Model IV: Spatially Explicit Harvest Scheduling with Difference Equations

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Abstract:

Spatially explicit harvest scheduling models optimize the layout of harvest treatments to best meet management objectives such as revenue maximization subject to a variety of economic and environmental constraints. A few exceptions aside, the mixed-integer programming core of every exact model in the literature requires one decision variable for every applicable prescription for a management unit. The only alternative to this "brute-force" method has been a network approach that tracks the management pathways of each unit over time via four sets of binary variables. Named after their linear programming-based aspatial predecessors, Model I and II, along with Model III, which has no spatial implementation, each of these models rely on static volume and revenue coefficients that must be calculated pre-optimization. We propose a fundamentally different approach, Model IV that defines stand volumes and revenues as variables and uses difference equations and Boolean algebra to transition forest units from one planning period to the next. We show via three sets of computational experiments that the new model is a computationally promising alternative to Models I and II.

Keywords: forestry; harvest scheduling; integer programming; spatial optimization

1. Introduction

The goal of this paper is to introduce a new integer programming model, Model IV, for spatial forest harvest scheduling. We show that the new model is compatible with three of the existing techniques that can capture maximum harvest opening size restrictions and that it can handle intermediate treatment decisions. We also provide empirical evidence of the favorable computational performance of Model IV relative to that of the benchmarks models (Models I-II).

Forest harvest scheduling models optimize the spatiotemporal layout of harvests to best meet management objectives such as revenue maximization or carbon sequestration subject to environmental, logistical or budgetary constraints. Due to the combinatorial complexity of assigning harvests to management units (contiguous groups of trees with similar silvicultural and operational attributes) across large areas and over long time horizons, harvest scheduling problems are typically cast as optimization models. The first linear programming (LP) models were introduced in the 1960s by Curtis (1962), Loucks (1964) and Kidd et al. (1966). These models were *aspatial* in that they only calculated the areas to cut from so called *analysis area*s in given time periods to maximize timber revenues (Johnson and Scheurman 1977). Analysis areas are parts of the forest that share key silvicultural characteristics important for management such as forest type, site class or initial age. Subsequent LPs can be classified in three categories, the first two of which are from Johnson and Scheurman (1977). Model I, first suggested by Kidd et al. (1966), is a "brute-force" method that requires the definition of one continuous decision variable for each analysis area and planning period representing the harvest area. Model II, in contrast, is a network formulation, first described by Nautiyal and Pearse (1967), that uses four sets of variables to track the management pathways for each analysis area (Fig. 1). The third LP-based model, Model III (Berck 1976, Johansson and Löfgren (1985), Gunn and Rai (1987), McDill (1989), also tracks forest areas by analysis area, planning period and age-class based on the age-classes and harvest decisions in the previous time period. The model simply multiplies the area harvested in each age-class and analysis area by a static volume coefficient to determine timber yield.

A common shortcoming of these LP-based harvest scheduling models was that they could not account for spatial concerns such as habitat fragmentation

(Franklin and Forman 1987) or harvest opening size restrictions (a.k.a., green-up or clear-cut size constraints) that are often present in forest regulations (e.g., U.S. Congress 1976). That said, the Model I construct is well suited for spatial optimization as analysis areas can easily be disaggregated into stands. While Bare and Norman's (1969) very early integer programming model did treat each forest stand as an indivisible decision unit, harvest opening size constraints were not added as such regulations had not been introduced until the mid 1970s.

The earliest models to respond to clear-cut size regulations were Kirby et al.'s (1980) and O'Hara et al.'s (1989) mixed integer programs (MIPs) that assumed that any adjacent pair of units in the forest had a combined area greater than the maximum opening size. A similar assumption was made by Snyder and ReVelle's (1996; 1997) network model, the only spatial Model II construct. All models using the above adjacency assumption are commonly referred to as Unit Restriction Models (URM, Murray 1999) in forestry, and are an instance of the Node Packing Problem, a.k.a. the Vertex Packing or the Maximal Weight Stable Set Problem in operations research. The Area Restriction Model (ARM, Murray 1999) is more general in that it allows groups of contiguous management units to be harvested concurrently as long as their combined area is below the maximum allowable clear-cut size. Apart from the numerous heuristics that have been proposed for the ARM (e.g., Lockwood and Moore 1993, Caro et al. 2003, Richards and Gunn 2003), so far only a few exact integer programming approaches have been documented in the refereed literature. These include McDill et al.'s (2002) Path Formulation, McDill et al.'s (2002) and Goycoolea et al.'s (2005) Cluster Packing Model, Gunn and Richards' (2005) Stand-Centred approach and Constantino et al.'s (2008) Bucket Formulation. While all of these models were introduced using the Model I form, we show that they are also compatible with the proposed Model IV construct.

Model IV is fundamentally different from the spatial versions of Model I and II in that it uses difference equations and Boolean algebra to transition the states of management units through time. While the use of difference equations to model stand development is not entirely new (e.g., Garcia 1979), this article presents the first linearized construct imbedded in a spatial harvest scheduling model. In Models I and II, unit attributes such as expected merchantable yield or harvest revenues were hard-wired in the programs a priori as coefficients. In contrast, the proposed model treats these quantities as variables and calculates them only during optimization. This affords a higher level of flexibility in model specifications than what had previously been possible. Inspired by Hof et al.'s (1995) pest management model that calculates the population of a pest in consecutive time periods using the derivative of an exponential invasion function, Model IV incorporates a sigmoid growth function to calculate timber volumes (or revenues). Sigmoid growth curves, which are standard in forestry (e.g., Pienaar and Turnbull 1973), have two distinct segments: an exponential and a tapering segment separated by an inflection point (Fig. 2, 3). To formulate Model IV as a linear program, we derive the rate of change between discrete time periods for both the exponential and tapering segments of the curve and use these linear expressions to describe the relations between the volume variables of Model IV. A binary switch is in place to track whether the value of a particular volume variable is below or above the inflection point. This switch allows the automation of the decision whether the difference equation of the exponential or the tapering segment should be used to calculate the volume of a management unit as a function of its volume in the previous period. If the unit is cut by the model in a given period, the harvest volume in the subsequent period is reset to zero, or to a small value depending on the length of the planning period. Model IV uses Boolean algebra to make sure that the value of the volume (or revenue) variables are properly set during optimization. A critical feature of this approach is that both the difference equations and the Boolean constructs are linear, enabling the use of efficient integer programming solvers.

2. Methods

After describing the key modeling assumptions, we provide a rigorous mathematical definition for Model IV. The benchmark models, Models I and II are described in Appendix A. Model III was not included in the tests, as it does not have a documented spatial version. The section ends with a description of experiments that were used to study the computational performance of Model IV.

2.1 Assumptions

For simplicity, the only silvicultural activity that we considered in the experiments was even-aged management with clear cutting. The incorporation of intermediate treatments such as thinning, pruning or fertilization is described in Appendix C. We assumed that clear-cuts occurred only in the midpoint of the planning periods. Whenever a management unit was cut, it was immediately replanted with the same species incurring a fixed per-hectare cost. As for timber prices and management expenses, we assumed they were constant in real terms. We also assumed that the goal of management was to maximize discounted net timber revenues over the entire planning horizon. Harvest flow constraints are in place to provide for a relatively smooth flow of harvest volume over time.

2.2 Benchmark Models

We used Models I and II for benchmarking Model IV. See Appendix A for the detailed mathematical formulations.

2.3 Model IV

We present Model IV in slightly different way than the benchmark models in the Appendix. We first define the difference equations that characterize the growth function, which is at the core of the new model, and then show how this function can be imbedded in an integer program using only linear inequalities. This imbedding plus the usual side constraints, such as harvest volume flow and clear-cut size restrictions, constitute Model IV. Given a set of management units S, indexed by *s*, and a set of planning periods T, indexed by *t*, we define binary variable x_{st} as the decision to harvest unit s in period t: $x_{st} = 1$ if unit *s* is to be cut in period *t*, 0 otherwise. Finally, we introduce v_{st} as an accounting variable that represents the unit area volume (or harvest revenue) in unit *s* in period *t*. We calculate the values of v_{st} based on the harvest decision and volume in period *t*-1 by imbedding the following function in the decision model:

$$v_{st} = \begin{cases} \phi_{\min}^{s}, & \text{if } x_{s,t-1} = 1; \\ v_{s,t-1}(1+\gamma_{\exp}^{s}), & \text{if } v_{s,t-1} < \beta^{s} \text{ and } x_{s,t-1} = 0; \\ v_{s,t-1}(1-\gamma_{\text{taper}}^{s}) + \phi_{\max}^{s}\gamma_{\text{taper}}^{s}, & \text{if } v_{s,t-1} \geq \beta^{s} \text{ and } x_{s,t-1} = 0; \end{cases} \qquad (1)$$

where parameter $\phi_{\min}^{s}$ denotes the unit area volume in unit *s* one period after it was cut, $\phi_{\max}^{s}$ denotes the asymptote of the growth curve, the maximum attainable volume of unit *s*, β^{s} the inflection point in terms of unit area volume separating the exponential segment of the curve with rate $\gamma_{\exp}^{s}$ from the tapering segment that has a rate $\gamma_{\text{taper}}^{s}$ (Fig. 2). To imbed Function (1) in Model IV as a set of linear inequalities, we first force the volume of unit s in period t+1 to be equal to $\phi_{\min}^{s}$ if it is cut in period t (i.e., if $x_{st}=1$)

$$v_{s,t+1} - \phi_{\max}^{s} \cdot \left(1 - x_{st}\right) \le \phi_{\min}^{s}, \qquad \forall s \in S, t \le T, \tag{2}$$

$$v_{s,t+1} \ge \phi_{\min}^{s}, \qquad \forall s \in S, t \le T. \tag{3}$$

Constraints (2)-(3) ensure that $v_{s,t+1} = \phi_{\min}^{s}$ if x_{st}=1. Otherwise, if x_{st}=0, neither constraint is binding with respect to $v_{s,t+1}$ because $\phi_{\min}^{s} \le \phi_{\max}^{s}$. If unit s is not cut in period t, then we need to determine the volume of unit s in period t+1 given its volume in period t. Since the functional relation between $v_{s,t+1}$ and v_{st} depends on whether $v_{st} < \beta^{s}$ (Eq. 1), we define a binary indicator variable y_{st} that tells the model whether the volume is above or below the inflection point in period t. If $v_{st} < \beta^{s}$, then $y_{st} = 0$, 1 otherwise. The behavior of y_{st} is defined by the following pair of inequalities:

$$v_{st} - \beta^{s} \le \phi_{\max}^{s} \cdot y_{st}, \qquad \forall s \in S, t \le T; \tag{4}$$

$$v_{st} \ge \beta^{s} \cdot y_{st}, \qquad \forall s \in S, t \le T. \tag{5}$$

If $v_{st} > \beta^{s}$, then, in order to satisfy constraint (4), y_{st} must be equal to 1. If $v_{st} < \beta^{s}$, constraint (5) forces y_{st} to be zero. If $v_{st} = \beta^{s}$, neither (4) nor (5) is binding with respect to the value of y_{st}. If $v_{st} > \beta^{s}$, the volume in period t+1 will be calculated using the rate of growth associated with the curve above the inflection point and the asymptote $\phi_{\max}^{s}$:

$$v_{s,t+1} - \phi_{\max}^{s} \cdot \left(1 - y_{st}\right) \le \left(1 - \gamma_{\text{taper}}^{s}\right) \cdot v_{st} + \phi_{\max}^{s} \cdot \gamma_{\text{taper}}^{s}, \qquad \forall s \in S, t \le T-1, \tag{6}$$

$$v_{s,t+1} + \phi_{\max}^{s} \cdot \left(1 - y_{st} + x_{st}\right) \ge \left(1 - \gamma_{\text{taper}}^{s}\right) \cdot v_{st} + \phi_{\max}^{s} \cdot \gamma_{\text{taper}}^{s}, \qquad \forall s \in S, t \le T-1. \tag{7}$$

If the volume of stand s in time t is below the inflection point, i.e., $y_{st} = 0$, Constraints (6)-(7) will always hold because $v_{s,t+1} \le \phi_{\max}^{s}$ by definition and because

$0 \le \gamma^s_{\text{taper}} \le 1$. Otherwise, i.e., if $y_{st} = 1$, $v_{s,t+1} = v_{st}(1-\gamma^s_{\text{taper}}) + \phi^s_{\max}\gamma^s_{\text{taper}}$. Note that if unit *s* is cut in time *t* ($x_{st} = 1$), then $v_{s,t+1} = \phi^s_{\min}$ by Constraints (2)-(3). At $x_{st} = 1$, Constraints (4)-(5) are non-binding regardless of what y_{st} is. Finally, if v_{st} is below the inflection point and no harvesting occurred ($x_{st} = 0$), the exponential rate is used to calculate the volume:

$$v_{s,t+1} - \phi^s_{\max} \cdot y_{st} \le \left(1+\gamma^s_{\exp}\right) \cdot v_{st}, \qquad \forall s \in S, t \le T-1, \tag{8}$$

$$v_{s,t+1} + \phi^s_{\max}(y_{st} + x_{st}) \ge \left(1+\gamma^s_{\exp}\right) \cdot v_{st}, \qquad \forall s \in S, t \le T-1. \tag{9}$$

If unit *s* is cut in period *t* ($x_{st} = 0$) and v_{st} is below the inflection point, then $v_{s,t+1} = \phi^s_{\min}$ by Constraints (2)-(3) and Constraints (6)-(7) are non-binding. Constraints (6)-(7) are also non-binding if the volume in time *t* is greater than the inflection point ($v_{st} > \beta^s$) and unit *s* was not cut in period *t* ($x_{st} = 0$). In this case, Constraints (4)-(5) become active and Function (1) is enforced. ■

The calculation of harvest volumes requires a nonlinear cross-product term: $v_{st}x_{st}$. We linearize this expression in order to avoid the computational difficulties that are associated with solving non-linear programs. To this end, we replace $v_{st}x_{st}$ with a new variable, Ω_{st}, that takes the value of v_{st} if $x_{st} = 1$, 0 otherwise. Variable Ω_{st} denotes the harvest volume in unit s in period t. To ensure that $\Omega_{st} = v_{st}$ iff $x_{st} = 1$, we add the following inequalities:

$$\Omega_{st} \le \phi^s_{\max} \cdot x_{st}, \qquad \forall s \in S, t \le T; \tag{10}$$

$$\Omega_{st} \le v_{st}, \qquad \forall s \in S, t \le T; \tag{11}$$

$$v_{st} - \Omega_{st} \le \phi^s_{\max} \cdot \left(1 - x_{st}\right), \qquad \forall s \in S, t \le T; \tag{12}$$

If $x_{st} = 0$, then Constraint (10) will force Ω_{st} to be 0. Constraints (11)-(12) are non-binding. If on the other hand, $x_{st} = 1$, Constraint (12) will reduce to $\Omega_{st} \ge v_{st}$. This, along with Constraint (11) will force Ω_{st} to be equal to v_{st}. Constraint (10) will be non-binding in this case.

With the linearization of $v_{st}x_{st}$ in place, we can formulate the objective function with c, a_s, e_s and i denoting the wood price per unit volume, the area and the regeneration costs of unit s, and the real discount rate, respectively:

$$\max \sum_{s\in S, t\le T} \left(c \cdot a_s \cdot \Omega_{st} - e_s \cdot x_{st}\right) \cdot \left(1+i\right)^{-t}. \tag{13}$$

Function (13) maximizes the net discounted timber revenues associated with managing a set of units S over T planning periods. Finally, we impose the minimum rotation age of k periods with logical constraint:

$$\sum_{t'=t-k}^{t} x_{st'} \le 1, \qquad \forall s \in S,\ k+1 \le t \le T; \tag{14}$$

and require that the harvest volumes (h_t) for the entire forest in adjacent periods stay between a specific upper ($f_{\max}$) and lower ($f_{\min}$) bound:

$$\sum_{s\in S} a_s \cdot \Omega_{st} = h_t, \qquad \forall t \le T; \tag{15}$$

$$\left(1 - f_{\min}\right) \cdot h_t \le h_{t+1}, \qquad \forall t \le T-1; \tag{16}$$

$$\left(1 + f_{\max}\right) \cdot h_t \ge h_{t+1}, \qquad \forall t \le T-1. \tag{17}$$

Harvest accounting constraint (15) and flow constraints (16)-(17) are analogous to constraints (A3)-(A5) in Model I and constraints (A16-18) in Model II. Constraints (18)-(19) define v_{st} and Ω_{st} as positive real and x_{st} and y_{st} as binary:

$$v_{st}, \Omega_{st} \in \mathbb{R}^+, \qquad \forall s \in S, t \le T; \tag{18}$$

$$x_{st}, \mathrm{y}_{st} \in \{0,1\}, \qquad \forall s \in S, t \le T. \tag{19}$$

Objective function (13) and constraints (2)-(12) and (14)-(19) define Model IV. The five parameters of Function (1) imbedded in Model IV, namely $\phi^s_{\min}$, $\phi^s_{\max}$, $\gamma^s_{\exp}$, γ^s_{taper} and β^s must be fitted to the original data or to the growth function that is to be used prior to optimization. In this study, we used Goal Programming (GP: Charnes and Cooper 1961), which is a standard curve fitting procedure (Williams 1999). For details about this GP, see Appendix B.

2.3.1 Simplified Model IV

Model IV can be greatly simplified if harvest activities can only begin at a volume beyond the inflection point. For example, if the volume at the minimum rotation age is greater than the volume at the inflection point, then only the taper segment of the logistic-like growth curve is needed (curve $\overline{BD}$ in Fig. 2). The volumes prior to the minimum rotation age are inconsequential because the management unit will not be touched during that time. For this reason, a modified growth curve can be used where the volumes prior to the minimum rotation age are set to be equal to the volume at the minimum rotation age. Since constraint set (14) in Model IV prevents the units from being cut prior to the minimum rotation age, this has no ill effects on the behavior of the model. After the minimum rotation age is reached, the volume increases according to the taper segment of the original logistic-like curve (1). The modified curve has fewer parameters to fit as the binary switch (y_{st}), the exponential growth rate ($\gamma^{s}_{\exp}$), and the seed volume parameters ($\phi^{s}_{\min}$) are no longer necessary. This allows for fewer constraints. To imbed the modified curve in Model IV, we let ϕ^{s}_{rot} denote the volume of management unit s at the minimum rotation age. Then, Constraints (2)-(3) can be modified to ensure that the volume in unit s remains constant at ϕ^{s}_{rot} until the minimum rotation age is reached:

$$v_{s,t+1} + \phi^{s}_{\max} \sum_{t'=t-k+1}^{t} x_{st'} \leq \phi^{s}_{\max} + \phi^{s}_{\text{rot}}, \qquad \forall s \in S, t \leq T-1. \tag{20}$$

$$v_{s,t+1} \geq \phi^{s}_{\text{rot}} \qquad \forall s \in S, t \leq T-1. \tag{21}$$

Since we do not need to consider the curve below the inflection point, Constraints (4) and (5) may be eliminated and Constraints (6)-(7) can be simplified to:

$$v_{s,t+1} \leq \left(1-\gamma^{s}_{\text{taper}}\right) \cdot v_{st} + \phi^{s}_{\max} \cdot \gamma^{s}_{\text{taper}}, \qquad \forall s \in S, t \leq T-1. \tag{22}$$

$$v_{s,t+1} + \phi^{s}_{\max} \sum_{t'=t-k+1}^{t} x_{st'} \geq \left(1-\gamma^{s}_{\text{taper}}\right) \cdot v_{st} + \phi^{s}_{\max} \cdot \gamma^{s}_{\text{taper}}, \qquad \forall s \in S, t \leq T-1. \tag{23}$$

Notice that if $\sum_{t'=t-k}^{t} x_{st'} = 1$, then $v_{s,t+1} = \phi^{s}_{rot}$ by Constraints (20)-(21) and Constraints (22)-(23) are non-binding. If on the other hand $\sum_{t'=t-k}^{t} x_{st'} = 0$, then $v_{s,t+1} = \left(1-\gamma^{s}_{\text{taper}}\right) \cdot v_{st} + \phi^{s}_{\max} \cdot \gamma^{s}_{\text{taper}}$ by Constraints (22)-(23).

While the above simplification does lead to smaller problem size, it precludes the preservation of volume information for units that are younger than the minimum rotation age. If such information is necessary, the full Model IV, objective function (13) and constraints (2)-(12) and (14)-(19), needs to be used.

2.3.2 ARM Compatibility of Model IV

In this section, we demonstrate that Model IV is compatible with three of the exact spatial models that are available from the literature: McDill et al.'s (2002) Path, Goycoolea et al.'s (2005) Cluster Packing and Constantino et al.'s (2008) Bucket Formulation.

McDill et al.'s (2002) Path Formulation with Model IV: The Path Formulation requires the *a priori* enumeration of *minimal covers* of management units (or minimally infeasible clusters). After Goycoolea et al. (2005), we let Λ^+ denote this set with C representing one particular element in Λ^+. A cover $C \in \Lambda^+$ is minimal if $\sum_{s\in C} a_s > A_{\max}$ holds, but $\sum_{s\in C\setminus\{l\}} a_s \le A_{\max}$ for any $l \in C$ such that set $C \setminus \{l\}$ is still a connected sub-graph. To enforce maximum harvest opening size restrictions, we add the following inequality:

$$\sum_{s\in C} x_{st} \le |C| -1, \qquad \forall C \in \Lambda^+, t \le T. \tag{24}$$

Since the Path Formulation tracks management units and not clusters of clear-cuts, it cannot be used to enforce average harvest opening size restrictions. These requirements can only be modeled with Cluster Packing (Murray et al. 2004).

Goycoolea et al.'s (2005) Cluster Packing with Model IV: To capture the Cluster Packing approach in Model IV, replacing decision variables x_{st} $\forall s \in S, t \le T$ with cluster variables as in Model II (Appendix A2) is not an option since the volume coefficients are unique to each unit. While the growth functions in Model IV could theoretically be refitted for each feasible cluster, such an effort might be costly due to the potentially enormous number of clusters. To avoid this scenario, we follow the same mapping approach as in Model I (Appendix A1: A8-9). The only difference is that here we map the Model IV decision variables, x_{st}, rather than Model I's prescription variables, x_{st}, to cluster variables $u_{\theta t}$ $\forall \theta \in \Theta, t \le T$:

$$\sum_{s\in\theta} x_{st} \geq |\theta| \cdot u_{\theta t}, \qquad \forall \theta \in \Theta, t \leq T; \tag{25}$$

$$\sum_{s\in\theta} x_{st} - \sum_{s\in A_\theta} x_{st} - u_{\theta t} \leq |\theta| - 1, \qquad \forall \theta \in \Theta, t \leq T; \tag{26}$$

As in (A8)-(A9), Constraint (25) allows, while constraint (26) forces cluster θ to be “declared” cut in period *t* if all the units in the cluster, but none adjacent to it, are cut in period *t*. To avoid double-counting the harvested areas, and to ensure that the maximum and the maximum average harvest opening sizes are never exceeded over the entire planning horizon, we add Model I’s constraints (A10)-(A11) to Model IV. Note that these constraints are identical in both models since they have the same function and use the same cluster variables. For easy reference, the full Model IV ARM with Cluster Packing is function (13) and constraints (2)-(12), (14)-(19), (A10)-(A11), and (25)-(26).

Constantino et al.’s (2008) Bucket Formulation with Model IV: The Bucket Formulation requires the definition of a class of *clear-cuts*, or *buckets* (Goycoolea et al. 2009), say set B, that are initially all empty. The model uses a set of assignment variables whose optimal values determine the composition of the buckets. Let variable $z_{st}^{b} \in \{0,1\}$ represent the decision whether management unit *s* should be assigned to bucket *b* in period *t*: $z_{st}^{b} = 1$ if it is, 0 otherwise. Since there cannot be more clear-cuts in a forest than there are management units, the number of units puts an upper bound on the number of buckets, i.e., $|B| \leq |S|$ which in turn restricts the number of assignment variables that are needed to capture maximum opening size constraints. To imbed the Bucket Formulation in Model IV, decision variables x_{st} $\forall s \in S, t \leq T$ need to be mapped to assignment variables z_{st}^{b} $\forall s \in S, b \in B, t \leq T$. This means that in objective function (13) and in constraints (2)-(3), (7), (9)-(10), (12) and (19), the terms x_{st} and $x_{st'}$ need to be replaced with $\sum_{b} z_{st}^{b}$ and $\sum_{b} z_{st'}^{b}$, respectively. All the other constraints in Model IV, constraints (4)-(6), (8), (11) and (13)-(18) remain intact since they do not carry decision variables. To put a limit of $A_{\max}$ on the total area of management units that can be assigned to the same clear-cut, we add:

$$\sum_{s} a_s z_{st}^b \le A_{\max}, \qquad \forall b \in S, t \le T. \tag{27}$$

Since adjacent clear-cuts can still have a combined area that exceeds $A_{\max}$, Constraint set (27) alone cannot prevent all clear-cut size violations. The Bucket Model makes use of two additional constraint sets, (28)-(29), along with a set of indicator variables to keep the clear-cuts disjoint. The indicator variables, $w_{bt}^{\kappa} \in \mathbb{R}^+, \forall \kappa \in \mathrm{K},\ b \in S,\ t \le T$, take the value of one if at least one unit in maximal clique κ is assigned to clear-cut *b* in period *t*. Then:

$$z_{st}^b \le w_{st}^{\kappa}, \qquad \forall \kappa \in \mathrm{K},\ b \le s,\ t \le T; \tag{28}$$

$$\sum_{b} w_{st}^{\kappa} \le 1, \qquad \forall \kappa \in \mathrm{K},\ t \le T. \tag{29}$$

Constraint set (28) defines the behavior of variables w_{bt}^{κ} based on the values of the assignment variables. Constraint set (29) prevents the management units in each clique from being assigned to more than one clear-cut. Finally, we note that incorporating maximum average clear-cut size restrictions in the Bucket Model is not possible as there is nothing in the formulation that prevents the formation of clear-cuts with disjoint management units.

2.4 Computational Experiments

The goal of this section is to describe the three experiments that serve to illustrate the computational performance of Model IV in comparison with Models I and II, with and without maximum and average harvest opening size restrictions. The first case study is a Radiata pine plantation in New Zealand where there are no clear-cut size restrictions. The second site is Loblolly pine located in the southeast United States where both maximum and average clear-cut size restrictions are present. The third experiment involves a larger forest, El Dorado (FMOS 2012), also located in the United States, that has 1,363 management units, and where only maximum harvest opening size restrictions are present.

We used MS Visual Basic 2008 to formulate and IBM ILOG CPLEX version 12.1.0 (64-bit) to solve the 3 models to optimality on a Dell Power Edge 510 Server with Intel Xeon CPU, X5670@2.93 GHz (2 processors) with 32 GB of RAM and a 64-bit Windows Server 2008 Operating System. Each model was run for three hours on 24 threads and five repetitions were carried out to account for the non-deterministic nature of parallel processing. For the smaller problems, provably optimal solutions were found in seconds, in which case the CPU times were recorded. For those problems that did not solve to optimality within three hours, the achieved optimality gaps were recorded. Optimality gaps are percentage gaps between the upper and lower bounds on objective function values that are established by the solver as it works to find better solutions. Tighter gaps indicate a higher likelihood of solution optimality.

2.4.1 Radiata pine, New Zealand

The 140-unit, 6132.6 ha Radiata pine plantation is located in the Taranga-Taupo catchment on the North Island of New Zealand (Fig. 3). The catchment is owned by the local Maori and is managed by New Zealand Forest Managers Ltd. The initial age of the management units ranged between 0 and 32 years (see Fig. 4). We set the length of the planning horizon to 25 years or five 5-year long planning periods, which corresponds to the minimum rotation age of Radiata pine. Since clear-cut size restrictions are not part of forest regulation in New Zealand, none of these constraints were modeled in this case study. For harvest volume fluctuations, we used 10% bounds for both the allowable increase and for the allowable decrease between adjacent periods. The representatives of the Maori owners, the Lake Taupo Forest Trust, and New Zealand Forest Managers Ltd. provided the spatial and economic input data such as harvest and silvicultural costs, initial ages, and the areas of the management units.

2.4.2 Loblolly pine plantation, Southeast U.S.

The second test forest was a 280-unit, 4,884 ha Lolblolly pine plantation in the southeastern United States. The initial age of the units ranged between 0 and 60 years (Fig. 5). Unlike Radiata pine in the New Zealand site, Loblolly pine is used for three different timber products in this plantation: pulpwood, sawtimber and chip-and-saw. While the volumes of these products do not all individually follow logistic-like growth curves, the sum of revenues associated with these volumes does (Fig. 3), thereby allowing a Model IV fit to timber revenues rather than to volumes. For the flow constraints, we defined the harvest variable h_t in all models as the harvest revenue generated at time t, rather than the volume harvested. As in the New Zealand case study, we used flow constraints with 10%

bounds. This time, however, the bounds were imposed on the maximum allowable increase and decrease in revenues from one period to the next instead of volumes.

A critical difference between the New Zealand and the U.S. case studies is that two different types of maximum clear-cut size restrictions had to be incorporated in the latter. The Sustainable Forestry Initiative certification scheme in the Southeast United States requires a 97-ha maximum harvest opening size restriction, as well as a 48.5-ha maximum average harvest opening size restriction. The Loblolly pine plantation in this study must comply with both of these regulations. To incorporate these restrictions, we imbedded Goycoolea et al.'s (2005) Cluster Packing Formulation in the benchmark models, Model I and II (Appendix A), as well as in Model IV as described in Section 2.3. Cluster Packing is currently the only available area restriction model that can account for both maximum and average clear-cut size restrictions. As in the Radiata test, the harvest allocations were optimized over 25 years or five 5-year planning periods.

2.4.3 El Dorado, U.S. (FMOS 2012)

Data for the third test forest, the 1,363-unit El Dorado (California, United States) was retrieved from a public forest data repository, a site maintained by the University of New Brunswick, Canada (FMOS 2012). As in the other test forests, the length of the planning horizon was 25 years comprising five 5-year long planning periods. A maximum harvest opening size of 120 ac (48.5 ha) and no maximum average clear-cut size restriction was present. We used McDill et al.'s (2002) Path Formulation to enforce the 48.5 ha maximum opening size. We chose the Path Model, as opposed to Cluster Packing, to demonstrate the computational viability of Model IV using this particular technique. In addition, there is some evidence (Tóth et al. 2012, Tóth et al. In Press) that the Path Formulation performs better than other ARMs when only maximum opening size restrictions are present.

3. Results and Discussion

3.1. Solution times and optimality gaps

Table 1 shows the solution times in seconds that were needed for each problem instance to reach proven optimality. If the predefined 3-hour time limit was exceeded without finding the optimal solution, we listed whatever optimality gap was achieved within this timeframe. The data in Table 1 strongly suggest that Model IV is a computationally viable approach for spatially explicit harvest scheduling. Of the 15 runs, where each of the three models were repeated 5 times for each test forest, Model IV performed the best in 14 instances (93%). While the advantage of the new model was less pronounced in the two smaller problems (in the Radiata and the Loblolly plantations), it achieved a mean optimality gap of 0.0168% on the larger El Dorado. Model I and II reached a gap of only 0.0386% and 0.035%, respectively, after 3 hours of run time. While these results are very encouraging, more extensive testing is necessary to establish if Model IV is a statistically superior formulation. Forest characteristics such as the initial age class distribution or the spatial configuration of the management units (e.g., adjacency) may play significant roles in just how well each model might perform. What is clear, however, is that Model IV is computationally tractable within 3 hours of runtime and that it *can* perform much better than the other models. In the light of this, we recommend that forest planners add Model IV to their list of tools that can be used to tackle hard, spatially explicit, forest planning problems. Having more options available increases the likelihood of finding a sufficiently optimal harvest schedule to the problem at hand even if it is unclear which one, if any, would perform the best.

Finally, we note that some might argue that the superior computational performance of Model IV is surprising given the complex logical structures that are involved. While the formulation is indeed complex mathematically, we suspect that integer programming models rich in Boolean constructs might in fact be well suited for the algorithms currently used by commercial solvers

3.2. Caveats

There are a few caveats in order for those who want to apply Model IV in practice. First, as we have already discussed in the Methods section, the Model IV construct requires a pre-optimization exercise to fit the parameters of the difference equations in Function (1). This is necessary so that the growth and yield of the units are properly accounted for. While this exercise has almost no computational cost, it does have a side-effect. That is, a perfect fit might not be available. As a result, the harvest volume (or revenue) projections associated with the solutions that are produced by Model IV might be slightly different from those produced by Models I or II where all the volume or revenue data are hard-wired. While these differences have little significance given that the original volume or revenue projections contain a degree of uncertainty anyway, they can, in the presence of harvest flow constraints, lead to optimal solutions in Model IV that are different from those in Models I and II. In fact, a harvest schedule that is optimal for Model IV can potentially violate the harvest flow constraints in Models I and II, and vice versa. This is due to the hard-fast margins that these, typically binding constraints impose on the allowable changes in harvest volumes between adjacent time periods. The problem can be eliminated by conducting sensitivity analyses on the harvest flow constraints, which we recommend regardless of the model being used. Another solution is to use soft instead of hard constraints that allow some deviations from the margins in volumes or revenues.

A second issue that needs to be raised is related to the simplified version of Model IV. We mentioned in Section 2.3.2. that the simplified model is applicable only if the volume or revenue projections associated with a unit are not needed below the inflection point. After reviewing a set of growth functions, however, we found that the inflection points were always below the volumes at which the first commercial treatments might potentially be applied in practice. While this finding does not eliminate the possibility that information about stand volumes might be needed below the inflection point even in the absence of silvicultural activities, it does suggest that the simplified model is likely to be viable in most practical cases.

Third, while it is relatively straightforward to aggregate multiple products or species into one growth curve for M-IV (as in the Loblolly case), modeling is more complex when these product flows need to be treated separately. If this is the case, additional decision variables are needed, along with additional curves to fit and additional constraints. This is a more involved exercise than in Model I where the prescription variables can have different coefficients for different purposes.

5. Conclusions

In this article, we introduced a new harvest-scheduling model for spatial forest planning. Model IV is fundamentally different from existing models, Models I and II (Johnson and Scheurman 1977), in that it that it uses variables and a set of linear difference relations rather than coefficients to carry information about the volumes or revenues of forest stands. We demonstrated that Model IV can capture a variety of important modeling concerns such as intermediate treatment decisions, and maximum and average clear-cut size restrictions. In addition, we described how Model IV is compatible with all three existing methods that can enforce maximum harvest opening size restrictions: McDill et al's (2002) Path, Goycoolea et al.'s (2005) Cluster and Constantino et al.'s (2008) Bucket Formulation. This compatibility is important because the different formulations offer different benefits and there are tradeoffs associated with them. Finally, we provided evidence with three experiments that the computational performance of Model IV is comparable with that of the existing methods.

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7. References

Acuna, M., Palma, C., Cui, W., Martell, D., Weintraub, A. (2010). Integrated spatial fire and forest management planning. *Canadian Journal of Forest Resources* 40(12), 2370-2383.

Bare, B.B., and Norman, E. (1969). An Evaluation of Integer Programming in Forest Production Scheduling Problems. *Purdue University Agricultural Experiment Station Research Bulletin* No. 847. 7p.

Barrett, T.M. (1997). Voronoi tessellation methods to delineate harvest units for spatial forest planning. *Canadian Journal of Forest Resources* 27(6), 903-910.

Berck, P. (1976). *Natural Resources in a Competitive Economy*. Ph.D. Thesis. Department of Economics, Massachusetts Institute of Technology.

Caro, F., Constantino, M., Martins, I. and Weintraub, A. (2003). A 2-opt tabu search procedure for the multiperiod forest harvesting problem with adjacency, greenup, old growth, and even flow constraints. *Forest Science* 49(5): 738-751.

Charnes, A., Cooper, W.W. (1961). *Management models and industrial applications of linear programming*. Wiley, New York.

Clark, M.M., Meller R.D., McDonald, T.P. (2000). A Three-Stage Heuristic for Harvest Scheduling with Access Road Network Development. *Forest Science* 46(2): 204-218.

Constantino, M., Martins, I., Borges, J. (2008). A new mixed-integer programming model for harvest scheduling subject to maximum area restrictions. *Operations Research* 56(3), 542-551.

Curtis, F. (1962). Linear programming: the management of a forest property. *Journal of Forestry* 60(9), 611-616.

Franklin, J., Forman, R. (1987). Creating landscape patterns by forest cutting: ecological consequences and principles. *Landscape Ecology* 1(1), 5-18.

FMOS- Forest Management Optimization Site. 2012. University of New Brunswick, Canada. http://ifmlab.for.unb.ca/fmos/datasets/. Retrieved: 6/1/2012.

Goycoolea, M., Murray, A., Barahona, F., Epstein, R., Weintraub, A. (2005). Harvest scheduling subject to maximum area restrictions: exploring exact approaches. *Operations Research* 53(3), 490-500.

Goycoolea, M., Murray, A., Vielma, J. P., Weintraub, A. (2009). Evaluating approaches for solving the area restriction model in harvest scheduling. *Forest Science* 55(2), 149-165.

Gunn, E. A., Rai, A. K. (1987). Modelling and decomposition for planning long-term forest harvesting in an integrated industry structure. *Canadian Journal of Forest Research* 17(12), 1507-1518.

Gunn, E.A., Richards, E.W. (2005). Solving the adjacency problem with stand-centred constraints. *Canadian Journal of Forest Research* 35:(4) 832-842.

Johansson, P.O., Löfgren, K.G. (1985). *The Economics of Forestry and Natural Resources*. Basil Blackwell. Oxford, UK.

Johnson, K. N., Scheurman, H. L. (1977). Techniques for prescribing optimal timber harvest and investment under different objectives- discussion and synthesis. *Forest Science Monograph 18*.

Jones, J.G., Hyde III, J.F.C., and Meacham, M.L. (1986). Four Analytical Approaches for Integrating Land Management and Transportation Planning on Forest Lands. Research Paper INT-361, USDA Forest Service, Intermountain Research Station. 33p.

Kidd, W., Jr., Thompson, E., Hoepner, P. (1966). Forest regulation by linear programming-a case study. *Journal of Forestry* 64(9), 611-613.

Kirby, M.W., P. Wong, W.A. Hager and M.E. Huddleston (1980). *Guide to the Integrated Resource Planning Model* (U.S. Department of Agriculture, Management Sciences Staff., Berkeley, CA. 212p.

Lockwood, C., and Moore, T. (1993). Harvest scheduling with spatial constraints: a simulated annealing approach. *Canadian Journal of Forest Research* 23: 468-478.

Loucks, D. (1964). The development of an optimal program for sustained-yield management. *Journal of Forestry* 62(7), 485-490.

McArdle, R.E., Meyer, W.H., and Bruce, D. (1961). The yield of Douglas-fir in the Pacific Northwest. *USDA Forest Service Technical Bulletin No. 201*. Washington, DC. 72 p. (rev.).

McDill, M. (1989). *Timber Supply in Dynamic General Equilibrium*. Ph.D. Thesis. Department of Forestry, Pennsylvania State University.

McDill, M., Rebain, S., Braze, J. (2002). Harvest scheduling with area-based adjacency constraints. *Forest Science* 48(4), 631-642.

Murray, A. T., Goycoolea, M., Weintraub, A. (2004). Incorporating average and maximum area restrictions in harvest scheduling models. *Canadian Journal of Forest Resources* 34(2), 456-464.

Murray, A. T. (1999). Spatial restrictions in harvest scheduling. *Forest Science* 45(1), 45-52.

Murray, A. T., and Church, R.L (1995). Heuristic Solution Approaches to Operational Forest Planning Problems. *OR Spektrum* 17: 193-203.

Nautiyal, J.C., and Pearse, P.H. (1967). Optimizing the Conversion to Sustained Yield – A Programming Solution. *Forest Science* 13(2):131-139.

O' Hara, A.J., Faaland, B.H., and Bare, B.B. (1989). Spatially Constrained Timber Harvest Scheduling. *Canadian Journal of Forest Research* 19:715-724.

Garcia, O.P. 1979. Modelling stand development with stochastic differential equations. In: Elliott,D.E. (ed) *Mensuration Systems for Forest Management Planning*. New Zealand Forest Service. Forest Research Institute Symposium No. 20, p.315-334. 1979.

Pienaar, L.V., Turnbull, K.J. (1973). The Chapman-Richards generalization of Von Bertalanffy's growth model for basal area growth and yield in even-aged stands. *Forest Science* 19(1), 2-22.

Rebain, S., and McDill, M.E. (2003a). Can mature patch constraints mitigate the fragmenting effect of harvest opening size restrictions? *International Transactions in Operational Research* 10(5): 499-513.

Rebain, S., and McDill, M.E. (2003b). A mixed-integer formulation of the minimum patch size problem. *Forest Science* 49(4): 608-618.

Richards, E.W., and Gunn, E.A. (2003). Tabu search design for difficult forest management optimization problems. *Canadian Journal of Forest Research* 33: 1126-1133.

Snyder, S., ReVelle, C. (1996). Temporal and spatial harvesting of irregular systems of parcels. *Canadian Journal of Forest Research* 26(6), 1079-1088.

Snyder, S., ReVelle, C. (1997). Dynamic selection of harvests with adjacency restrictions: the SHARe model. *Forest Science* 43(2), 213-222

Timber Mart-South. (2008). Warnell School of Forest Resources, University of Georgia, Athens, GA. URL: http://www.timbermart-south.com/ . Last accessed: 10/4/2011.

Tóth, S.F., M.E. McDill, N. Könnyű, and S. George. In Press. Testing the Use of Lazy Constraints in Solving Area-Based Adjacency Formulations in Harvest Scheduling Models. *Forest Science*.

Tóth, S.F., M.E. McDill, N. Könnyű, and S. George. 2012. A Strengthening Procedure for the Path Formulation of the Area-Based Adjacency Problem in Harvest Scheduling Models. *Mathematical and Computational Forestry and Natural Resources Sciences* 4(1), 16-38.

Tóth, S.F., and McDill, M.E. (2008). Promoting large, compact mature forest patches in harvest scheduling models. *Environmental Modeling and Assessment* 13(1), 1-15.

United States Congress. 1976. National Forest Management Act of 1976. 16 U.S.C. 1600.

Williams, H.P. (1999). *Model Building in Mathematical Programming* (4th Edition). John Wiley and Sons, New York. 354p.

Table 1. Solution times and optimality gaps. The problems were solved to CPLEX's default optimality gap of 0.01%. A time limit of 3 hours was imposed and if the target optimality gap was not reached during this time frame, then the best gap is reported (see El Dorado runs). The cells that correspond to model instances requiring the shortest solution times or smallest optimality gaps are in grey highlight.

		Solution times (s) / Optimality gaps (%)					
		Run 1	Run 2	Run 3	Run 4	Run 5	Mean
Radiata pine New Zealand	M-I	0.39s	2.40s	1.31s	2.37s	2.00s	1.69s
	M-II	1.47s	0.45s	2.11s	0.78s	2.03s	1.37s
	M-IV	0.91s	0.93s	0.78s	0.72s	0.97s	0.86s
Loblolly pine United States	M-I	4.65s	3.15s	6.50s	3.80s	4.52s	4.53s
	M-II	5.55s	3.09s	9.91s	3.62s	3.09s	5.41s
	M-IV	1.69s	1.62s	2.06s	1.90s	1.61s	1.80s
El Dorado United States	M-I	0.0408 %	0.0518 %	0.0344 %	0.0294 %	0.0363 %	0.0386 %
	M-II	0.0323 %	0.0360 %	0.0363 %	0.0428 %	0.0274 %	0.0350 %
	M-IV	0.0083 %	0.0158 %	0.0196 %	0.0261 %	0.0143 %	0.0168 %

FIGURE CAPTIONS

Fig. 1: Network flow representation of spatial Model II where the arrows correspond to the binary decision variables whose values define the harvest schedule for management unit *s*.

Fig. 2: An example of a logistic-like growth curve ($\overline{ABCD}$ in black) that was fitted using Function (1). The curve describes the merchantable yield of a Douglas fir (*Pseudotsuga mensiesii*) stand in the Pacific Northwest United States, with a site index of 42.672 meters (140 ft) and a base age of 100 years (Source: McArdle et al. 1961). The fitted values of the five parameters of the curve are listed in the five small boxes. Black curve $\overline{OBCD}$ corresponds to the simplified Model IV where the yield below the inflection point is not accounted for. Black curve $\overline{ABCEF}$ (or $\overline{OBCEF}$) represents the yield trajectory given that the stand is thinned at age 60. Lastly, the original growth curve to which Function (1) was fitted is in grey. Please note that the fit in this diagram was not optimized so that the two curves can be distinguished. For an optimal fit of Function (1), see Fig. 3.

Fig. 3: The fitted Function (1) for the Loblolly pine experiment. The solid curve represents the original growth function and the dashed grey curve represents Function (1). The fitted values of the five parameters of the curve are listed in boxes. The sum of deviations over time steps 25, 30,…, and 60 years was $128.35, or 0.27% of the total revenue. Deviations were minimized only above the minimum rotation age (25 yrs). A site index of 21.336 meters (70 ft) and a base age of 25 years were used, along with a pulpwood price of US$8.38/ton, a chin-and-saw price of US$17.64 and a sawtimber price of US$27.62 (Source: Timber Mart-South, Q4 2008) to generate the revenue curve.

Fig. 4: The 140-unit Radiata pine (*Pinus radiata*) plantation in the Tauranga-Taupo Catchment of New Zealand's North Island that was used for the first computational experiment. Darker shades of grey represent older stands.

Fig. 5: The 280-unit Loblolly pine (*Pinus taeda*) plantation in the Southeast U.S. was used for testing M-IV. Darker shades of grey represent older stands.

APPENDIX A:
Benchmark Models

We present Model I, the first benchmark model, differently from existing literature in that we use a prescription-based rather than harvest timing-based formulation. The decision variables denote the choice whether a sequence of actions (e.g., harvests) should be applied to a management unit or not. Traditionally, integer versions of Model I have been presented with variables that represented "cut or not cut" decisions for each unit. The more general, prescription-based formulation was needed in our experiments because both Models II and IV allow any number of harvests to be applied to a given stand over a particular planning horizon. Among other things, one consequence of this generalization of Model I is that the prescription variables need to be mapped to harvest timing-based cluster variables for the Cluster Packing formulation to work (see Constraints A8-A9). Finally, our presentation of Model II is also new in that here, an ARM-based extension is used. Snyder and ReVelle's (1996; 1997) Model II construct was URM-based.

A1. Model I

To define the integer version of Johnson and Scheurman's (1977) Model I, we let S denote the set of management units, $t = 1, 2, ..., T$ the time periods in the planning horizon, and k the minimum rotation age. We note that based on the assumption that harvests can occur only in the midpoints of the planning periods, the number of times a unit can potentially be harvested over the planning horizon is T / k. Model I requires the definition of the set of all possible prescriptions that can be assigned to a management unit: $P = \{(0,...,0),(1,0,...,0),...\}$, where every prescription $p \in P$ is a vector of length T. The elements of vector p represent the binary decisions of whether the management unit should be harvested in a particular planning period or not. The first element of the vector corresponds to period 1, the second to period 2, and so on. A value of one indicates that a harvest is to occur in the corresponding period, whereas zero indicates that no harvest should occur. Let 0-1 variable x_{sp} represent the decision whether unit s should follow prescription p. If it should, $x_{sp} = 1$, 0 otherwise. Decision variables are created only for those prescriptions that would not lead to premature harvests. In other words, all prescription variables with first harvests that occur before the unit reaches its minimum rotation age are excluded from the model during pre-processing.

Further, for management unit $s \in S$, we let a_s denote the area, e_s the regeneration cost, and v_{sp}^t the volume per unit area in period t given prescription p. We use h_t as an accounting variable for the volume harvested from the entire forest in period t, we use c for unit volume timber price, i for the real interest rate, and r_{sp} for the revenues associated with harvesting unit s according to prescription p. Finally, bounds $f_{\min}$ and $f_{\max}$ denote the allowable percent decrease and increase in harvest volume in consecutive planning periods. Such bounds are often used by practitioners to promote some level of operational sustainability. Then,

$$\max \sum_{s,p} r_{sp} x_{sp} \tag{A1}$$

st:

$$\sum_{p} x_{sp} = 1, \qquad \forall s \in S; \tag{A2}$$

$$\sum_{s,p} a_s \cdot v_{sp}^t \cdot x_{sp} = h_t, \qquad \forall \mathrm{t} \leq \mathrm{T}; \tag{A3}$$

$$\left(1 - f_{\min}\right) \cdot h_t \leq h_{t+1}, \qquad \forall t \leq T-1; \tag{A4}$$

$$\left(1 + f_{\max}\right) \cdot h_t \geq h_{t+1}, \qquad \forall t \leq T-1; \text{ and} \tag{A5}$$

$$x_{sp} \in \{0,1\}, \qquad \forall s \in S, p \in P \tag{A6}$$

defines the spatial version of Model I . The objective function (A1) maximizes the net discounted timber revenues associated with the entire forest across the planning horizon of T periods with the revenue coefficients being

$$r_{sp} = \sum_{t} \left(c \cdot a_s \cdot v_{sp}^t - e_s\right) \cdot \left(1+i\right)^{-t}, \qquad \forall s \in S, p \in P. \tag{A7}$$

Constraint (A2) ensures that each unit is assigned exactly one prescription. Constraint (A3) calculates the harvest volume in each time period and Constraints (A4) and (A5) ensure that the harvest volume in adjacent planning periods does not fluctuate by more than a lower and an upper bound, $f_{\min}$ and $f_{\max}$, respectively. Finally, Constraint (A6) defines the decision variables as binary.

Model I can easily incorporate spatial constraints such as maximum or average harvest opening size restrictions. All of the three existing ARMs, McDill et al.'s (2002) Path, Goycoolea et al.'s (2005) Cluster Packing and Constantino et al's (2008) Bucket formulation, were introduced in the literature based on the Model I construct. Goycoolea et al.'s (2005) Cluster Packing is the only ARM, however, that can handle maximum average clear-cut size restrictions (Murray et al. 2004), which are often present in forest regulations. Cluster Packing requires an *a priori* enumeration of all contiguous clusters of management units whose combined area does not exceed the maximum allowable clear-cut size. Formally, a set of management units $s \in \theta$ that forms a connected sub-graph of graph $G(S,E)$, where E denotes the edges representing the adjacency among the units (S), and for which inequality $\sum_{s\in\theta} a_s \le A_{\max}$ ($A_{\max}$ = the maximum harvest opening size) holds, is called a feasible cluster. Let Θ denote the set of all feasible clusters that arise from a given problem instance. To incorporate Goycoolea et al.'s (2005) Cluster Packing approach in Model I, the management unit-based prescription variables $x_{sp} \in \{0,1\}$ must be mapped to cluster variables to account for the feasible clusters. Let $u_{\theta t} \in \{0,1\}$ denote the decision whether all management units in cluster θ should be cut in period t:

$$\sum_{s\in\theta} \sum_{p\in P_{\theta t}} x_{sp} \ge |\theta| \cdot u_{\theta t}, \qquad \forall \theta \in \Theta, t \le T; \tag{A8}$$

$$\sum_{s\in\theta} \sum_{p\in P_{\theta t}} x_{sp} - \sum_{s\in A_\theta} \sum_{p\in P_{\theta t}} x_{sp} - u_{\theta t} \le |\theta| - 1, \qquad \forall \theta \in \Theta, t \le T; \tag{A9}$$

where $P_{\theta t}$ is the set of all prescriptions applicable to cluster θ that involve a harvest in period t and A_θ is the set of management units that are adjacent to cluster θ. Constraint (A8) allows, while constraint (A9) forces cluster θ to be "declared" cut in period *t* if all the units in the cluster, but none adjacent to it, are assigned prescriptions that involve a harvest in period *t*. To avoid double-counting the harvested areas, and to ensure that the maximum harvest opening size is never exceeded in any of the solutions, one constraint needs to written for each *maximal clique* of management units in each planning period. A set of mutually adjacent management units forms a maximal clique if no other unit exists that is adjacent to every member of the clique. Let K denote the set of all maximal cliques that arise from a problem with κ being one such clique. To prevent maximum harvest opening size violations, it is necessary that

$$\sum_{\theta \in \Theta_\kappa} u_{\theta t} \le 1, \qquad \forall \kappa \in \mathrm{K}, t \le T. \tag{A10}$$

where Θ_κ is the set of feasible clusters that contain at least one management unit that is also a member of maximal clique κ. Constraint (A10) requires that only one feasible cluster containing one or more units in clique κ is assigned to be clear-cut in period *t*. This precludes harvest arrangements where adjacent units are cut as parts of two or more feasible clusters of units whose combined area exceeds the maximum harvest opening size. Constraint (A10), which is the multi-harvest version of the original Cluster Packing adjacency constraint (Goycoolea et al. 2005), also prevents a unit from being part of multiple clusters in a given period. Finally, to enforce that the average harvest opening size over the entire planning horizon does not exceed $\bar{A}_{\max}$, we add the following forest-wide inequality (Murray et al. 2004):

$$\sum_{\theta, t} \left(a_\theta - \bar{A}_{\max} \right) \cdot u_{\theta t} \le 0. \tag{A11}$$

In constraint (A11), a_θ is the area of cluster θ and $\bar{A}_{\max}$ is the maximum allowable average clear-cut size.

If only maximum harvest opening size restrictions are present, McDill et al.'s (2002) Path Formulation can also be used by replacing A8-11 with the following set of adjacency constraints:

$$\sum_{s \in C, p \in P_{st}} x_{sp} \le |C| \text{-} 1, \qquad \forall C \in \Lambda^+, t \le T. \tag{A12}$$

where P_{st} is the set of all prescriptions applicable to unit s that involve a harvest in period t. In the computational experiments, A1-7 were used for the Radiata problem, A1-11 for the Loblolly, and A1-7 and A12 for El Dorado.

A2. Model II

To give a formal definition of Model II, we let variable sets b_{st}, $\ell_{st} \in \{0,1\}$ $\forall s, t$ denote the decision whether unit *s* should be cut in period *t* the first time and whether it should be cut in period *t* the last time, respectively. If unit *s* is to be cut the first time in period *t*, then $b_{st} = 1$, 0 otherwise. If unit *s* is to be cut the last time in period t, then $\ell_{st} = 1$, 0 otherwise. Further, we let variable set $g_{s,t',t} \in \{0,1\}$ $\forall s,t$ denote the decision whether unit s should be harvested in period t after it had last been cut in period t'. If it is, then $g_{s,t',t} = 1$, 0 otherwise. Lastly, we let $n_s \in \{0,1\}$ $\forall s,t$ represent the "do-nothing" scenario. If unit s is not to be cut during the planning horizon, then $n_s = 1$, 0 otherwise. Strictly speaking, variable set $n_s \in \{0,1\}$ is not needed for Model II except for the convenience of keeping track of uncut forest tracts.

As in Model I, we seek to maximize net timber revenues over the planning horizon subject to logical constraints that allow exactly one first harvest or no harvest of a unit, and harvest volume flow constraints (A16-18) that are analogous to inequalities (A3-5) in Model I. We calculate the revenue coefficient associated with the first harvest of unit s in period t using relation $r_{st}^{b} = \left(c \cdot a_s \cdot v_{st}^{b} - e_s\right) \cdot (1+i)^{-t}$, where v_{st}^{b} is the harvest volume of unit s in period t given that this is the first time unit s is cut. Similarly, we use formula $r_{s,t',t}^{g} = \left(c \cdot a_s \cdot v_{s,t',t}^{g} - e_s\right) \cdot (1+i)^{-t}$ to calculate the revenues associated with cutting unit s in period t after having cut it already in period t'. Parameter $v_{s,t',t}^{g}$ represents the unit area volume associated with the harvest of unit s in period t given that the previous harvest occurred in period t'. Then, the following function and set of inequalities define Model II:

$$\max \sum_{s,t} \left[r_{st}^{b} \cdot b_{st} + \sum_{t' \le t-k} r_{s,t',t}^{g} \cdot g_{s,t',t} \right], \tag{A13}$$

st.

$$n_s + \sum_{t} b_{st} = 1, \qquad \forall s \in S; \tag{A14}$$

$$b_{st} + \sum_{t' \le t-k} g_{s,t',t} = \sum_{t^* \ge t+k} g_{s,t,t^*} + \ell_{st}, \qquad \forall s \in S, t \le T; \tag{A15}$$

$$\sum_{s} \left[a_s \cdot v_{st}^{b} \cdot b_{st} + \sum_{t' \le t-k} a_s \cdot v_{s,t',t}^{g} \cdot g_{s,t',t} \right] = h_t, \qquad \forall t \le T; \tag{A16}$$

$$\left(1 - f_{min}\right) \cdot h_t \le h_{t+1}, \qquad \forall t \le T-1; \tag{A17}$$

$$\left(1 + f_{\max}\right) \cdot h_t \ge h_{t+1}, \qquad \forall t \le T-1; \tag{A18}$$

$$b_{st}, \ell_{st} \in \{0,1\}, \qquad \forall s \in S, t \le T; \tag{A19}$$

$$g_{s,t',t} \in \{0,1\}, \qquad \forall s \in S, t \le T, t' \le T-k; \quad \text{(A20)}$$

$$n_s \in \{0,1\}, \qquad \forall s \in S; \quad \text{(A21)}$$

where Identity (A15) is a "network flow" constraint that ensures whenever a management unit is cut in a particular period, whether it is the first, second or an intermediate harvest, there must be another variable that either declares this harvest to be the last for that unit or it forces it to be cut again during the planning horizon. In other words, this constraint forces each management unit to have a sound prescription plan with no discrepancies. Along with the logical constraints (A14), the flow constraints create a network representation of the harvest scheduling problem with nodes representing the starting (source), the ending (sink) and the intermediate states of the units (Fig. 1). Whatever harvest decision takes the unit to a given harvest-period state, there has to be another decision that moves it along to another state. This decision is either a declaration that this harvest was the unit's last, or the unit could be cut again in a subsequent period. Finally, constraints (A19-21) define the four sets of decision variables as binary.

Model II is also compatible with all the three exact ARMs: McDill et al.'s (2002) Path, Goycoolea et al.'s (2005) Cluster Packing and Constantino et al.s (2008) Bucket Formulation. Path constraints can be added as:

$$\sum_{s \in C}\left(b_{st} + \sum_{t' \le t-k} g_{s,t',t}\right) \le |C|\text{-}1, \qquad \forall C \in \Lambda^+, t \le T. \quad \text{(A22)}$$

Objective (A13) and Constraints (A14-22) were used to model the El Dorado problem with Model II in the computational experiment.

To incorporate Goycoolea et al.'s (2005) Cluster Packing Formulation, the decision variables in Model II are redefined to handle Cluster Packing. Variable sets b_{st}, ℓ_{st}, and $g_{s,t',t}$ are replaced with $b_{\theta t}$, $\ell_{\theta t}$, and $g_{\theta,t',t}$ to represent the decision whether cluster θ should be cut in period *t* first, whether it should be cut in period *t* last, and whether it should be cut in period *t* after it had already been cut in period *t'*, respectively. Variable $b_{\theta t}$ takes the value of one if cluster θ is to be cut for the first time in period *t*, 0 otherwise. Variable $\ell_{\theta t}$ takes the value of one if cluster θ is to be cut for the last time in period *t*, 0 otherwise. Lastly, $g_{\theta,t',t} = 1$ if cluster θ is to be cut in period *t* after it had been cut in period *t'*, 0 otherwise. To incorporate harvest opening size restrictions using Goycoolea et al.'s (2005) model, objective function (A13) and constraints (A14)-(21) remain the same except that all decision variables are replaced with the cluster variables as described above. To account for the logical condition that a given management unit can only be assigned to at most one feasible cluster that is cut in a particular period for the first time, we replace the logical constraint set (A14) with

$$n_s + \sum_{t,\theta:s\in\theta} b_{\theta t} = 1, \qquad \forall s \in S. \qquad \text{(A23)}$$

Variable n_s is added to the sum of cluster assignments to account for the option of not cutting unit *s* during the planning horizon. We also replace the network flow constraints to allow a unit to be assigned to different clusters in each period:

$$\sum_{\theta:s\in\theta} b_{\theta t} + \sum_{t'\le t-k,\theta:s\in\theta} g_{\theta,t',t} = \sum_{t^*\ge t+k,\theta:s\in\theta} g_{\theta,t,t^*} + \sum_{\theta:s\in\theta} \ell_{\theta t}, \qquad \forall s \in S, t \le T; \qquad \text{(A24)}$$

To prevent adjacent management units from being assigned to multiple clusters that are cut at the same time, we add inequality set

$$\sum_{\theta\in\Theta_\kappa} \left(b_{\theta t} + \sum_{t'\le t-k} g_{\theta,t',t} \right) \le 1, \qquad \forall \kappa \in \mathrm{K}, t \le T. \qquad \text{(A25)}$$

Constraints (A25) are analogous to constraints (A10) in Model I; they ensure that maximum harvest opening size restrictions are never violated. Finally, to enforce the forest-wide maximum allowable average harvest opening size restriction ($\overline{A}_{\max}$), we require that:

$$\sum_{\theta,t} \left[\left(a_\theta - \overline{A}_{\max} \right) \cdot \left(b_{\theta t} + \sum_{t'\le t-k} g_{\theta,t',t} \right) \right] \le 0. \qquad \text{(A26)}$$

APPENDIX B:

Goal Programming to fit Model IV

We let ω_a denote the unit area volume of a management unit at age a (in periods) available from the original data, and we let ω_a^* denote the volume at age a coming from the logistic curve (1). For simplicity, we assume that the growth of each management unit is the same: volumes ω_a, accounting variables ω_a^* and the five parameters that are optimized for best fit have no *s* sub- or superscripts. This, of course, does not mean that Function (1) cannot be fitted for each management unit individually.

The objective of the GP, Function (A1), minimizes the sum of absolute deviations, δ_a, between the fitted, ω_a^*, and the original data, ω_a:

$$min \sum_a \delta_a, \qquad \text{(B1)}$$

st:

$$\omega_a - \omega_a^* \le \delta_a, \qquad \forall a \in A; \qquad \text{(B2)}$$

$$\omega_a^* - \omega_a \le \delta_a, \qquad \forall a \in A; \qquad \text{(B3)}$$

$$\omega_1^* = \phi_{\min}; \qquad \text{(B4)}$$

$$\omega_a^* \ge \beta \cdot y_a, \qquad \forall a \in A; \qquad \text{(B5)}$$

$$\omega_a^* - \beta \le \phi_{\max} \cdot y_a, \qquad \forall a \in A; \qquad \text{(B6)}$$

$$\omega_{a+1}^* - \phi_{\max} \cdot y_a \le \left(1+\gamma_{\exp}\right)\omega_a^*, \qquad \forall a \in A; \qquad \text{(B7)}$$

$$\omega_{a+1}^* \ge \left(1+\gamma_{\exp}\right)\omega_a^* - \phi_{\max} \cdot y_a, \qquad \forall a \in A; \qquad \text{(B8)}$$

$$\omega_{a+1}^* - \left(1-y_a\right)\cdot\phi_{\max} \le \left(1-\gamma_{\text{taper}}\right)\cdot\omega_a^* + \phi_{\max}\cdot\gamma_{\text{taper}}, \quad \forall a \in A; \qquad \text{(B9)}$$

$$\omega_{a+1}^* \ge \left(1-\gamma_{\text{taper}}\right)\cdot\omega_a^* + \phi_{\max}\cdot\gamma_{\text{taper}} - \left(1-y_a\right)\cdot\phi_{\max}, \quad \forall a \in A; \qquad \text{(B10)}$$

$$\beta, \phi_{\min}, \phi_{\max}, \gamma_{\exp}, \gamma_{\text{taper}} \in \mathbb{R}^+; \qquad \text{(B11)}$$

$$\omega_a^*, \delta_a \in \mathbb{R}^+, \qquad \forall a \in A; \qquad \text{(B12)}$$

$$y_a \in \{0,1\}, \qquad \forall a \in A. \qquad \text{(B13)}$$

Constraints (B2) and (B3) calculate the absolute deviation, δ_a between ω_a^* and ω_a with the help of objective function (B1), which maximizes the sum of the deviations. Constraint (B4) sets the value of the fitted ω_1^* to be equal to $\phi_{\min}$. Constraints (B5) and (B6) work together to determine if the fitted volume at a particular age is above or below the inflection point β. If the volume is strictly above the inflection point, then $y_a = 1$. Indicator y_a is zero otherwise. Note that the goal program will not allow the inflection point to be equal to the volume in any particular period, unless using the difference equation for the taper (B9)-(B10) vs. the exponential segment of the curve (B7)-(B8) does not make any difference in the objective function value. Recall that in the goal program, both the inflection point and the fitted volumes are variables. Constraints (B5) through (B10) establish the relationship between the five function parameters ($\phi_{\min}$, $\phi_{\max}$, $\gamma_{\exp}$, γ_{taper} and β) that are to be optimized and the fitted volumes: $\omega_a^* \;\forall a \in A$. Along with inequalities (B5)-(B6), constraints (B7)-(B10) capture Function (1). Lastly, constraints, (B11)-(B13) define the domains of the variables.

While GP (B1)-(B13) is a quadratically constrained, non-convex[1], and non-smooth[2] optimization problem, it is trivial to solve since it has only five decision variables ($\phi_{\min}$, $\phi_{\max}$, $\gamma_{\exp}$, γ_{taper} and β) and only $|A|$ accounting variables for the fitted volumes (ω_a^*), the deviations (δ_a) and the indicators y_a, each. Moreover, there is only one pair of goal constraints per data point, (B2)-(B3), and only a few extra constraints to account for the relationship between the volumes in consecutive age classes (B4)-(B10). GP (B1)-(B13) can be converted to a smooth optimization problem by optimizing for each binary combination of y_a's and selecting the combination that leads to the smallest total deviation. In this study, we used MS Excel Solver to fit Function (1).

[1] The quadratic coefficient matrix of B2-B10 is not positive semidefinite.

[2] Due to the binary y_a's in B1-B13.

APPENDIX C:

Modeling intermediate treatments with Model IV

In this Appendix, we discuss how Model IV can incorporate intermediate treatment decisions that are hard to capture in Models I and II parsimoniously: Should a management unit be thinned in a particular planning period with a given intensity or not? Should fertilization be applied to increase site productivity or should the trees be pruned? Thinning reduces the volume of the unit at the time of treatment and puts it on a steeper growth trajectory for merchantable timber. Thinings are applied in an attempt to maximize revenues or to improve forest health, or both. In addition to stimulating growth, fertilization can also increase site productivity, and pruning can change the quality of timber products.

The potential timings of treatments introduce additional prescriptions that are applicable to a management unit leading to an explosion of 0-1 variables in Model I. While the explosion is less dramatic in Model II, it is still an exponential function of the number of periods where intermediate treatments can occur. This is because additional sets of “first-”, “last-” and “intermediate harvest” variables are needed to link harvest decisions to intermediate treatments in prior or subsequent periods. If, in addition to the timing, the intensity of the treatments is also variable, then the combinatorial explosion is even more significant.

In Model IV, the number of extra decision variables that are needed to capture intermediate treatments increases only linearly as a function of the number of planning periods. To illustrate how Model IV can keep model size small, we consider a binary thinning decision τ_{st} in management unit *s* in period *t* with a preset thinning intensity of $\alpha \in [0,1]$. Let variable τ_{st} take the value of one if unit *s* is to be thinned in period *t*, 0 otherwise. We assume that the expected post-thinning growth rates of the unit are known: $\gamma^s_{\exp'}$ in the exponential and $\gamma^s_{\text{taper}'}$ in the taper section with an inflection point of β^s. First, Constraints (6)-(9) in Model IV are modified so that they would inactivate should thinning occur in period *t*:

$$v_{s,t+1} - \phi^s_{\max} \cdot \left(1 - y_{st} + \tau_{st}\right) \le \left(1 - \gamma^s_{\text{taper}}\right) \cdot v_{st} + \phi^s_{\max} \cdot \gamma^s_{\text{taper}}, \qquad \forall s \in S, t \le T-1, \qquad \text{(C1)}$$

$$v_{s,t+1} + \phi^s_{\max} \cdot \left(1 - y_{st} + x_{st} + \tau_{st}\right) \ge \left(1 - \gamma^s_{\text{taper}}\right) \cdot v_{st} + \phi^s_{\max} \cdot \gamma^s_{\text{taper}}, \qquad \forall s \in S, t \le T-1, \qquad \text{(C2)}$$

$$v_{s,t+1} - \phi^s_{\max}\left(y_{st} + \tau_{st}\right) \le \left(1 + \gamma^s_{\exp}\right) \cdot v_{st}, \qquad \forall s \in S, t \le T-1, \qquad \text{(C3)}$$

$$v_{s,t+1} + \phi^{s}_{\max}(y_{st} + x_{st} + \tau_{st}) \geq \left(1+\gamma^{s}_{\exp}\right)\cdot v_{st}, \qquad \forall s \in S, t \leq T-1. \tag{C4}$$

If $\tau_{st} = 1$, the value of $v_{s,t+1}$ is unrestricted in inequalities (C1)-(C4) regardless of the value of indicator y_{st} because $v_{s,t+1} \leq \phi^{s}_{\max}$ and because $0 \leq \gamma^{s}_{\text{taper}} \leq 1$. If $\tau_{st} = 0$, Constraints (C1)-(C4) work the same way as Constraints (6)-(9). To define the post-thinning growth of the units, we add:

$$v_{s,t+1} - \phi^{s}_{\max}\cdot\left(1+y_{st}-\tau_{st}\right) \leq \alpha\left(1+\gamma^{s}_{\exp'}\right)\cdot v_{st}, \qquad \forall s \in S, t \leq T-1; \tag{C5}$$

$$v_{s,t+1} + \phi^{s}_{\max}\cdot\left(1+y_{st}-\tau_{st}\right) \geq \alpha\left(1+\gamma^{s}_{\exp'}\right)\cdot v_{st}, \qquad \forall s \in S, t \leq T-1; \tag{C6}$$

$$v_{s,t+1} - \phi^{s}_{\max}\cdot\left(2-y_{st}-\tau_{st}\right) \leq \alpha\left(1-\gamma^{s}_{\text{taper}'}\right)\cdot v_{st} + \phi^{s}_{\max}\cdot\gamma^{s}_{\text{taper}}, \qquad \forall s \in S, t \leq T-1. \tag{C7}$$

$$v_{s,t+1} + \phi^{s}_{\max}\cdot\left(2-y_{st}-\tau_{st}\right) \geq \alpha\left(1-\gamma^{s}_{\text{taper}'}\right)\cdot v_{st} + \phi^{s}_{\max}\cdot\gamma^{s}_{\text{taper}}, \qquad \forall s \in S, t \leq T-1. \tag{C8}$$

These constraints are structurally similar to (6)-(9) or (C1)-(C4). The only difference is in the post-thinning growth rates. If $\tau_{st} = 1$, it is the value of y_{st} that determines whether the $v_{s,t+1}$ is going to be equal to the right-hand-side of inequality (C5)-(C6) or (C7)-(C8). Note that the volume in period *t* is adjusted based on the intensity of thinning in period *t*: α. If $\tau_{st} = 0$, the value of $v_{s,t+1}$ becomes unrestricted regardless of the value of y_{st}. In this case, Constraints (C1)-(C4) become active and work the same way as the original Constraints (6)-(9).

The number of decision variables per unit that need to be introduced in Model IV to account for thinning decisions with set intensities is equal to the number of planning periods that are eligible for thinning. The number of extra constraints, Constraints (C5)-(C8), is four times the number of eligible periods. If the range of volumes that correspond to planning periods eligible for thinning falls exclusively below or above the inflection point, then either (C5)-(C6) or (C7)-(C8) may be dropped. Lastly, alternative thinning intensities, as well as fertilization and pruning decisions may be modeled the same way as thinning decisions. For fertilization and pruning, coefficient α will be one since no merchantable volume is removed from the unit. The post-treatment growth rates in volume (for fertilization) or revenues (for pruning) will drive the transition of the units from one period to the next in accordance with Constraints (C5)-(C8).

Figure 1
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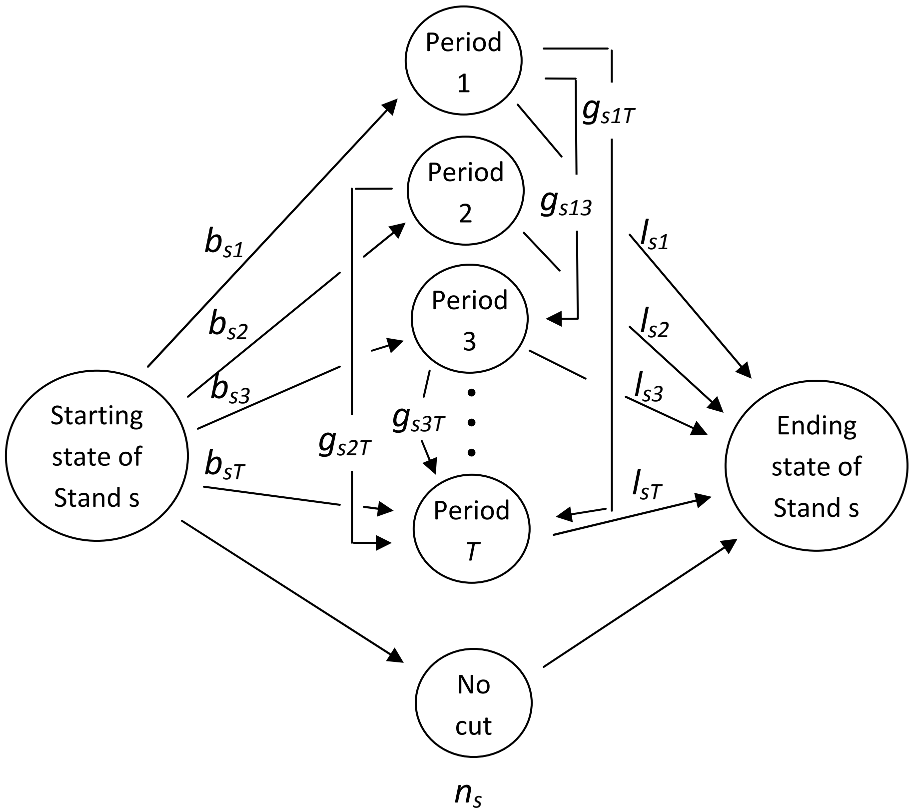

Figure 2
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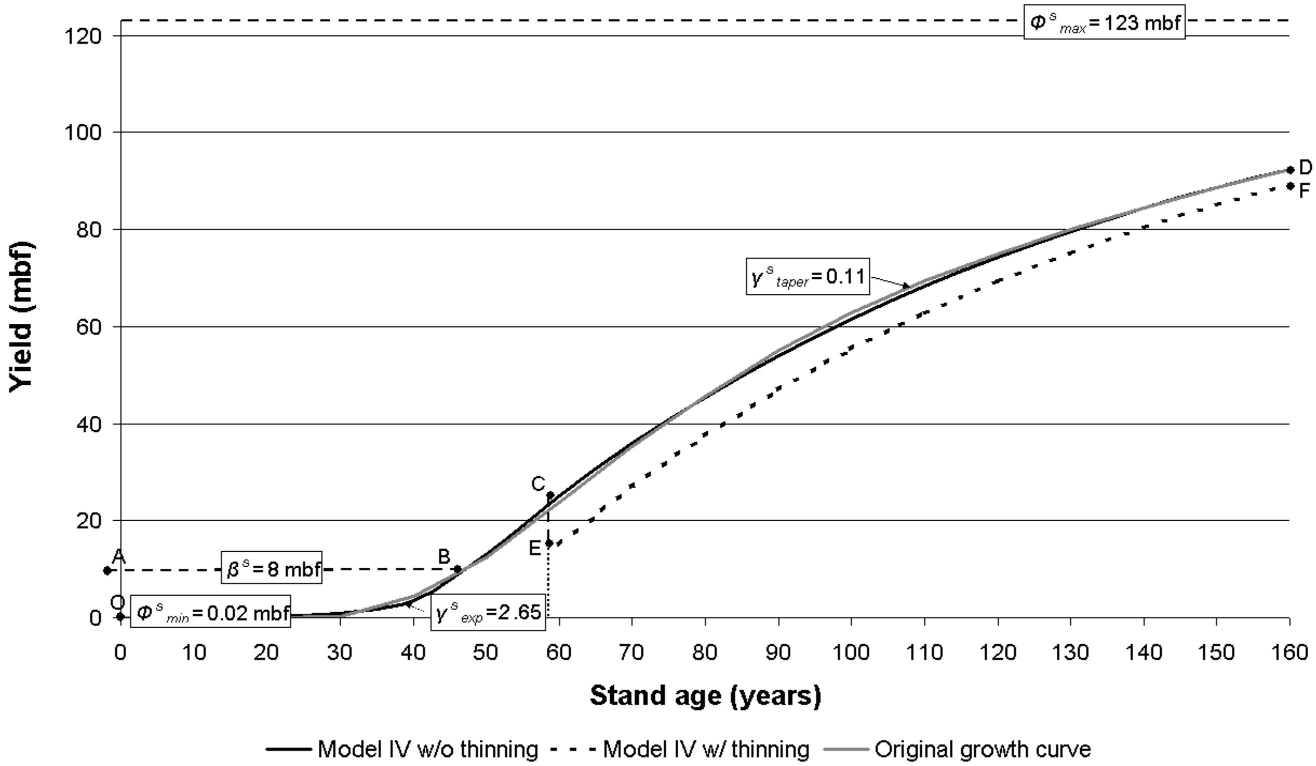

Fig. 2

Figure 3
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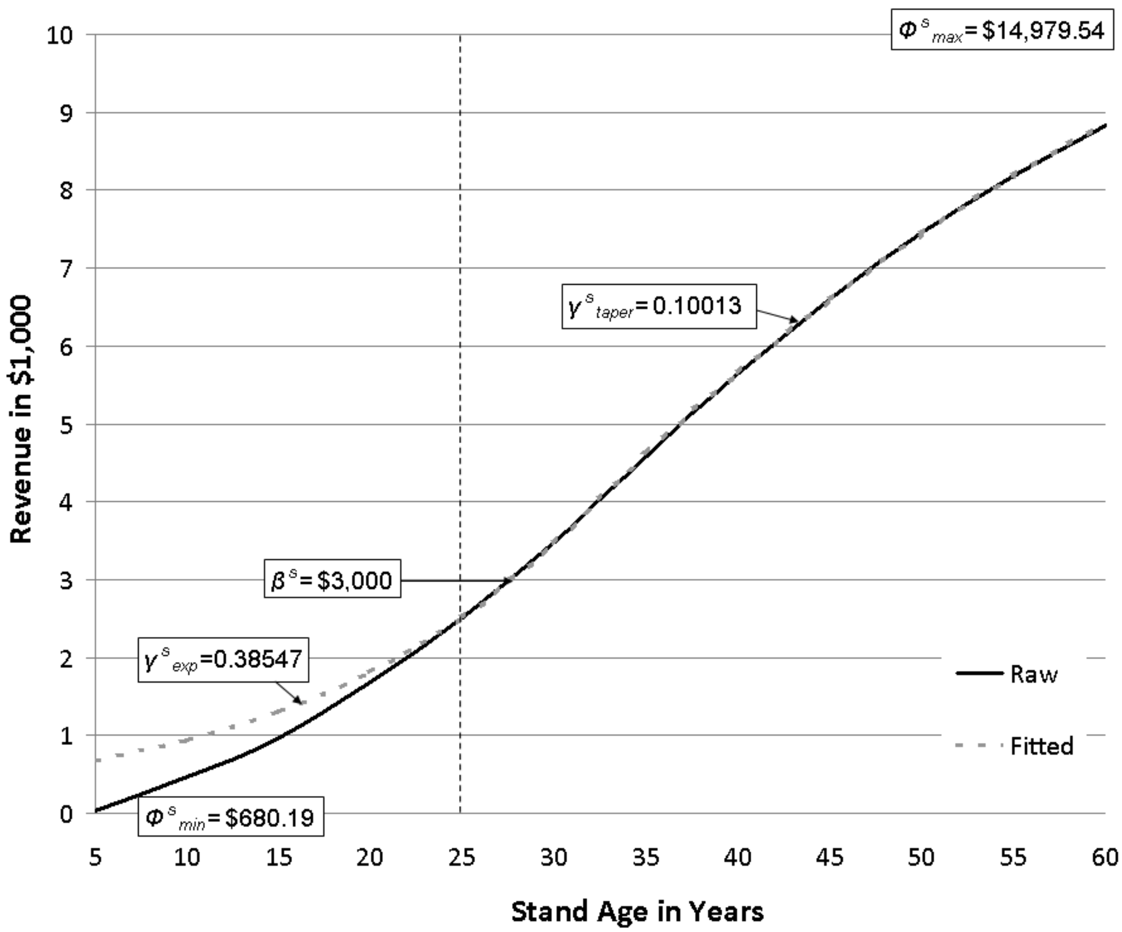

Fig. 3

Figure 4
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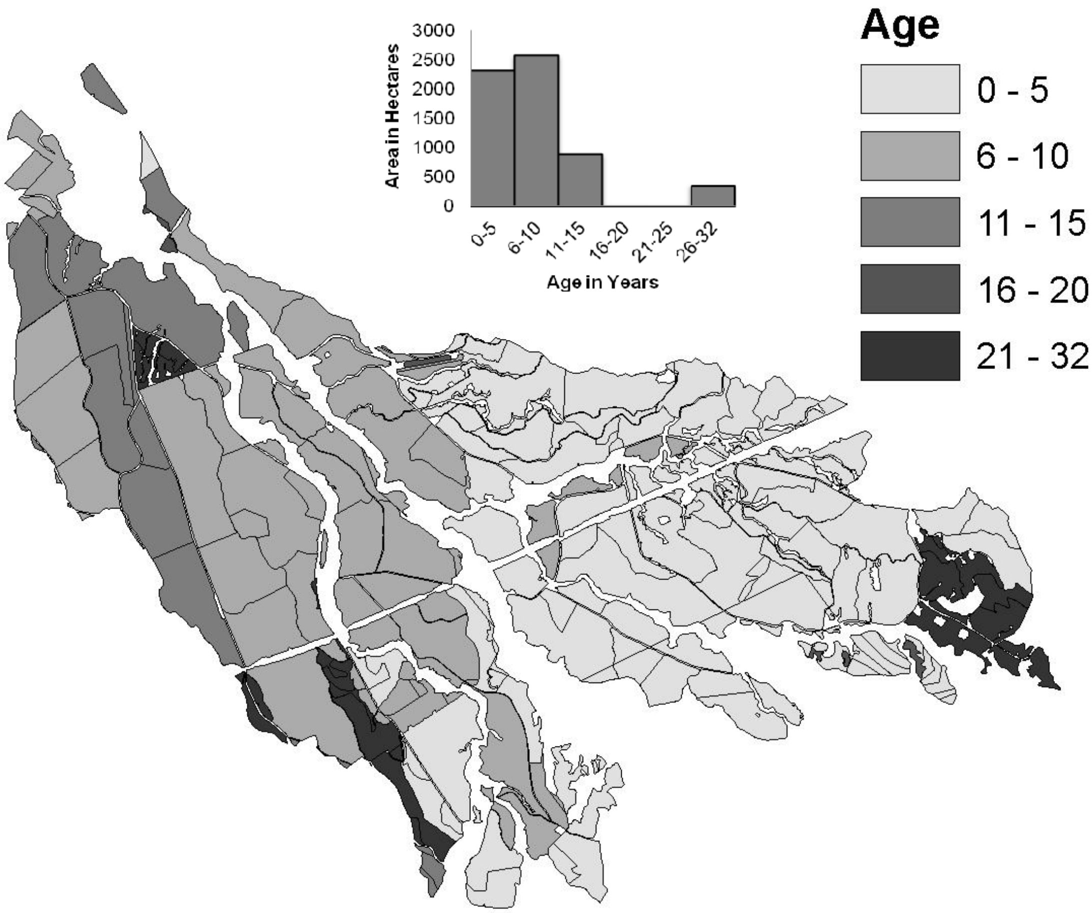

Fig. 4

Figure 5
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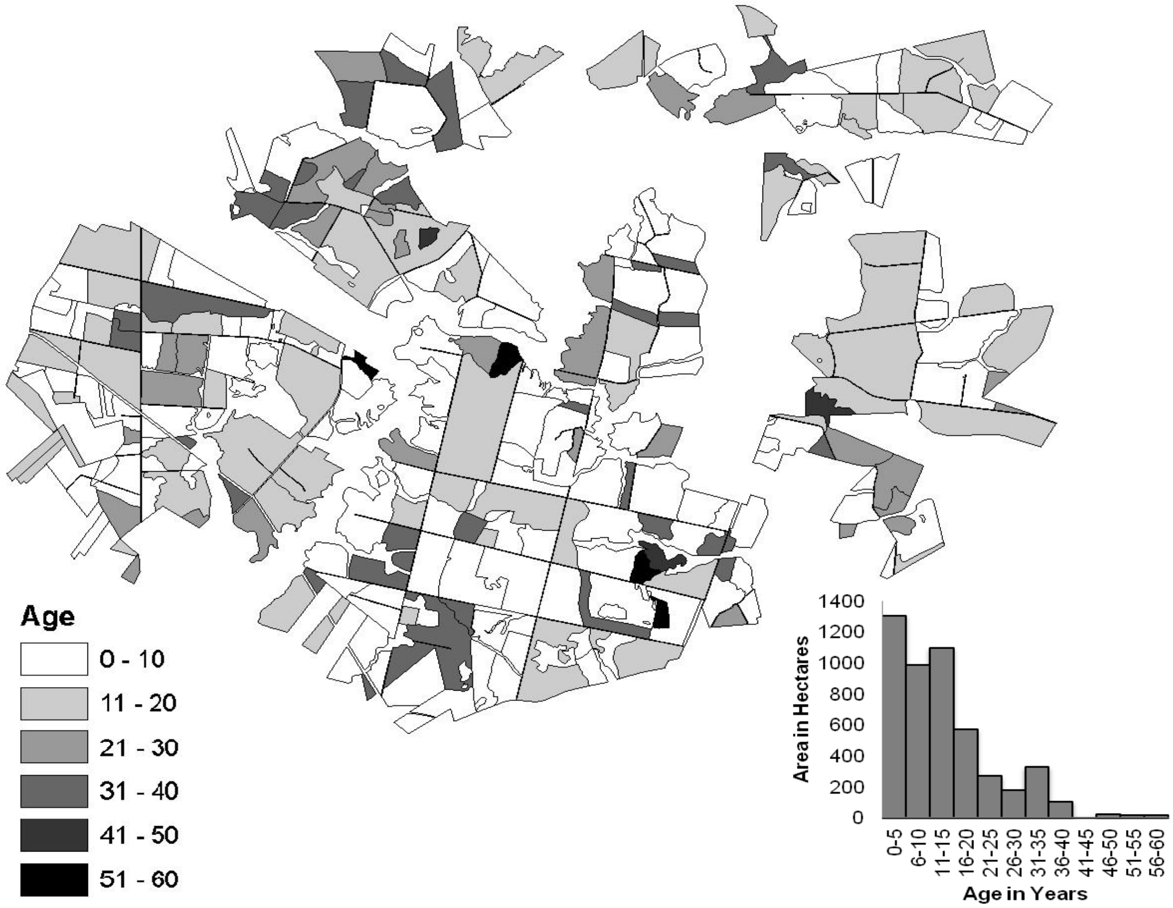

Fig. 5