i\vec{v}_i .$$

is known as a <u>linear</u> <u>combination</u> of

$\vec{v}_1, \dots, \vec{v}_K$.

Their <u>span</u> is the subset

$W = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\} \subset V$ consisting of

all possible linear combinations.

- $\text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is a subspace of V.

$$\vec{v} = \sum_{i=1}^{k} c_i\vec{v}_i , \quad \vec{w} = \sum_{i=1}^{k} d_i\vec{v}_i$$

$$a\vec{v} + b\vec{w} = a\sum_{i=1}^{k} c_i\vec{v}_i + b\sum_{i=1}^{k} d_i\vec{v}_i$$

$$= \sum_{i=1}^{k} ac_i\vec{v}_i + \sum_{i=1}^{k} bd_i\vec{v}_i = \sum_{i=1}^{k} \underbrace{(ac_i + bd_i)}_{e_i}\vec{v}_i$$

$$= \sum_{i=1}^{k} e_i\vec{v}_i \in \text{span}\{\vec{v}_1, \dots, \vec{v}_k\} . \checkmark$$

Let's look at $\mathbb{R}^3$. Pick two vectors:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

By the previous arguement about the types of subspaces of $\mathbb{R}^3$ that exist, we know

$W = \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$ must be either a plane or a line.

It is not too hard to see W is a plane.

Q: How do we find if $\vec{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \in W$?

Find constants c_1, c_2 (if they exist) such that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$ (weighted sum)

This can be expressed in matrix form:

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

Apply GE:

$$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{array}\right). \implies \begin{array}{l} c_2 = -2 \\ c_1 = 2 \end{array}$$

Luckily, the last two equations are redundant.

$$\implies \vec{v}_3 \in W.$$

Change $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{array}\right) \leftarrow \text{inconsistent} \quad c_2 = 0 \ \& \ c_2 = -1$$

$\Rightarrow \vec{v}_3 \notin W.$

<u>Definition</u>: Vectors $\vec{v}_1, \dots, \vec{v}_k \in V$ are said to be linearly dependent if there exists scalars $c_1, \dots, c_k$, NOT ALL ZERO, such that

$$c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k = 0.$$

Vectors that are not linearly dependent are called linearly independent.

How do we check if vectors are linearly independent?

<u>Theorem</u>: Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ and let

$$A = (\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_k)$$ be the $n \times k$

matrix. Then

1) The vectors $\vec{v}_1, \dots, \vec{v}_k$ are LD if and only if $Ac = 0$ has a non-zero solution.

2) The vectors $\vec{v}_1, \dots, \vec{v}_k$ are LI if and only if $Ac = 0$ has only the trivial solution, $c = 0$.

3) A vector $\vec{b}$ lies in the span of $\vec{v}_1, \dots, \vec{v}_k$ if and only if the linear system

$$A\vec{c} = \vec{b}$$

is compatible (has at least one solution).

<u>Ex</u> Are the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

LD or LI? (Linearly dependent or linearly independent).

Construct the matrix (no RHS since it is zero)

$$A = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 0 & -4 & 2 \\ -1 & 4 & 6 & 3 \end{pmatrix} \xrightarrow[r_1 + r_3 \to r_3]{-2r_1 + r_2 \to r_2} \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & -6 & -6 & -6 \\ 0 & 7 & 7 & 7 \end{pmatrix}$$

$$\tfrac{7}{6} r_2 + r_3 \to r_3 \quad \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & -6 & -6 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ (row echelon form)

$Ac = 0$:

$$\to \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$c_1 + 3c_2 + c_3 + 4c_4 = 0$$
$$c_2 + c_3 + c_4 = 0$$

$$\vec{c} = \begin{pmatrix} -3c_2 - c_3 - 4c_4 \\ -c_3 - c_4 \\ c_3 \\ c_4 \end{pmatrix}$$ ← many solutions. LD!

Facts:

- Any collection of $k > n$ vectors in $\mathbb{R}^n$ is linearly dependent.

- A set of k vectors in $\mathbb{R}^n$ is linearly independent if and only if the corresponding matrix A has rank k. (This requires $k \leq n$).

[rank — number of non-zero pivots.

A – has n rows – maximum of n pivots.
– maximum rank of n.]

- A set of k vectors spans $\mathbb{R}^n$ iff and only if their $n \times k$ matrix has rank n.

(This requires $k \geq n$).

Note the difference between Linear independence and spanning:

LI: Does each vector encode information not encoded by any other vector?

Span: Do all of the vectors encode the whole space even if some vectors are redundant?

Basis and Dimension

Definition A basis of a vector space V is a finite collection of elements $v_1, \dots, v_n \in V$ such that

a) spans V, and

b) is linearly independent.

Ex

$$\left\{ \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is}$$

a basis of $\mathbb{R}^3$. (A basis is a set).

Important theorem:

- Every basis of $\mathbb{R}^n$ consists of exactly n vectors.
- A set of n vectors $\vec{v}_1, \dots, \vec{v}_n$ is a basis of $\mathbb{R}^n$ if and only if the matrix $A = (\vec{v}_1 \dots \vec{v}_n)$ is non-singular (rank n).

A bunch more facts:

- Suppose V has a basis $\vec{v}_1, \dots, \vec{v}_n$ then every other basis has the same number of elements. This number is called the dimension.

Suppose V is an n-dimensional vector space.

a) Every set of more than n elements is linearly dependent.

Why? Too many elements - can't be a basis.

If $W = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\} \subset V$, $k > n$.

By definition $\vec{v}_1, \dots, \vec{v}_k$ spans W

$$\dim W \le \dim V = n,$$

but it is not a basis. (must have exactly n elements)

If $\vec{v}_1, \dots, \vec{v}_k$ fails to be a basis it must either not span or not be linearly independent.

But it spans W so it must fail the linear independence!

b) No set of less than n elements spans V.

Why?

Assume $\vec{v}_1, \dots, \vec{v}_n$ is a basis for V.

Also Assume $\vec{w}_1, \dots, \vec{w}_{n-1}$ spans V.

Then each $\vec{v}_j$ can be written as a sum of $\vec{w}_1, \dots, \vec{w}_{n-1}$.

$\Rightarrow \dim V \le n-1$, which is not true!

c) A set of n elements forms a basis if and only if it spans V.

d) A set of n elements forms a basis if and only if it is linearly independent.

Assume $\vec{v}_1, \dots, \vec{v}_n$ spans V. What can we say about dim V?

$$\dim V \leq n \;!$$

Fundamental Matrix Subspaces

Definitions:

- The range of a matrix A (rng A) is the span of its columns, also called the column space.
- The kernel of a matrix A (ker A) is the set of vectors

$$\ker A = \{\vec{z} : A\vec{z} = 0\}.$$

If A is m×n. for $A\vec{v}$ to make sense $\vec{v} \in \mathbb{R}^n \leftarrow (n\times 1)$

$$\underset{m\times n}{A}\;\underset{n\times 1}{\vec{v}} = \underset{m\times 1}{\vec{b}} \implies \begin{array}{l} \text{rng } A \subset \mathbb{R}^m \\ \ker A \subset \mathbb{R}^n \end{array}$$

Let us verify that rng A and ker A are indeed subspaces.

let $\vec{x}, \vec{y} \in \text{rng } A \subset \mathbb{R}^m$, check that $c\vec{x} + d\vec{y} \in \text{rng } A$.

$$\Rightarrow \underset{\in \mathbb{R}^n}{\vec{v}, \vec{w}} \quad \vec{x} = A\vec{v}, \ \vec{y} = A\vec{w}$$

Consider

$$\underset{\in \mathbb{R}^n}{A(c\vec{v} + d\vec{w})} = A(c\vec{v}) + A(d\vec{w}) = \underset{\text{This is in the range!}}{c\vec{x} + d\vec{y}}$$

let $\vec{v}, \vec{w} \in \ker A \subset \mathbb{R}^n$, check that $c\vec{v} + d\vec{w} \in \ker A$

$$A(c\vec{v} + d\vec{w}) = c\cancelto{0}{A\vec{v}} + d\cancelto{0}{A\vec{w}} = 0.$$

this is in the kernel!

<u>Ex</u> Compute the kernel of

$$A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & -1 & -4 \\ 3 & -5 & -1 & -1 \end{pmatrix}.$$

Q: Is there a kernel? $A : \mathbb{R}^4 \to \mathbb{R}^3$ (lose a dimension, must be a kernel).

$A = LU$, $\quad U = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & -1 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$x_1 - 2x_2 + 3x_4 = 0 \quad \Rightarrow \quad x_1 = +2x_2 - 3x_4$$

$$x_2 - x_3 - 10x_4 = 0 \quad \Rightarrow \quad x_2 = x_3 + 10x_4$$

$$x_1 = 2(x_3 + 10x_4) - 3x_4$$

$$x_1 = 2x_3 + 17x_4$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_3 + 17x_4 \\ x_3 + 10x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 17 \\ 10 \\ 0 \\ 1 \end{pmatrix}$$

How do we write the kernel?

$$\operatorname{Ker} A = \left\{ x_3 \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 17 \\ 10 \\ 0 \\ 1 \end{pmatrix} : x_3, x_4 \in \mathbb{R} \right\}$$

OR

$$\operatorname{Ker} A = \operatorname{span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 17 \\ 10 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Now compute the range:

$$A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & -1 & -4 \\ 3 & -5 & -1 & -1 \end{pmatrix} \xmapsto{LU} \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & -1 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If I have time later, I will prove this result but to find the range (column space), first look at what columns of $\mathcal{U}$ have a pivot.

- Columns 1 & 2

Then the same colums of A span the range.

$$\operatorname{ran} A = \operatorname{rng} A = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ -5 \end{pmatrix} \right\}.$$

<u>Theorem</u> The linear system $A\vec{x} = \vec{b}$ has a solution $\vec{x}^*$ if and only if $\vec{b}$ lies in the range of A.

If this occurs, then $\vec{x}$ is a solution to the linear system if and only if

$$\vec{x} = \vec{x}^* + \vec{z}$$

where $\vec{z} \in \ker A$.

<u>Proof</u>: The statement $A\vec{x} = \vec{b}$ implies that $\vec{b}$ must be in the range of A.

1) $A\vec{x} = A(\vec{x}^* + \vec{z}) = A\vec{x}^* + \cancel{A\vec{z}}^{\,0} = A\vec{x}^* = \vec{b}$ ✓

2) Let $\vec{x}_1$ and $\vec{x}_2$ be any two solutions:

then

$$A(\vec{x}_1 - \vec{x}_2) = \vec{b} - \vec{b} = 0 \Rightarrow \vec{x}_1 - \vec{x}_2 \in \ker A$$

Let $\vec{x}_1 = \vec{x}^*$. Any other solution $\vec{x}_2$ must differ from $\vec{x}^*$ by an element of the kernel:

$$\vec{x}_2 - \vec{x}^* = \vec{z} \in \ker A.$$

$$\Rightarrow \vec{x}_2 = \vec{x}^* + \vec{z} \quad \checkmark.$$

<u>Remark</u>: To find the most general solution we need to find $\underline{\underline{a}}$ solution of $A\vec{x} = \vec{b}$ and then characterize the kernel of A. Of course, if such a solution exists.

<u>Ex</u> Consider

$$A\vec{x} = \vec{b} \quad \text{when} \quad a) \ \vec{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad b) \ \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{pmatrix}.$$

Find all solutions $\vec{x}$ if they exist.

a)
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 1 & - & 3 & 0 \end{array}\right)$$

$-r_1 + r_3 \to r_3$
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array}\right)$$

$2r_2 + r_3 \to r_3$
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \leftarrow \text{compatible.}$$

- Find a solution:

$$\vec{x}^* = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

- Charaterize kernel: zero right-hand side:

$$x - z = 0$$
$$y - 2z = 0$$

$$\vec{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1z \\ 2z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\ker A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

$\Rightarrow$ Every solution is of the form:

$$\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad c \in \mathbb{R}.$$

b) $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -2 & 3 & 1 \end{array}\right)$

$\xrightarrow{GE} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right) \rightarrow$ incompatible, no solution.

The Superposition Principle

Suppose we know solutions $\vec{x}_1^*, \ldots, \vec{x}^*_k$ to each of the systems

$$A\vec{x} = \vec{b}_1, \quad \ldots, \quad A\vec{x} = \vec{b}_k. \quad (\text{implying } \vec{b}_j \in \operatorname{rng} A).$$

Then a solution of

$$A\vec{x} = c_1\vec{b}_1 + \ldots + c_k\vec{b}_k \quad \text{is}$$

$$\vec{x}^* = c_1\vec{x}_1^* + \ldots + c_k\vec{x}_k^*,$$

and the most general solution is

$$\vec{x} = c_1\vec{x}_1 + \ldots + c_k\vec{x}_k^* + \vec{z}$$

where $\vec{z} \in \ker A$.

Adjoint Systems, Cokernel and Corange.

The adjoint to a linear system $A\vec{x} = \vec{b}$ of m equations in n unknowns is the linear system

$$A^T y = f$$

consisting of n equations in m unknowns.

Definition The corange of an $m \times n$ matrix is the range of its transpose,

$$\operatorname{corng} A = \operatorname{rng} A^T = \{ A^T y : y \in \mathbb{R}^m \} \subset \mathbb{R}^n.$$

The cokernel, or left null space of a matrix A is the kernel of its transpose:

$$\operatorname{coker} A = \ker A^T = \{ w \in \mathbb{R}^m : A^T w = 0 \}.$$

$$\begin{cases} \operatorname{rng} A \text{ - column space - subspace of } \mathbb{R}^m \text{ spanned by the colums of } A \\ \operatorname{corng} A \text{ - row space - subspace of } \mathbb{R}^n \text{ spanned by the rows of } A. \end{cases}$$

The Fundamental Theorem of Linear Algebra

Let A be an $m \times n$ matrix of rank r. Then

$$\dim \operatorname{corng} A = \dim \operatorname{rng} A = \operatorname{rank} A = \operatorname{rank} A^T = r$$

$$\dim \ker A = n - r$$

$$\dim \operatorname{coker} A = m - r$$

How to remember / interpret this?

- We think of A as a transformation of the space $\mathbb{R}^n$ to $\mathbb{R}^m$

$$A : \mathbb{R}^n \mapsto \mathbb{R}^m.$$

- The rank is a measure of how much of the information in the domain is preserved to the range.

- A dimension in the domain must be either mapped to zero or something "useful", r is the dimension of what is "useful."

$$n \to r \quad \text{useful} \leftarrow \text{rank}$$

$$n \to n-r \quad \text{zero} \leftarrow \text{kernel.}$$

$$\dim \operatorname{rng} A + \dim \ker A = n \qquad A : \mathbb{R}^n \to \mathbb{R}^m$$

$$\begin{cases} \dim \operatorname{rng} A^T + \dim \ker A^T = m & A^T : \mathbb{R}^m \to \mathbb{R}^n \\ \dim \operatorname{corng} A + \dim \operatorname{coker} A = m & \end{cases}$$