CONVERGENCE SPEEDS AND TRANSITIONAL DYNAMICS IN NON-SCALE GROWTH MODELS*

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1. Introduction

Neoclassical and endogenous growth models yield strikingly different predictions regarding the determinants of long-run growth rates and their implications for long-run cross-country convergence characteristics. On the one hand, Mankiw, Romer, and Weil (1992) and Barro and Sala-i-Martin (1995) have shown that countries converge to identical growth rates, but to distinct income levels.[1] Since these empirical findings run counter to the predictions of endogenous growth models, they have cast doubt on the relevance of such models to explain long-run cross-country convergence and transition paths. On the other hand, while the empirical evidence confirms the implications of the traditional neoclassical in terms of cross-country convergence, calibrations show that the neoclassical model's implied convergence speed of about 7 percent, greatly exceeds the empirical estimates of approximately 2 percent. This excessive speed of convergence is also accompanied by implausibly high rates of return (in the standard model) or by implausible rates of investment (in models with human capital).[2] In addition, Bernard and Jones (1996a) maintain that the neoclassical convergence approach overemphasizes capital accumulation at the expense of technological change. They document that, at least since the 1970s, there exists little evidence for cross-country convergence of manufacturing technologies within the OECD.

In this paper we seek to reconcile these empirical findings by using a two-sector model of capital accumulation that incorporates endogenous technological change (knowledge). To do so, we examine the transition dynamics and convergence characteristics of a new class of *non-scale* growth models.[3] In many respects these models are a hybrid of endogenous and neoclassical models, and indeed the traditional Solow-Swan model is a special example. Technology is endogenous as in Romer (1990), and emerges as the outcome of agents' optimizing behavior, while the dynamic

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[1] These analyses controlled for parametric differences across countries such as savings rates.

[2] The introduction of adjustment costs can slow the speed of convergence, while factor mobility increases it, see King and Rebelo (1993), Ortigueira and Santos (1997), and Barro and Sala-i-Martin (1995).

[3] *Non-scale* refers to the characteristic that variations in the size or scale of the economy do not permanently alter its long-run equilibrium *growth rate*. For example, R&D-based growth models that follow Romer (1990) are *scale* models since they imply that an increase in the level of resources devoted to R&D should increase the growth rate proportionately; see Jones (1995b)

characteristics are similar to those of the neoclassical model. But in contrast to the latter, as our calibration exercises highlight, the two-sector non-scale model generates remarkably plausible convergence speeds, without having to introduce adjustment costs as in Ortigueira and Santos (1997).

One further advantage of the non-scale model is that very general production structures are compatible with balanced growth paths. Previous models of endogenous growth require production functions to exhibit constant returns to scale in all accumulated factors to ensure balanced growth. This strong requirement, which imposes a strict knife edge restriction on the production structure, has been the source of criticism; see Solow (1994).[4] If the knife-edge restriction that generates traditional endogenous growth models is not imposed, then any stable balanced growth equilibrium is characterized by the absence of scale effects. From this standpoint, non-scale growth equilibria should be viewed as being the norm, rather than the exception, and consequently this class of models merits serious investigation.

Examples of non-scale models have been introduced by Jones (1995a), Segerstrom (1995) and Young (1995). Eicher and Turnovsky (1996) have since provided a general characterization of non-scale, balanced growth equilibria in two-sector models. But no comprehensive analysis of the transitional dynamics of this class of models exists, and without an understanding of the underlying dynamics, it is by no means clear that the economy will reach its equilibrium.[5] Furthermore, even if the system is stable, the relevance of the steady-state balanced growth path depends upon how rapidly the economy converges to the steady state along the transitional path. This aspect is especially crucial for neoclassical and non-scale models: even if government policy is irrelevant in determining long-run growth rates, as these models suggest, it may nevertheless be crucial in determining the transitional path for substantial periods of time.

[4] Mulligan and Sala-i-Martin (1993) established a slightly weaker condition for balanced growth for a Lucas type endogenous growth model.

[5] Jones (1995a) sketched the transitional dynamics of a simplified non-scale model. To reduce the dimensionality of his system, he assumes that sectoral labor allocation and investment rates are exogenous constants. A complete analysis requires these variables to be endogenously determined as part of the dynamic equilibrium.

Our analysis of the dynamics reveals that the progression from endogenous to non-scale growth models raises its dimensionality. While the standard one-sector AK model has no transitional dynamics, the non-scale growth model is described by a second-order system. This implies that a saddlepath stable balanced growth equilibrium is approached along a one-dimensional locus. Likewise, Bond, Wang and Yip (1996) proved that the dynamics of the two-sector Lucas (1988) model can be expressed as a third order system, also having a single stable root and a one-dimensional stable manifold. In the latter two cases all variables converge to their respective steady-state equilibria at *identical and constant rates*; the economy possesses a unique speed of convergence.

By contrast, we show that the dynamics of the two-sector non-scale models lead to a fourth order system in appropriately scaled variables. A saddlepath stable system now has two negative eigenvalues, so that the stable manifold is a two-dimensional locus, thereby introducing important flexibility to the convergence and transition characteristics. In contrast to the neoclassical and Lucas two-sector models, there is no longer a unique constant speed of convergence. Instead, two-dimensional manifolds imply that the convergence speeds will vary through time and across sectors, often dramatically so. Furthermore, the speeds of convergence of different inputs exhibit distinct time profiles, which in turn reflect the differential characteristics of their respective transitional paths. As a result, the convergence speed and transition path of output is also time varying, since its dynamics are simply a composite of the transitional characteristics of the underlying factors of production, capital, and technology.[6]

These properties are consistent with the suggestion offered by Bernard and Jones (1996b) that the process of convergence is more complex than that indicated by changes in any single aggregate measure alone. These authors show that different sectors exhibit distinctly different convergence time profiles. But as noted, such diversities of convergence speeds cannot be generated by standard neoclassical or endogenous growth models. Consequently, since our analysis produces neither

[6] The empirical evidence on the constancy convergence rates is mixed. Barro and Sala-i-Martin (1995), who abstract from technological change, reject constancy in Japan, but not in the US and Europe. Nevertheless, all reported rates of convergence (0.4-3 percent, 0.4-6 percent, and 0.7-3.4 percent for Japan, the US, and Europe respectively) are similar to the range that the non-scale model generates.

common transition paths and rates of convergence, nor common balanced growth rates for all variables, we conclude that the non-scale model thus addresses the empirical concerns of Bernard and Jones (1996a, 1996b).

While the various transitional characteristics of specific variables are informative, we also suggest one comprehensive measure that summarizes the speed with which the overall economy is converging to its long-run growth path. For this purpose, the percentage change in the Euclidean distance of the two state variables, capital and technology, from their steady state serves as a natural measure of the economy's speed of convergence. This measure indicates that at any instant of time, the speed of convergence is a weighted average of the speeds of convergence of the two stocks, the weights being the relative square of their distance from equilibrium. In general, the system converges asymptotically to the new equilibrium at the rate of the slower growing stable eigenvalue.

To obtain an idea about the implied speed of convergence, we calibrate a general version of the two-sector non-scale model. Interestingly, the one-sector non-scale model's speed of convergence is slightly greater than that of the neoclassical model. Moving from one to two sectors, and introducing endogenous technological change, leads to a drastic reduction in the speed of convergence. A key result of the calibration exercises is that the magnitude of the speed of adjustment is robust throughout, and conforms closely to that observed in the data. Essentially, the accumulation of knowledge and the role it plays in slowing down investment in physical capital is similar to that played by the adjustment costs in the Ortigueira-Santos model.[7] Our results also contrast with Jones' (1995b), preliminary examination of non-scale transitional dynamics, in which he found adjustment to be excessively slow with half-lives ranging from 62 to 674 years for per capita output.[8]

For completeness, we report three adjustment speeds: the time profiles of capital, technology, and the overall comprehensive distance measure. Our calibration results document the wide range of transitional adjustment paths that may result. We show that a necessary condition for monotonic

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[7] Ortigueira and Santos examine an endogenous growth model where only preferences but not technology determine the balanced growth rate. The exact opposite is true in our non-scale model.
[8] A half life of 35 years implies a convergence rate of 2 percent per year.

transitions is that the steady state levels of both variables change in the same direction (i.e., both capital and knowledge increase or decrease). But even if both variables converge monotonically, this does not imply that their *speeds* of adjustment are either constant or identical. Especially interesting are non-monotonic paths that involve “overshooting," in that variables along their transition paths *exceed* their new long run equilibrium value. For example, a shock may induce a transition during which the economy decumulates more than the necessary amount of some quantity in the intermediate term. The excessive decumulation subsequently requires an accumulation of that quantity during the final stages of transition. This contrasts with overshooting familiar from the one-dimensional transitional path, which always occurs only on impact with the arrival of new information and is therefore not generated by the system's internal dynamics.

Another interesting, but seemingly underappreciated, consequence of the two-dimensional transitional path is that it implies a potentially asymmetric adjustment with respect to positive and negative shocks of equal magnitude. In the familiar case of the one-dimensional stable manifold, the transitional adjustment to a negative shock is just the mirror image to that of the corresponding positive shock. That is not necessarily the case here. We will present an example where a positive productivity shock in the technology sector leads to a monotonic adjustment in both capital and technology, whereas a subsequent reversal of that shock may be associated with highly nonmonotonic behavior.

The rest of the paper is organized as follows. Section 2 presents the general two-sector model of economic growth. The two-sectors we consider are output and a knowledge-producing sector, also referred to as technology and R&D. We begin by deriving the equilibrium conditions and by briefly characterizing the balanced growth path. In Section 3 we analyze the transitional dynamics of a one-sector non-scale model, in which the only good is final output. This serves as a starting point in that the transitional dynamics can be easily characterized and compared to the standard one-sector AK model. The speeds of adjustment in the neoclassical and in the one-sector non-scale model are shown to be similar. Section 4 lays out the formal dynamic structure of the two-sector non-scale model. Because a complete formal analysis of the fourth order system is virtually intractable, our analysis relies on numerical calibration methods, the results of which are presented in Section 5. In that Section we first identify a plausible benchmark set of parameter values. Then we proceed to examine the dynamic adjustment of the economy to various kinds of shocks and structural changes, such as variations in the parameters relating to technological change, returns to scale, preference parameters, as well as population growth.

2. A General Two-Sector Model of Growth

We begin by outlining the structure of a general two-sector non-scale model that features exogenous population growth and endogenously capital and technology. The properties of this base model have been discussed extensively in Eicher and Turnovsky (1996) and so our discussion can be brief, focusing only on those aspects that are most relevant to the dynamics. The model is general in the sense that we do not restrict the magnitude of the parameters ex ante, thus allowing us to replicate the features of a large variety of growth models.

We focus on a centrally planned economy and use social production functions in which externalities are internalized. The population, *N*, is assumed to grow at the steady rate $\dot{N}/N = n$. The objective of the planner is to maximize the intertemporal utility of the representative agent:

$$\frac{1}{1-g}\int_0^\infty (C/N)^{1-g} e^{-rt} dt \qquad r > 0; \;\; g > 0 \tag{1a}$$

where C/N denotes per capita consumption and $1/g > 0$ is the intertemporal elasticity of substitution.

The economy consists of two sectors, one produces final output, *Y*, the other technological change (new knowledge), *A*. The final good is produced utilizing the social stocks of technology, labor, *N*, and physical capital, *K*, according to[9]

[9]We use the term "social stocks" to refer to the amalgam of private stocks and those representing possible social spillovers. This allows us to specify decreasing, constant, or increasing returns to scale for F(.) without having to worry about issues pertaining to market structure. As in Mulligan and Sala-i-Martin (1993), the elasticities derived from (1b) refer to the sum of private and social elasticities.

$$Y = F[A, qN, fK] \qquad 0 \le q \le 1;\ 0 \le f \le 1 \qquad \text{(1b)}$$

The fractions of labor and capital devoted to the production of the final good are q and f, respectively.

Physical capital accumulates residually, after aggregate consumption needs, *C*, have been met.

$$\dot{K} = Y - C - d_K K \qquad \text{(1c)}$$

where d_K denotes the (constant) rate of physical capital depreciation. Technology is produced in an alternative sector in accordance with the production function:

$$\dot{A} = J[A, (1-q)N, (1-f)K] - d_A A \qquad \text{(1d)}$$

using the same three factors of production, the common stock of existing technology and the remaining fractions of labor and capital, $(1-q)$ and $(1-f)$, respectively, and depreciates at the constant rate d_A. As discussed in Eicher and Turnovsky (1996), equation (1d) encompasses a broad range of previously specified models of knowledge accumulation.[10]

The planner's problem is to maximize the intertemporal utility function, (1a), subject to the production and accumulation constraints, (1b) - (1d), and the usual initial conditions. His decision variables are: (i) the rate of per capita consumption; (ii) the fractions of labor and capital to devote to each activity; (iii) the rate of accumulation of physical capital and technology. The optimality and transversality conditions to this central planning problem can be summarized as follows:

$$C^{-g} = nN^{1-g} \qquad \text{(2a)}$$

$$nF_N = mJ_N \qquad \text{(2b)}$$

$$nF_K = mJ_K \qquad \text{(2c)}$$

[10]Eicher and Turnovsky (1996) abstract from physical depreciation. We introduce it here, since it is quite important for the purposes of numerical calibration and simulation. The structure (1b) - (1d) was originally investigated within a traditional Ramsey framework by Shell (1967).

$$F_K f + \frac{m}{n} J_K (1-f) - d_K = r - \frac{\dot{n}}{n} \qquad (2d)$$

$$\frac{n}{m} F_A + J_A - d_A = r - \frac{\dot{m}}{m} \qquad (2e)$$

$$\lim_{t\to\infty} nKe^{-rt} = \lim_{t\to\infty} mAe^{-rt} = 0 \qquad (2f)$$

where n, m are the respective shadow values of physical capital and knowledge. These conditions are standard and have been discussed in Eicher and Turnovsky (1996).

2.1. Balanced Growth Equilibrium

Before examining the dynamics, we characterize the balanced growth equilibrium. We define a balanced growth equilibrium to be a growth path along which all variables grow at constant, but possibly different, rates. In accordance with the stylized empirical facts (Romer 1989), we assume that the output/capital ratio, Y/K, is constant. Taking the differentials of the production functions (1b) and (1c), leads to the following homogeneous system of linear equations in $\hat{A}, \hat{K}, n$:

$$s_A \hat{A} + s_N n + (s_K - 1)\hat{K} = 0 \qquad (4a)$$

$$(h_A - 1)\hat{A} + h_N n + h_K \hat{K} = 0 \qquad (4b)$$

where: $s_x = F_x x/K \geq 0$ and $h_x = J_x x/J \geq 0$; $x = A, N, K$ denote the structural elasticities in the production and knowledge sectors, respectively. These two equations represent the heart of the model in that they determine the long-run sectoral balanced growth rates and are the source of their non-scale properties and policy independence. In general, these elasticities are functions of all variables in the two production functions, except in the Cobb-Douglas case, when they are exogenous constants. Eicher and Turnovsky (1996) discuss the tradeoff between: (i) the flexibility of the production function and (ii) the generality of returns to scale, consistent with non-scale growth. They show that the Cobb-Douglas imposes the fewest restrictions on sectoral returns to scale for a balanced growth equilibrium to prevail. Since we shall employ numerical computations to examine the dynamic properties of the model, henceforth we will assume Cobb-Douglas functional forms.

For $n > 0$, equations (4a) and (4b) jointly determine the rates of growth of physical capital and knowledge as functions of the population growth rate and the various production elasticities in the two sectors:

$$\hat{A} = \frac{n\left[h_N(1 - s_K) + h_K s_N\right]}{\Delta} \equiv nb_A \tag{5a}$$

$$\hat{K} = \frac{n\left[(1 - h_A) s_N + h_N s_A\right]}{\Delta} \equiv nb_K \tag{5b}$$

where $\Delta \equiv (1 - h_A)(1 - s_K) - h_K s_A > 0$. Eicher and Turnovsky (1996) show that $1 > s_K$ or $1 > h_A$ is a necessary and sufficient condition to attain positive growth rates. It is evident from (5a) and (5b) that the relative sectoral growth rates depend upon the assumed production elasticities.

2.2 Returns to Scale and Balanced Growth

Since the relationship between returns to scale and relative growth rates is of interest to us in our calibrations, we briefly review some of these relationships that were previously examined by Eicher and Turnovsky (1996). We assume that the two production functions *F* and *J* are homogeneous of degrees *k* and *a* in the three factors *A*, *N*, and *K*, separately, so that $s_A + s_K + s_N = s + s_N \equiv k$ and $h_A + h_K + h_N = r + h_N \equiv a$ Combining these definitions with the solutions (5a) and (5b), we find that relative sectoral growth rates and returns to scale are related by:

$$\hat{A} \gtrless \hat{K} \text{ according as } \frac{1-s}{s_N} \gtrless \frac{1-r}{h_N} \Leftrightarrow \frac{1-k}{s_N} \gtrless \frac{1-a}{h_N} \tag{6}$$

The effect of returns to scale on the growth rate of the endogenous factors thus depends on the quantities $(1-s)/s_N$ and $(1-r)/h_N$. They represent the ratio of deviation from constant returns to scale of the *endogenously* growing factors (*K* and *A*), to the returns to scale of the *exogenously* growing factor in the respective sectors. These quantities may either be positive, if there are decreasing returns in *K* and *A*, or negative in the case of increasing returns. Capital grows faster than technology if this ratio of returns to scale in endogenous versus exogenous factor is larger in the technology sector than it is in the output sector. This will be so if: (a) economies of scale of the endogenous factors in the output sector are sufficiently greater than they are in the technology sector, or (b) the returns to scale to labor in the output sector is sufficiently greater than those in the technology sector. In addition, output and capital grow faster than knowledge if the final goods sector is subject to increasing returns to scale in all three factors $(k > 1)$, while knowledge is subject to corresponding decreasing returns to scale $(a < 1)$. The contrary applies if these returns to scale are reversed.[11] Moreover, Eicher and Turnovsky (1996) show that it is even possible to find degrees of decreasing returns to scale in both sectors may be consistent with positive balanced growth.

3. Dynamics in the One-sector Non-scale Model

We start our examination of the dynamics by considering the simplest non-scale model, one with a distinct AK character. The main virtue of this is pedagogic, in that analytical results are easily obtained. In Section 4 below, we examine the full fledged two-sector non-scale model. However, the complexity of closed form analytical solutions in that case requires us to resort to numerical computations in order to characterize the transition paths.

The generic one-sector non-scale model can be parameterized conveniently by setting $s_A = 0,\ h_x = 0, x = A, N, K$ (no technology sector), so that output is determined by:

$$Y = N^{s_N} K^{s_K} \tag{7a}$$

Given this production function, the optimal path for aggregate consumption, obtained by differentiating (2a) and combining with (2c) and (2d) yields:

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[11]The insight that output and technology need not grow at identical rates is simple, but clearly differentiates this model from Jones (1995b). This aspect will have important implications for our simulation results. Most notably, to calibrate the model to the US economy, Jones takes the population and per capita growth rates of 2% as given. In his model this immediately yields a linear relationship between admissible magnitudes of h_A and h_N. Both elasticities are unconstrained in our framework.

$$\frac{\dot{C}}{C} = \frac{1}{g}\left(s_K N^{s_N} K^{s_K - 1} - r - (1-g)n - d_K\right) \qquad (7b)$$

while goods market clearance, (1c) is given by

$$\dot{K} = N^{s_N} K^{s_K} - C - d_K K \qquad (7c)$$

The long-run balanced growth path in this one-sector economy is given by

$$\hat{K} = \hat{C} = \left(\frac{s_N}{1-s_K}\right)n;\ \hat{K} - n = \hat{C} - n = \left(\frac{s_N + s_K - 1}{1-s_K}\right)n \qquad (8)$$

Our objective is to derive the transitional dynamics around the long-run balanced growth equilibrium (8) and to do so it is convenient to transform the system in terms of the stationary variables $k \equiv K \big/ N^{(s_N/(1-s_K))}$; $c \equiv C \big/ N^{(s_N/(1-s_K))}$. The stationary quantities k, c can be characterized as "scale adjusted" per capita quantities and in the case that the social production function, (7a), has constant returns to scale, they reduce to standard per capita quantities.

Noting that $\dot{k}/k = \dot{K}/K - (s_N/(1-s_K))n$; $\dot{c}/c = \dot{C}/C - (s_N/(1-s_K))n$, and combining with (7b) - (7c), we may express the dynamics of the system in the form:

$$\dot{k} = k^{s_K} - c - \left(\frac{s_N}{1-s_K}\right)nk - d_K k \qquad (9a)$$

$$\dot{c} = \frac{c}{g}\left\{s_K k^{s_K - 1} - r - d_K + \left[g\left(1 - \frac{s_N}{1-s_K}\right) - 1\right]n\right\} \qquad (9b)$$

The steady-state values of the transformed variables, $\tilde{k}, \tilde{c}$, are given by:

$$\tilde{k} = \left[\frac{1}{s_K}\left(r + d_K + \left(1 - g\left[1 - \frac{s_N}{1-s_K}\right]\right)n\right)\right]^{\frac{1}{s_K - 1}} \qquad (10a)$$

$$\tilde{c} = \frac{1}{s_K}\left(r + d_K(1-s_K) + \left[(1-g) + (g - s_K)\left(\frac{s_N}{1-s_K}\right)\right]n\right)\tilde{k} \qquad (10b)$$

Thus, linearizing (9a), (9b) around (10a), (10b), the transitional dynamics may be approximated by:

$$\begin{pmatrix}\dot{k}\\ \dot{c}\end{pmatrix} = \begin{pmatrix} r+(1-g)\left[1-\left(s_N/(1-s_K)\right)\right] & -1 \\ \left(s_K/g\right)(s_K-1)\tilde{c}\tilde{k}^{s_K-2} & 0\end{pmatrix}\begin{pmatrix}k-\tilde{k}\\ c-\tilde{c}\end{pmatrix} \tag{11}$$

From (11) it is clear that the system is saddlepath stable if and only if $s_K < 1$. In that case, the stable saddlepath in *c*-*k* space is described by

$$k(t)-\tilde{k} = (k_0-\tilde{k})e^{m_1 t} \tag{12a}$$

$$c(t)-\tilde{c} = \frac{s_K(s_K-1)\tilde{c}\tilde{k}^{s_K-2}}{m_1}(k(t)-\tilde{k}) \tag{12b}$$

where $m_1 < 0$ represents the stable eigenvalue and the locus is attained by an initial jump in the consumption-capital ratio, *c*. Equations (12) imply that along the stable saddlepath consumption increases with capital, just as it does in the conventional one-sector Ramsey growth model.

Since our focus is on growth rates it is convenient to focus on the transitional adjustment of the growth rate of capital, $\dot{K}/K$, which from (9) and (12) can be expressed in terms of the evolution of the state variable, *k*, in the form:

$$\frac{\dot{K}}{K}-\left(\frac{s_N}{1-s_K}\right)n = \frac{m_1}{\tilde{k}}(k-\tilde{k}) \tag{13}$$

As *k*, increases, the marginal productivity of capital declines, thereby leading to a decline in the growth rate of capital, as illustrated by the stable adjustment path illustrated in Fig. 1a.

To illustrate the dynamics and to contrast the behavior of the non-scale growth model to the standard *AK* model, consider an increase in r, the rate of time preference. In the conventional one-sector *AK* model of Barro (1990) and Rebelo (1991), for example, this causes an immediate increase in the consumption-wealth ratio and the growth rate falls instantaneously and permanently.

In the non-scale growth model the higher rate of time preference has no permanent effect on the growth rate, though there is a temporary effect. The adjustment is illustrated in Fig. 1a. The increase in the rate of time preference causes the stable locus XX to shift down. This leads to an immediate reduction in the growth rate of capital, after which it gradually increases back to its original level. Intuitively, the increase in the rate of time preference causes an immediate reduction in the shadow value of capital, causing agents to begin decumulating their stock of capital. As the capital stock declines, the marginal physical product of capital increases, causing the growth rate in the economy gradually to be restored to its steady-state balanced growth rate. It is this transitional adjustment in the capital-labor ratio, that restores the growth rate in the non-scale growth model that is absent from the more rigid technology of the AK model.

An important issue for the one-sector model concerns its speed of convergence, as parameterized by the stable eigenvalue m_1. Taking the following standard parameter values:

$$\boxed{s_N = 0.65,\ s_K = 0.35,\ r = 0.04,\ g = 1,\ d_K = 0.05,\ n = 0.128,}$$

we find that the stable eigenvalue $m_1 = -0.1058$, implying that the rate of adjustment is over 10% per annum.[12] This speed, while characteristic of other numerical simulations of the one-sector growth model, is implausibly rapid, as previous authors have noted.

4. Dynamics of a Two-Sector Model

We now turn to the dynamics of the two-sector model to ascertain qualitative insights into the transition in non-scale models. Following most of the literature, we shall assume that capital enters only the final goods sector. Thus, imposing the Cobb-Douglas specification, the production functions are of the form:

$$Y = a_F A^{s_A}[qN]^{s_N} K^{s_K} \tag{14a}$$

$$J = a_J A^{h_A}[(1-q)N]^{h_N} \tag{14b}$$

[12] In all of our simulations the choice of parameters is based on the unit time interval being one year.

where a_F, a_J represent exogenous technological shift factors to the production functions. This modifies equations (5a), (5b) such that the balanced growth rates of output, capital, technology consumption, and the respective shadow values are given by

$$\hat{A} = b_A n = \frac{h_N(1-s_K)n}{(1-h_A)(1-s_K)} = \frac{h_N n}{(1-h_A)} \tag{15a}$$

$$\hat{K} = \hat{Y} = \hat{C} = b_K n = \frac{\left[(1-h_A)s_N + h_N s_A\right]n}{(1-h_A)(1-s_K)} \tag{15b}$$

$$\hat{n} - \hat{m} = \left(b_A - b_K\right)n \tag{15c}$$

If the production function for knowledge has constant returns to scale (as we shall assume for the benchmark case in our numerical analysis), the growth rate of the final output sector becomes:

$$\hat{K} = \hat{Y} = \hat{C} = b_K n = \frac{\left(s_N + s_A\right)n}{(1-s_K)} \tag{17b'}$$

and depends upon production elasticities alone. Thus $b_K \gtrless 1$ according to whether there are increasing or decreasing returns to scale in producing output.[13]

To derive the dynamics about the balanced growth path we define the following stationary variables: $y \equiv Y/N^{b_K}$; $k \equiv K/N^{b_K}$; $c \equiv C/N^{b_K}$; $a \equiv A/N^{b_K}$; $j \equiv J/N^{b_K}$; $q \equiv n/mN^{(b_A - b_K)}$. These are analogous to k and c introduced in Section 3, except that labor is now scaled in accordance with the equilibrium growth factor in that sector. For convenience, we shall refer to $y, k, c,$ and a as per capita quantities. Using these variables allows us to rewrite per capita output and technology as

$$y = a_F q^{s_N} a^{s_A} k^{s_K} \tag{18a}$$

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[13]This technology is somewhat more general than Jones, who in terms of our notation specifies that the production functions for final output is constant returns to scale in physical capital and knowledge-adjusted labor, *AN*: $Y = (AqN)^s K^{1-s;}$ i.e. $s_A = s_N = 1 - s_K \equiv s$ and $J = ((1-q)N)^{h_N} A^{h_A}$; $h_K = 0$ resulting in a growth rate of $\hat{K} - n = \hat{A} = h_N n/(1-h_A)$. The striking feature of the equilibrium growth rate in the Jones model, is that per capita consumption, per capita output and capital, and technology must all grow at a common rate determined by: (i) the growth rate of labor, and (ii) the elasticities of labor and knowledge in the R&D sector alone; the characteristics of the final output sector are irrelevant.

$$j = a_J (1-q)^{h_N} a^{h_A} \tag{18b}$$

The optimality conditions then enable the dynamics to be expressed in terms of these scale-adjusted variables, as follows. First, the labor allocation condition (2b) can be expressed in the form

$$a_F q s_N q^{s_N - 1} a^{s_A} k^{s_K} = a_J h_N (1-q)^{h_N - 1} a^{h_A}$$

enabling us to solve for the fraction of labor allocated to output:

$$q = q(q, a, k); \quad \P q / \P q > 0, \ \P q / \P k > 0, \ sgn(\P q / \P a) = sgn(s_A - h_A) \tag{19}$$

Intuitively, an increase in the relative value of capital attracts labor to the output (capital-producing) sector. An increase in the stock of physical capital raises the productivity of labor in producing final output and thus also attracts labor to that sector. By contrast, an increase in technology, being an input in both sectors, will raise the productivity of labor in both sectors and cause a net shift in employment toward the sector in which knowledge has the greater production elasticity (is more productive).

The shadow values of capital and technology, determined by (2c) and (2d) can be expressed as:

$$\frac{\dot{n}}{n} = r + d_K - s_K a_F q^{s_N} a^{s_A} k^{s_K - 1} \tag{20a}$$

$$\frac{\dot{m}}{m} = r + d_A - a_J (1-q)^{h_N} a^{h_A - 1} \left[h_A + \frac{s_A h_N}{s_N} \frac{q}{1-q} \right] \tag{20b}$$

Taking the time derivative of (2a) and combining with (20a), the growth rate of aggregate consumption is given by

$$\frac{\dot{C}}{C} = \frac{1}{g} \left[s_K a_F q^{s_N} a^{s_A} k^{s_K - 1} - \left((1-g)n + d_K + r \right) \right] \tag{20c}$$

Using these first order conditions the dynamic system can be expressed in terms of the redefined stationary variables by:

$$\dot{k} = k\left[q^{s_N} a_F a^{s_A} k^{s_K-1} - b_K n - d_K - \frac{c}{k}\right] \tag{21a}$$

$$\dot{a} = a\left[(1-q)^{h_N} a_J a^{h_A-1} - b_A n - d_A\right] \tag{21b}$$

$$\dot{q} = q\left\{a_J (1-q)^{h_N} a^{h_A-1}\left[h_A + \frac{s_A h_N}{s_N}\frac{q}{1-q}\right] - s_K a_F q^{s_N} a^{s_A} k^{s_K-1} - (b_A - b_K)n - (d_A - d_K)\right\} \tag{21c}$$

$$\dot{c} = \frac{c}{g}\left\{s_K a q^{s_N} a^{s_A} k^{s_K-1} - (r + d_K) + [g(1-b_K) - 1]n\right\} \tag{21d}$$

where q is determined by (19). To the extent that were are interested in the growth rates of capital and knowledge, themselves, they are given by $\dot{K}/K = \dot{k}/k + b_K n$; $\dot{A}/A = \dot{a}/a + b_A n$.

The steady state to this system, denoted by "~" superscripts, can be summarized by:

$$\frac{\tilde{y}}{\tilde{k}} - \frac{\tilde{c}}{\tilde{k}} = b_K n + d_K \tag{22a}$$

$$\frac{\tilde{j}}{\tilde{a}} = b_A n + d_A \tag{22b}$$

$$\left[h_A + \frac{s_A h_N}{s_N}\frac{\tilde{q}}{1-\tilde{q}}\right]\frac{\tilde{j}}{\tilde{a}} - b_A n - d_K = s_K \frac{\tilde{y}}{\tilde{k}} - b_K n - d_K \tag{22c}$$

$$s_K \frac{\tilde{y}}{\tilde{k}} - b_K n - d_K = r + (1-g)(1-b_K)n \tag{22d}$$

together with the two production functions (18a, 18b), and the labor allocation condition (19). These seven equations determine the steady-state equilibrium in the following sequential manner. First, (22d) determines the output-capital ratio, so that the long-run net return to capital equals the rate of return on consumption. Notice that the elasticity g raises or lowers the long-run output-capital ratio depending upon whether $b_K \gtrless 1$. If the production function *J* has constant returns to scale, $\tilde{y}/\tilde{k}$ is independent of the production elasticities, h_N, h_A, of the knowledge-producing sector. Likewise, (22b) yields the *gross* equilibrium growth rate of knowledge, $\tilde{j}/\tilde{a} = \tilde{J}/\tilde{A}$, in terms of the returns to scale, b_A, and the rates of population growth and depreciation. Having obtained the output-capital ratio, (22a) determines the consumption-capital ratio consistent with the growth rate of capital necessary to equip the growing labor force and replace depreciation. Given $\tilde{y}/\tilde{k}$ and $\tilde{j}/\tilde{a}$, (22c) implies the sectoral allocation of labor, $\tilde{q}$, at which the rates of return to investing in the two sectors are equalized.

Given $\tilde{q}$, and $\tilde{j}/\tilde{a}$, the production function for knowledge determines the stock of knowledge, $\tilde{a}$, while the production function for output then yields the stock of capital, $\tilde{k}$. Finally, having derived $\tilde{q}, \tilde{a}, \tilde{k}$, the labor allocation condition determines the long-run equilibrium relative shadow value of the two assets, $\tilde{q}$.

Linearizing around the steady state denoted by $\tilde{k}, \tilde{a}, \tilde{q}, \tilde{c}$, the dynamics may be approximated by the following fourth order system:

$$\begin{pmatrix} \dot{k} \\ \dot{a} \\ \dot{q} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} s_K \frac{\tilde{y}}{\tilde{k}} + s_N \frac{\tilde{y}}{\tilde{q}} \frac{\P q}{\P k} - b_K n - d_K & s_A \frac{\tilde{y}}{\tilde{a}} + s_N \frac{\tilde{y}}{\tilde{q}} \frac{\P q}{\P a} & s_N \frac{\tilde{y}}{\tilde{q}} + \frac{\P q}{\P q} & -1 \\ \frac{-h_N \tilde{j}}{1-\tilde{q}} \frac{\P q}{\P k} & (h_A - 1)\frac{\tilde{j}}{\tilde{a}} - \frac{h_N \tilde{j}}{1-\tilde{q}} \frac{\P q}{\P a} - b_A n - d_A & -\frac{h_N \tilde{j}}{1-\tilde{q}} \frac{\P q}{\P q} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ \frac{\tilde{c} s_K}{g\tilde{k}}\left(a_{11} - \frac{\tilde{x}}{\tilde{k}}\right) & \frac{\tilde{c} s_K a_{12}}{g\tilde{k}} & \frac{\tilde{c} s_K a_{13}}{g\tilde{k}} & 0 \end{pmatrix} \begin{pmatrix} k - \tilde{k} \\ a - \tilde{a} \\ q - \tilde{q} \\ c - \tilde{c} \end{pmatrix} \quad (23)$$

where we define:

$$a_{31} \equiv \left[\frac{d}{\tilde{a}} a_{21} + \frac{s_A h_N}{s_N} \frac{\tilde{j}}{a(1-q)^2} \frac{\P q}{\P k} - \frac{g}{\tilde{x}} a_{41}\right]\tilde{q}; \quad a_{32} \equiv \left[\frac{d}{\tilde{a}} a_{22} + \frac{s_A h_N}{s_N} \frac{j}{\tilde{a}(1-\tilde{q})^2} \frac{\P q}{\P a} - \frac{g}{\tilde{x}} a_{42}\right]\tilde{q},$$

$$a_{33} \equiv \left[\frac{d}{\tilde{a}} a_{23} + \frac{s_A h_N}{s_N} \frac{j}{\tilde{a}(1-\tilde{q})^2} \frac{\P q}{\P k} - \frac{g}{x} a_{43}\right]\tilde{q}, \text{ and } d \equiv h_A + \frac{s_A h_N}{s_N}\left(\frac{\tilde{q}}{1-\tilde{q}}\right)$$

It is straightforward (but tedious) to show that the determinant of this matrix is proportional to $(h_A - 1)(s_K - 1)$. Imposing the condition that $s_K < 1,\ h_A < 1$ implies that the determinant is positive which means that it has either 0, 2, or 4 positive roots. Under weak conditions it is possible to rule the first case. However, we are unable to rule out the exploding system with four unstable roots, although none of simulations yielded such a case. Indeed, all of our numerous calibrations that generated meaningful steady-state equilibria yielded two-dimensional saddlepaths with 2 positive and 2 negative eigenvalues. Since the system features two state variables, *k* and *a*, and two jump variables, *c* and *q*, the equilibrium yields a unique stable adjustment path.

4.1 Characterization of Transitional Dynamics

Henceforth we assume that the stability properties are ensured so that we can denote the two stable roots by m_1, m_2, with $m_2 < m_1 < 0$. The key variables of interest are physical capital, and technology. The generic form of the stable solution for these variables is given by:

$$k(t) - \tilde{k} = B_1 e^{m_1 t} + B_2 e^{m_2 t} \tag{24a}$$

$$a(t) - \tilde{a} = B_1 n_{21} e^{m_1 t} + B_2 n_{22} e^{m_2 t} \tag{24b}$$

where B_1, B_2 are constants and the vector $\begin{pmatrix}1 & n_{2i} & n_{3i} & n_{4i}\end{pmatrix}'$ $i = 1,2$ (where the prime denotes vector transpose) is the normalized eigenvector associated with the stable eigenvalue, m_i. The constants, B_1, B_2, appearing in the solution (24) are obtained from initial conditions, and depend upon the specific shocks. Thus suppose that the economy starts out with given initial stocks of capital and knowledge, k_0, a_0 and through some policy shock converges to $\tilde{k}, \tilde{a}$. Setting $t = 0$ in (24a), (24b) and letting $d\tilde{k} \equiv \tilde{k} - k_0$, $d\tilde{a} \equiv \tilde{a} - a_0$, B_1, B_2 are given by:

$$B_1 = \frac{d\tilde{a} - n_{22} d\tilde{k}}{n_{22} - n_{21}}; \quad B_2 = \frac{n_{21} d\tilde{k} - d\tilde{a}}{n_{22} - n_{21}} \tag{25}$$

In studying the dynamics, we are interested in characterizing the slope along the transitional path in *a*-*k* space. In general, this is given by:

$$\frac{da}{dk}=\frac{B_1 n_{21} m_1 e^{m_1 t}+B_2 n_{22} m_2 e^{m_2 t}}{B_1 m_1 e^{m_1 t}+B_2 m_2 e^{m_2 t}} \tag{26}$$

and is time varying. Note that since $0 > m_1 > m_2$, as $t \to \infty$ this converges to the new steady state along the direction $(dz/dk)_{t\to\infty} = n_{21}$, for all shocks. The initial direction of motion. obtained by setting $t = 0$ in (26) and depends upon the source of the shock.

It is convenient to express the dynamics of the state variables in phase-space form:

$$\begin{pmatrix}\dot{k}\\ \dot{a}\end{pmatrix}=\begin{pmatrix}\frac{(m_1 n_{22}-m_2 n_{21})}{n_{22}-n_{21}} & \frac{(m_2-m_1)}{n_{22}-n_{21}}\\ \frac{(m_2-m_1)n_{21}n_{22}}{n_{22}-n_{21}} & \frac{(m_2 n_{22}-m_1 n_{21})}{n_{22}-n_{21}}\end{pmatrix}\begin{pmatrix}k-\tilde{k}\\ a-\tilde{a}\end{pmatrix} \tag{27}$$

By construction, the trace of the matrix in (27) $= m_1 + m_2 < 0$ and the determinant = $m_1 m_2 > 0$, so that (27) describes a stable node. The dynamics expressed in (24) and (27) are in terms of the scale adjusted per capita quantities, from which the growth rates of capital and knowledge, themselves, these can be derived.

Equations (24a) and (24b) highlight the fact that with the transition path in *k* and *a* being governed by two stable eigenvalues, the speeds of adjustment for capital and knowledge are neither constant nor equal over time. In addition, with output being determined by capital and technology, the transition of output is also not constant over time, but is a simple composite of the transition characteristics of *a* and *k* as determined in (27).

Fig. 1b illustrates the phase diagram in the case that prevailed in all of our simulations where the $\dot{a} = 0$, $\dot{k} = 0$ loci are both upward sloping, with the latter having the steeper slope. Six types if transition paths are illustrated, indicating the distinct possibilities of overshooting. If we start from a point such as E or F, both capital and technology converge monotonically, which implies that the transition path for output is also monotonic. Note that this does not imply that the *speeds* of adjustment are either constant or identical for capital and technology, Hence the speed of transition of output also varies over time. If we start from points A or C (B or D) it is clear that technology (capital) overshoots its long run equilibrium level during transition. This leads to strong variation in the rates of growth of technology (capital) during the transition, featuring both above and below long run growth rates during the adjustment. The asymmetry of the transitional paths to positive and negative shocks occurs if the positive shock occurs when the economy starts from a point such as E, relative to the new equilibrium, while the reverse negative shock starts if the new equilibrium is at a point such as B relative to the original equilibrium.[14] An example of this is provided in Fig. 5a and Fig. 5e.

Much of the literature emphasizes the speed of convergence; see Barro and Sala-i-Martin (1992) and Ortigueira and Santos (1997). In the neoclassical growth model, for example, or in the two-sector Lucas model, where the stable manifold is a one-dimensional locus, the transitional adjustment of a typical variable, x, is described by $\dot{x} = m(x(t) - \tilde{x})$, so that the speed of adjustment, $\dot{x}/\left(x(t) - \tilde{x}\right) = m$, is parameterized unambiguously by the magnitude of the unique stable eigenvalue.

By contrast, in the present example where the stable transitional path is a two-dimensional locus, the speed of convergence in general varies over time and across variables. Although each specific speed and corresponding transition path may be informative, it is nevertheless desirable to have one comprehensive measure that summarizes the speed of convergence of the overall economy. For this purpose the percentage change in the Euclidean distance:

$$V(t) \equiv \sqrt{(k(t) - \tilde{k})^2 + (a(t) - \tilde{a})^2}$$

serves as a natural summary measure of the speed of convergence. Thus

$$\frac{\dot{V}(t)}{V(t)} = \left(\frac{(k(t) - \tilde{k})^2}{(k(t) - \tilde{k})^2 + (a(t) - \tilde{a})^2} \right) \left(\frac{\dot{k}(t)}{(k(t) - \tilde{k})} \right) + \left(\frac{(a(t) - \tilde{a})^2}{(k(t) - \tilde{k})^2 + (a(t) - \tilde{a})^2} \right) \left(\frac{\dot{a}(t)}{(a(t) - \tilde{a})} \right) \quad (28)$$

and is seen to be a direct generalization of the one-dimensional measure. In that case, all variables converge at the same rate, and (28) reduces to the (single) eigenvalue. In the present example, (28) indicates that at any instant of time the generalized speed of convergence is a weighted average of the

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[14]We should emphasize the reason why this may occur is because the $\dot{k} = 0$, $\dot{a} = 0$ lines that identify the various regions in the phase diagram Fig. 1b are subject to nonparallel shifts between equilibria.

speeds of convergence of the two stocks, the weights being the relative square of their distance from equilibrium. It is straightforward to establish that $\lim_{t\to\infty}\left(\dot{V}(t)/V(t)\right)=m_1$, so that asymptotically the system will converge to the new equilibrium at the rate of the slower growing stable eigenvalue.

5. Numerical Analysis of Transitional Paths

We turn now to the numerical analysis of the transitional paths. There are several objectives that we seek to achieve. First, we employ standard parameter values to establish a benchmark economy. This simulated economy provides insights into how well the general non-scale model replicates key variables of actual economies. We focus on three aspects: *growth rates*, *output shares*, and *speeds of convergence*. Second, we conduct sensitivity analysis, to test the robustness of our results to shocks in the underlying parameters. Finally, we characterize the transitional paths of both the levels and growth rates of capital and technology, and introduce our Euclidean distance measure of the overall economy-wide speed of convergence.

5.1 Benchmark Economy

To establish benchmark values for our economy we employ values for our fundamental parameters that are essentially identical to those suggested by previous calibration exercises; see, for example, Prescott (1986), Lucas (1988), Ortigueira and Santos (1997), and Jones (1995b):

Production parameters:	$a_F = 1,\ s_N = 0.6,\ s_K = 0.3,\ s_A = 0.2;\ \ a_J = 1,\ h_N = 0.5,\ h_A = 0.5$
Preference parameters:	$r = 0.04,\ g = 1.5$
Depreciation and population parameters:	$d_K = 0.05,\ d_A = 0.015,\ n = 0.0128$

The economy is one in which the production of final output is subject to mildly decreasing returns to scale in capital and labor, but subject to mildly increasing returns to scale with the inclusion of knowledge. The production of technology is subject to constant returns to scale. The rate of time preference is 4 per cent, while the intertemporal elasticity of substitution is 0.67. Physical capital is assumed to depreciation at 5 percent, while knowledge depreciates at a slower rate of 1.5 percent. Population is assumed to grow at 1.28 percent, the current U.S. growth rate. The only area of ambiguity is the specification of the production function for research. Given our reading of the data, we felt most comfortable assuming constant returns to scale in research.[15] All assumptions on the magnitudes of parameters and returns to scale are challenged in our sensitivity analysis below. Our initial objective is simply to establish a "consensus benchmark."

We group the resulting endogenous variables into three groups. The *balanced growth rates* of capital (output), and technology; *key equilibrium ratios*, including the output-capital ratio, the share of consumption in output, and the share of labor employed in the output sector; the *convergence speed*. All turn out to be remarkably plausible.

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[15] The empirical literature on research functions is sparse, especially if one requires separate elasticities for labor and technology. Adams (1990) and Caballero and Jaffee (1993) are examples of thorough empirical investigations that are ultimately unsuccessful in reporting separate elasticities for labor and technology. Kortum (1993) derives values of about .2 by extrapolating results from aggregate patent data. Jones and Williams (1995) obtain estimates between .5 and .75, these however, are a function of their assumed rate of growth and the assumed share of technology in research. In summary, the magnitude of the elasticities seem to be clustered around .5, and contained with a range of .1 to .9. Beyond that, hopes to obtain exact estimates are unrealistic, since all depends on the specification of the research function in the regression and on the type of R&D data employed.

Growth Rates	$\hat{K} = 0.0146;\ \hat{A} = 0.0128,\ \frac{\tilde{j}}{\tilde{a}} = 0.0278$, $(b_K = 1.143,\ b_A = 1)$
Equilibrium Ratios	$\frac{\tilde{Y}}{\tilde{K}} = 0.352;\ \frac{\tilde{C}}{\tilde{Y}} = 0.816;\ \tilde{q} = 0.964$
Convergence Speeds	$m_1 = -0.0239,\ m_2 = -0.1053$

Given the slight increasing returns in the final good sector, the growth rate of capital and output of 1.46 percent exceeds that of technology, which grows at the rate of population growth (see equation 6). The capital-output ratio is approximately 3, while around 81 percent of output is devoted to consumption. Slightly more than 96 percent of the work force is employed in the output sector, with the balance of around 4 percent employed in producing knowledge. As indicated in Section 4, the larger of the two stable eigenvalues, m_1, implies that the system adjusts asymptotically at an annual rate of about 2.4 percent, again consistent with the empirical evidence.

5.2 Alternative Specifications

The benchmark is specific to our assumptions on the fundamental parameters and to determine the robustness of its characteristics requires a sensitivity analysis. This serves two additional purposes. First, it shows the wide range of values that are consistent with balanced growth paths (as pointed out in Section 3). Second, the parametric changes provide us with examples of the types of shocks that generate distinct qualitative transition paths, the general form of which was outlined analytically in Fig. 1b.

We conduct a number of experiments in which we choose to vary one or more parameters contained in the following three categories: (i) productivity shocks to output (reported in Table 2), (ii) productivity shocks to knowledge (in Table 3), (iii) changes in structural parameters (Tables 4-6). Since our analytical results in Section 3 highlighted that the structural elasticities are of key importance in the non-scale model, group (iii) is further subdivided into two sub-categories. First, we examine variations in these parameters while retaining the returns to scale assumptions of the benchmark model (Tables 4 and 5). Second, we allow for variations in the returns from scale (Table 6), to address the relationships summarized in equation (6).

Aside from the tables that report the changes in the key variables, we also provide the locus of the transition paths for technology and capital (levels and growth rates) and the economy-wide Euclidean speed of convergence (Figs. 2-8).

5.3 Balanced Growth Rates

Two issues are of interest when investigating long-run growth rates in a non-scale model. First, we verify the invariance of long-run growth rates with respect to parameters that are known to influence long-run growth rates in previous endogenous growth models. Tables 1 - 6 document that changes in key productivity or preference parameters, such as a_F, a_J, r have no influence on long-run growth rates. This is a manifestation of the non-scale aspects of the model, since these can be viewed as embodying scale effects. Instead, from Tables 1 - 6, we verify that the long-run growth rates are determined exclusively by the magnitudes of the assumed production elasticities in technology and output, in conjunction with the population growth rate; equation (5) implies that changes in n, increase the growth rates in both sectors proportionately. Per capita growth rates in the benchmark economy are relatively low (zero for knowledge). This is a direct consequence of the chosen production specifications, and Table 1a shows that it is easy to increase these rates by increasing the external effects in knowledge (R&D) slightly.

Second, we provide examples for the unique feature of the model where capital (output) and technology grow at differential rates along the balanced growth path. The analytics in Section 3 established a general relationship between returns to scale, the endogenous and exogenous factors' structural elasticities and the relative magnitude of the growth rates. From equation (6), the steady-state capital and output growth rate of 1.46 percent in the benchmark economy exceeds the growth rate of knowledge and population, due to the increasing returns in production and constant returns in R&D. Alternatively, if we assume greater returns to scale in the knowledge sector (Table 6, row 3 is the only example) we find that the growth rate of technology exceeds that of output and capital.

5.4 Output-Capital, Consumption-Output, Employment Ratios

A surprising feature of the model is not only how well the capital, consumption and labor shares in output reflect real world magnitudes, but also how robust the results are to significant changes in the underlying parameters. With only a few exceptions, all simulations produce an output-capital ratio of about 0.35, a consumption share in output of about 80 percent, and 96 percent share of the labor force employed in the output sector.

The exceptions to the above values are intuitive. Higher output-capital ratios are generated by high rates of population growth (2.5% in Tables 2 and 3), high rates of time preference (10% in Tables 2 and 3). Also, the greater the returns to scale in the technology sector and the greater the share of technology in the R&D sector relative to the output sector, the higher the output-capital ratio (Tables 5 and 1a). The consumption to output ratio is sensitive to the rate of time preference, and we show that the specified increase in the rate of time preference raises the consumption to output ratio by about 10 percent (Tables 2 and 3). The C/Y ratio is also sensitive to the rate of depreciation of capital (Table 4). The share of labor employed in the final good sector, q, is robust, remaining around 96 percent across changes in the fundamental parameters, a_F, a_J, r, n. This share is, however, sensitive to changes in the returns to scale, and especially to the share of technology in the final output and technology sectors (Table 6, 5 and 1b), which increases the return to technology and induces the employment of a larger share of labor in that sector.

5.5 The Speeds of Convergence

5.5.1 Asymptotic Convergence

As indicated in Section 4, the larger stable eigenvalue, m_1, determines the asymptotic speed of adjustment. In the benchmark economy this convergence speed is around 2.4 percent, consistent with the empirical evidence. One striking feature of the benchmark results is that the introduction of the knowledge-producing sector dramatically slows the asymptotic speed of convergence from over 10 percent (in the one-sector model) to about 2 percent once knowledge production is considered.

This crucial result is remarkably robust across variations in parameters and policy shocks. Tables 1 through 6 show that the asymptotic speed of adjustment ranges from around 0.8 to 5.6 percent for the cases when the shares of labor in the R&D sector are dramatically decreased or increased (Tables 5 and 6). In general, all other variations of the underlying base parameters, a_F, a_J, r, n, yield a range of the speed of convergence of between 1.6 and 3.3 percent, which represents a remarkably tight fit.

5.5.2 Transition speeds

To obtain a complete picture of the transition speed, it is important to look beyond the asymptotic speed of adjustment. For this purpose we consider our comprehensive Euclidean measure of convergence (Figs. 2b-8b), from which two important observations follow. First the speed of adjustment is not constant, since the fourth order system possesses two stable roots. Second, the adjustment toward the steady state might involve accumulation of technology accompanied by decumulation of capital, as the transitional path loops around, so either one measure of convergence would be misleading.

The figures show that the speed of transition never exceeds 10 percent, even in the initial phases of transition despite the fact that we allow for significant production and technology shocks. Frequently, our simulations produced transition speed around 3% initially. Interestingly enough, even the Euclidean adjustment measure need not be monotonic (Figs. 5g and 7b) if either capital or technology contracts excessively during transition and then recovers.

5.6 Transitional Adjustment Paths

The adjustment paths for capital and technology are illustrated in Figs 2a - 8a, Fig. 5e by the lines with arrows. They turn out to be remarkably different across changes in parameters. This highlights the crucial difference between steady states and transition in the non-scale model. We present a number of examples of possible transition paths that are generated by alternative classes of parametric changes

Productivity Increase in Final Goods Sector: The dynamic adjustment to a simple increase in the productivity in the final sector ($a_F = 1$ to $a_F = 2$ Fig. 2a) is an example of the path described by a transition from (A) in Fig. 1b. Essentially same shape of the transition path applies if we also lower the intertemporal elasticity of substitution (Fig. 3a), by increasing g from 1.5 to 5. In the latter case, however, the higher rate of return to consumption, and the resulting increase in steady-state consumption, leads to less expansion in the long-run capital stock and prevents the stock of knowledge from being restored to its original level.

The intuition underlying these two transition path is as follows. In both cases, the increase in productivity of the output sector attracts resources to that sector and away from the knowledge-producing sector. The initial accumulation of the per capita capital stock is accompanied by a reduction in knowledge to a degree that its rate of falls short of the depreciation rate. The decline in knowledge and the increase in capital raises the relative return to investing in knowledge, relative to the return on capital. Thus, after following a path of initially declining knowledge and increasing capital, the return to investing in knowledge rises sufficiently to induce the central planner to begin reinvesting in the accumulation of knowledge. The speed of capital accumulation slows dramatically during the last stages of transition as it declines to its steady state growth rate, while the speed of technology accumulation increases to return to its long run growth rate.

Fig. 4 illustrates the case where the productivity increase is accompanied by in the population growth rate from 1.28% to 2.5%. It highlights how the transition is now distinctly different from the previous cases and is an example for transition path (B) in Fig. 1b. In contrast to Figs. 2 and 3 where the stock of knowledge overshoots during the transition, it now declines monotonically, while the stock of capital now overshoots its long-run response. Capital is being accumulated during the early and middle stages of transition and then decumulated during the latter stages of the transition. As in Figs. 2a and 3a, the productivity shock to the final output sector induces an initial reduction in investment in knowledge and increase in investment in capital. As k increases and a declines, the productivity of knowledge rises, but now by an insufficient amount to offset the higher population growth rate, so that a declines monotonically. Likewise, the accumulation of capital reduces the productivity of capital such that given the higher population growth rate, it eventually declines as well.

It is important to note that technology and capital grow at different speeds, both compared to each other and over time. Most interestingly, in the case of a productivity shock alone, and also when this is accompanied by an increase in the intertemporal elasticity of substitution, the growth rate of technology declines initially, and then increases *above* its long run level before approaching its stationary state from above (Figs. 3c, 3c). This sectoral and intertemporal variation in adjustment speeds highlights the importance of having a summary measure of the aggregate rate of convergence, such as the distance measure $\dot{V}(t)/V(t)$.

Productivity Increase in Knowledge Sector: The second group of experiments consists of various parametric changes associated with changes in the productivity of the knowledge sector, $a_J = 1$ to $a_J = 2$. The parameter changes are summarized in Table 3, while Figs. 5 and 6 illustrate the corresponding transitional dynamic adjustments. This group of simulations provides a particularly instructive example of the significance of two stable roots and the importance of two-dimensional transition paths.

Analogous to the productivity shock in the output sector which increases the steady-state stock of capital, we now find that the productivity shock in the knowledge sector raises the equilibrium stock of knowledge, $\tilde{a}$. However, now we find an additional effect. Since the production function for output, (20a) implies that an increase in $\tilde{a}$ raises the productivity of physical capital, capital must increase as well, in order for its average product, $\tilde{y}/\tilde{k}$, to remain constant. Thus in contrast to a productivity shock in the output sector, an increase in the productivity of the knowledge-producing sector leads to long-run increases in the stocks of *both* knowledge and physical capital, although the increase is heavily biased toward the latter.

The transitional adjustments are markedly different from our previous cases. An *increase* in the productivity in the knowledge sector causes monotonic adjustment, in contrast to the overshooting during the transition in the case when the productivity in the final goods sector rises. The adjustment path when only the productivity of the research sector is increased (Fig. 5a) serves as an example for a transition from point E in Fig. 1b.

The increase in productivity in the technology sector attracts resources to that sector, so that the growth rate of knowledge initially rises (Figs. 5c, 6c). Since physical capital is not an input into knowledge production, even its initial decline (as in Fig. 6a), does not have an adverse effect on the productivity of new knowledge. The growth rate of capital falls below that of technology, due to the fact that the expansion in capital occurs as a result of the higher stock of knowledge, which takes time to accumulate. In the intermediate term, however, the acceleration in the rate of capital peaks and then declines back to its long run value in what is a very slow transition (Figs. 5d, 6d).

The initial decline of the capital stock in Fig. 6a, when the technology shock to knowledge is accompanied by a higher intertemporal elasticity of substitution, generates an example of (D) in Fig. 6a. The intuition for the initial decline is as before. With $b_K > 1$, the decline in the elasticity of intertemporal substitution raises the long-run rate of return on consumption, and induces a long-run substitution toward consumption and away from the accumulation of both physical capital and knowledge; see (22d). While the shift toward consumption reduces the long-run stocks of both technology and capital, it is sufficient to cause overshooting during the transition adjustment phase.

To highlight the role of the two-dimensional transition path, the panel on the right hand side of Fig. 5 illustrates the case of a fall in knowledge-producing productivity from $a_J = 1$ to $a_J = 0.5$. The contrast in the transitional path in this case from the corresponding increase (given in the left hand side panel) highlights how different the speed of and transition adjustment may be, depending on the initial conditions and depending on the type of shock. Initially one might think that halving the productivity would simply imply a reverse monotonic transition to doubling it, as in Fig. 5a. But that is not the case, as the transition path is now similar to (B) in Fig. 1b. The growth rate of capital overshoots its long run growth rate significantly, and leads to a non monotonic adjustment not only of capital but also of the aggregate speed of adjustment as given in Fig. 5f.

Returns to Scale: The final set of transition results reported here pertains to returns to scale. Thus far, we have held the returns to scale fixed at 1.1 in the case of output, and to 1 for the production of knowledge. Table 6 considers variations in the degrees of returns to scale. First, we subject both sectors to mildly diminishing returns to scale. As pointed out in Section 3, the model is extremely flexible and allows for a wide range of parameters consistent with decreasing returns *in both sectors* to generate balanced growth.

With decreasing returns in output and technology both the long-run capital and knowledge grow slower than the population growth rate. The transition from the benchmark case to diminishing returns in both sectors involves substantial reductions in the capital stock and knowledge (Fig. 7). Both these reductions occur monotonically at a balanced rate and the overall speed of adjustment in the economy is virtually constant at around 2.7 percent. Constant returns to both sectors lead to similar patterns of long-run responses and transitional adjustment paths. Both are examples of (F)-type adjustments in Fig. 1b.

Increasing returns to scale in both sectors leads to long-run growth rate of capital and labor which greatly exceed the growth rate of population. The transition is essentially the reverse of the first case two cases presented in Table 3. Figure 8 is an example of (E)- type adjustment in Fig 1b. Both capital and technology increase monotonically. Because of the extreme returns to scale being assumed in the production of knowledge, technology experiences a bigger boost in its growth rate during early stages than does capital. The transition to the steady-state rate of convergence of around 1.23 percent occurs quite rapidly. The higher productivity of knowledge in producing new knowledge also attracts more labor to that sector.

Decreasing returns to scale in output accompanied by increasing returns to scale in knowledge leads to long-run per capita growth in both capital and knowledge, consistent with the formal propositions of Eicher and Turnovsky (1996). Because the returns to scale being assumed for knowledge dominate, the long-run response and transitional adjustment are essentially similar to the case of uniformly increasing returns.

6. Conclusions

The determinants of long run growth rates and the characteristics of economies' transitions to their balanced growth paths are central to theories of economic growth. The traditional one-sector neoclassical growth model and the two-sector endogenous growth models imply a uniform speed of convergence, both through time and across different sectors of the economy. As a consequence, much empirical research has focused on monotonic convergence in per capita output only, with the implication that this was an adequate representation of the economy-wide speed of convergence. Recent work has called into question the empirical validity of this approach, suggesting that a more flexible view to convergence issues is required; see e.g. Barro and Sala-i-Martin (1995, Chapter 8) and Bernard and Jones (1996a, 1996b) . In this paper we have provided such an approach by introducing a hybrid two-sector non-scale growth model that possesses features of both neoclassical and endogenous growth models. We show that the non-scale model easily accounts for reasonable speeds of convergence, in contrast to neoclassical models, but it can also account for conditional convergence, in contrast to endogenous growth models.

One of the advantages of the non-scale growth model resides in its generality. Balanced growth is consistent with increasing and/or decreasing returns in one or both sectors. In addition, the model allows for differential long-run sectoral growth rates. Previous models rely on knife edge conditions to attain their balanced growth paths. The neoclassical model required constant returns to scale in capital and exogenously growing labor, while endogenous growth models require constant returns to scale in the factors being accumulated.

With respect to the transitional characteristics, the key difference from previous growth models is that non-scale models raise the dimensionality of the dynamics. In contrast to AK models that always lie on their balanced growth path, the one-sector non-scale model is characterized by a one-dimensional stable locus. In contrast to two-sector Lucas-type endogenous growth models, the stable manifold of the two-sector non-scale model is not one, but two dimensional. This enriches the transitional dynamics, allowing for variable speeds of convergence over time and across different sectors in the economy. With the wide range of adjustment paths that are now possible it is necessary to consider a single summary measure of the economy-wide adjustment speed and for this purpose we adopted the percentage rate of change of the conventional Euclidean distance measure.

The price of the added flexibility of the higher order dynamic system is that to analyze its dynamics requires the use of numerical simulation methods. Our simulations of the non-scale model constitute the first comprehensive dynamic analysis of this class of models, although several examples of their balanced growth properties have been presented earlier. Parameter values of generally accepted magnitudes generate surprisingly plausible results, easily replicating the salient features of advanced economies, a finding that is robust to extensive sensitivity analysis. We also provide examples of the important implications of two-dimensional transition paths, which imply, for example, that a positive exogenous technology shock may generate qualitatively different adjustment paths than does a negative shock of the same magnitude.

In our view, the general non-scale model, by allowing for endogenous capital and technology, non-constant returns to scale, and for distinct and variable speeds of adjustment, may provide for a more general approach for future empirical investigations into cross country convergence. In addition, the framework provides a promising starting point for analyzing other aspects of the dynamics of non-scale economies. An interesting extension may be recover the distinction between private and social factors and returns. Then one could then shed light on the role of fiscal policy on the characteristics of the transition path.

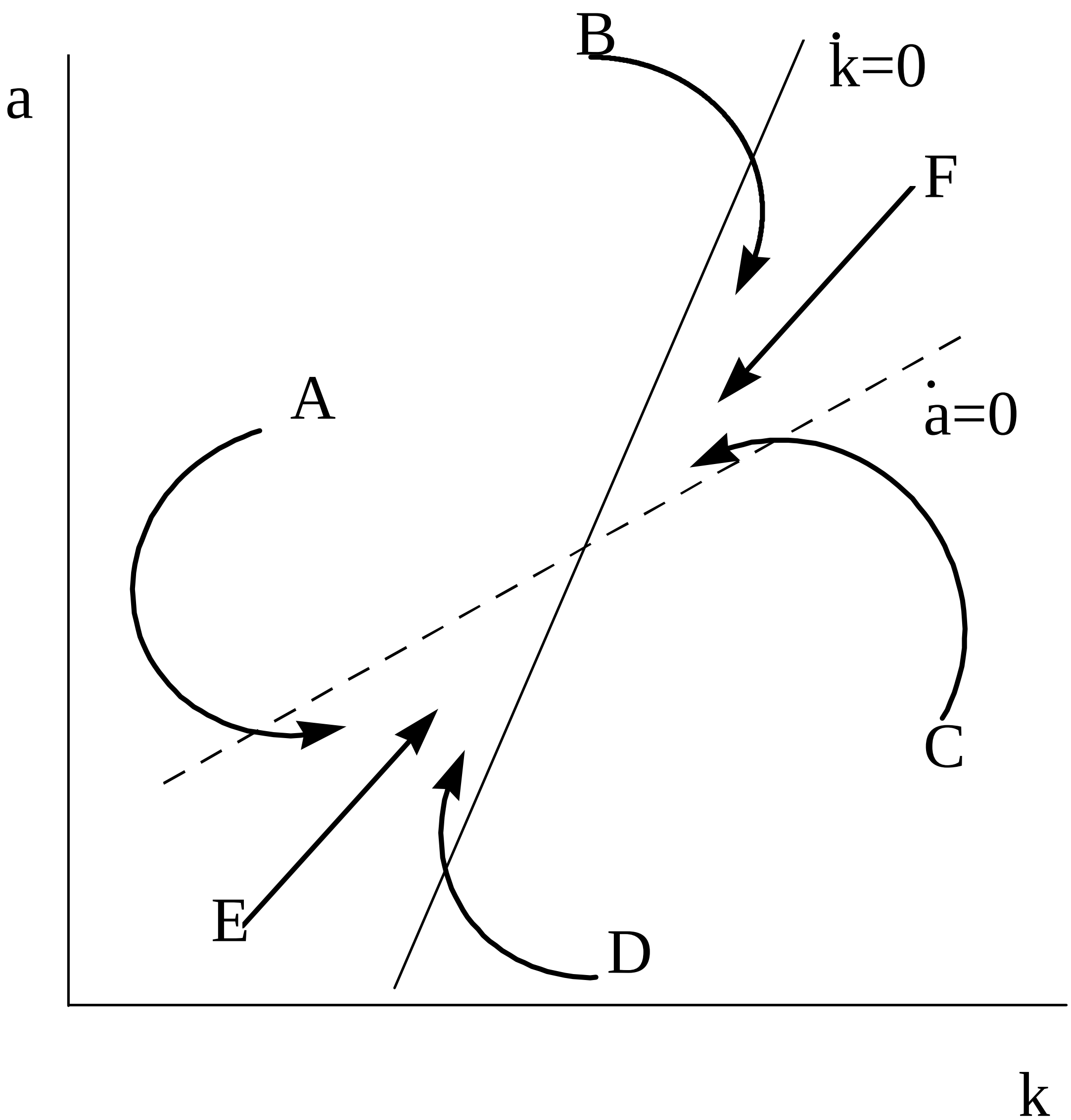


Fig 1b: Two dimensional transition paths in two-sector non-scale models

Benchmark Parameters

Production parameters: $a_F = 1,\ s_N = 0.6,\ s_K = 0.3,\ s_A = 0.2;\ \ a_J = 1,\ h_N = 0.5,\ h_A = 0.5$

Preference parameters: $r = 0.04,\ g = 1.5$

Depreciation and population parameters: $d_K = 0.05,\ d_A = 0.015,\ n = 0.0128$

Table 1
Benchmark Equilibrium Values

$\hat{K}$	$\hat{A}$	$\frac{\tilde{j}}{\tilde{a}}$	$\frac{\tilde{Y}}{\tilde{K}}$	$\frac{\tilde{C}}{\tilde{Y}}$	$\tilde{q}$	m_1	m_2	$\tilde{k}$	$\tilde{a}$
0.0146	0.0128	0.0278	0.352	0.816	0.964	-0.0239	-0.1053	12.90	46.31

Table 1a
Alternative Benchmarks

	$\hat{K}$	$\hat{A}$	$\frac{\tilde{j}}{\tilde{a}}$	$\frac{\tilde{Y}}{\tilde{K}}$	$\frac{\tilde{C}}{\tilde{Y}}$	$\tilde{q}$	m_1, m_2	$\tilde{k}$	$\tilde{a}$
$s_N = .7, s_K = .3,$ $s_A = .6, h_N = .6,$ $h_A = .8$	0.0457	0.0384	0.0534	0.5072	0.8113	0.8331	-0.01669 -0.18726	6244	10696
$s_K = .4, s_A = .5,$ $h_N = .65$	0.0266	0.0166	0.0316	0.309	0.8518	0.8856	-0.0283 -0.1084	189.0	59.59

Table 2
I. Increase in a_F from 1 to 2

	$\hat{K}$	$\hat{A}$	$\frac{\tilde{j}}{\tilde{a}}$	$\frac{\tilde{Y}}{\tilde{K}}$	$\frac{\tilde{C}}{\tilde{Y}}$	$\tilde{q}$	m_1, m_2	$\tilde{k}$	$\tilde{a}$
$a_F = 2$	0.0146	0.0128	0.0278	0.352	0.816	0.964	-0.0239 -0.1053	34.72	46.31
$g = 5$ $a_F = 2$	0.0146	0.0128	0.0278	0.373	0.827	0.966	-0.0195 -0.0623	31.53	44.13
$n = .025$ $a_F = 2$	0.0286	0.025	0.04	0.401	0.804	0.952	-0.0329 -0.1233	25.16	30.09
$r = 0.10$ $a_F = 2$	0.0146	0.0128	0.0278	0.552	0.883	0.976	-0.0250 -0.1552	16.53	31.65

Table 3
Increase in a_J from 1 to 2

	$\hat{K}$	$\hat{A}$	$\frac{\tilde{j}}{\tilde{a}}$	$\frac{\tilde{Y}}{\tilde{K}}$	$\frac{\tilde{C}}{\tilde{Y}}$	$\tilde{q}$	m_1, m_2	$\tilde{k}$	$\tilde{a}$
$a_J = 2$	0.0146	0.0128	0.0278	0.352	0.816	0.964	-0.0239 -0.1053	19.16	185.3
$a_J = .5$	0.0146	0.0128	0.0278	0.352	0.816	0.964	-0.0239 -0.1053	8.6788	11.5784
$g = 5$ $a_J = 2$	0.0146	0.0128	0.0278	0.373	0.827	0.966	-0.0195 -0.0623	17.4	176.5
$n = .025$ $a_J = 2$	0.0286	0.025	0.04	0.401	0.804	0.952	-0.0329 -0.1233	13.89	120.4
$r = 0.10$ $a_J = 2$	0.0146	0.0128	0.0278	0.552	0.883	0.976	-0.0250 -0.1552	19.13	126.6

Table 4
Variations in Depreciation Rates

	$\hat{K}$	$\hat{A}$	$\frac{\tilde{j}}{\tilde{a}}$	$\frac{\tilde{Y}}{\tilde{K}}$	$\frac{\tilde{C}}{\tilde{Y}}$	$\tilde{q}$	m_1, m_2	$\tilde{k}$	$\tilde{a}$
$d_A = .03$	0.0146	0.0128	0.0428	0.352	0.816	0.935	-0.0333 -0.1100	11.65	35.58
$d_K = .025$	0.0146	0.0128	0.0278	0.268	0.852	0.942	-0.0221 -0.0810	21.38	75.46

Table 5
Variations in Knowledge and Output Technologies

	$\hat{K}$	$\hat{A}$	$\frac{\tilde{j}}{\tilde{a}}$	$\frac{\tilde{Y}}{\tilde{K}}$	$\frac{\tilde{C}}{\tilde{Y}}$	$\tilde{q}$	m_1, m_2	$\tilde{k}$	$\tilde{a}$
$h_N = .25$ $h_A = .75$	0.0146	0.0128	0.0278	0.352	0.816	0.981	-0.0089 -0.1019	84.94	32273
$h_N = .75$ $h_A = .25$	0.0146	0.0128	0.0278	0.352	0.816	0.950	-0.0508 -0.1182	7.084	5.949
$s_N = .4$ $s_A = .4$	0.0146	0.0128	0.0278	0.352	0.816	0.900	-0.0222 -0.1091	67.49	129.7
$s_N = .4$ $s_K = .2$ $s_N = .5$	0.0144	0.0128	0.0278	0.526	0.878	0.878	-0.0219 -0.1548	49.55	158.2

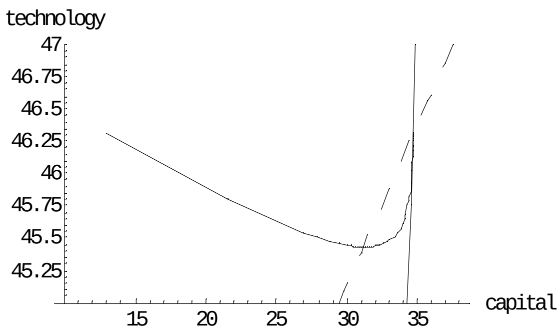
Table 6
Variations in Returns to Scale

	$\hat{K}$	$\hat{A}$	$\frac{\tilde{j}}{\tilde{a}}$	$\frac{\tilde{Y}}{\tilde{K}}$	$\frac{\tilde{C}}{\tilde{Y}}$	$\tilde{q}$	m_1, m_2	$\tilde{k}$	$\tilde{a}$
DRS in Y and A $s_N = .55, s_K = .25$ $s_A = .15, h_A = .4$	0.0115	0.0107	0.0257	0.404	0.848	0.973	-0.0269 -0.1169	6.113	22.27
DRS in Y, IRS in A $s_N = .55, s_K = .25$ $s_A = .15, h_A = .9$	0.0222	0.064	0.0790	0.468	0.846	0.919	-0.0129 -0.1429	33.53	363187
IRS in Y and A $h_N = .9$	0.01755	0.0230	0.0380	0.3664	0.8156	0.9201	-0.0566 -0.1640	6.898	7.3149
IRS in Y and A $h_A = .9$	0.0292	0.064	0.079	0.424	0.813	0.905	-0.0123 -0.1357	151.4	796942
$s_A = s_N = .67$ $s_K = 1 - s_N$	0.0244	0.0128	0.0278	0.401	0.814	0.903	-0.0217 -0.1305	333.6	125.3
CRS in Y and A $s_A = .15$	0.0128	0.0128	0.0278	0.343	0.817	0.987	-0.0246 -0.1001	7.147	23.75

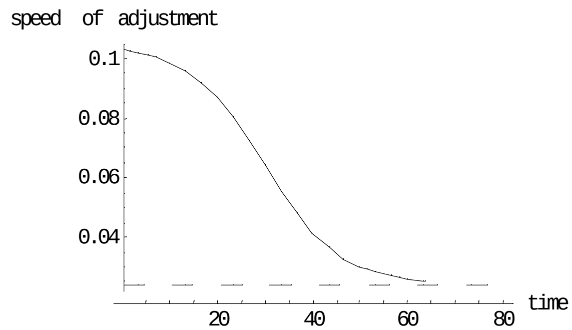
Figure 2

Increasing α_F from 1 to 2

a) Transition Paths

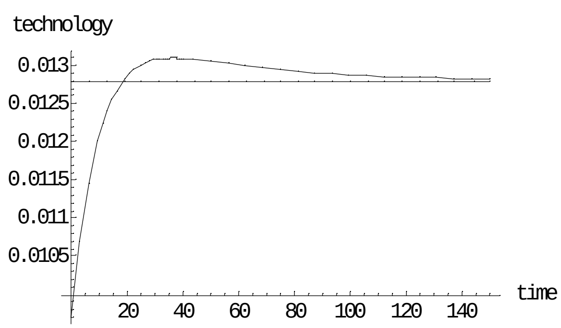


b) Aggregate Speed of Adjustment



Time profile of growth rates

c) Growth Rate of Technology



d) Growth Rate of Capital

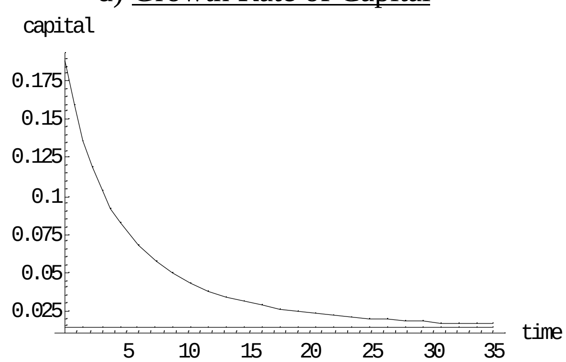
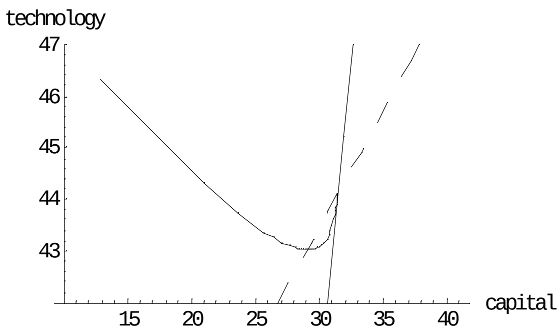


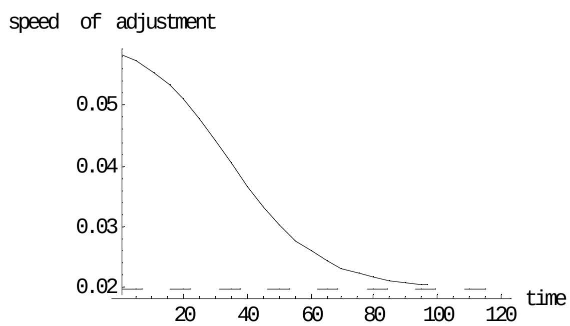
Figure 3

Increasing α_F from 1 to 2 and γ from 1.5 to 5

a) Transition Paths

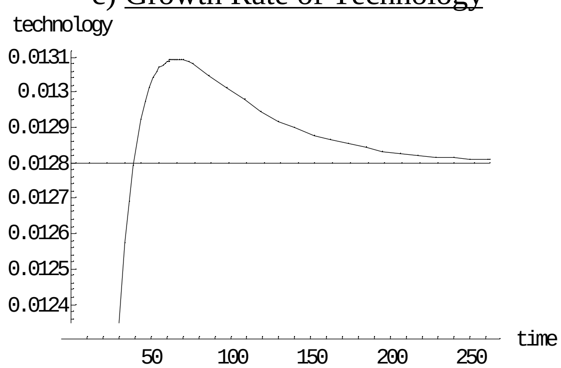


b) Aggregate Speed of Adjustment



Time profile of growth rates

c) Growth Rate of Technology



d) Growth Rate of Capital

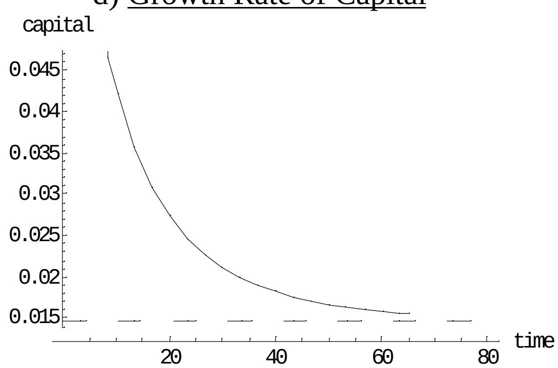
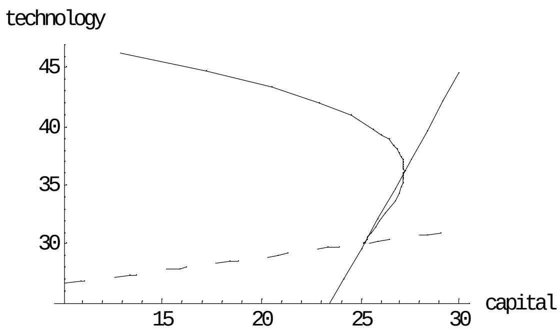


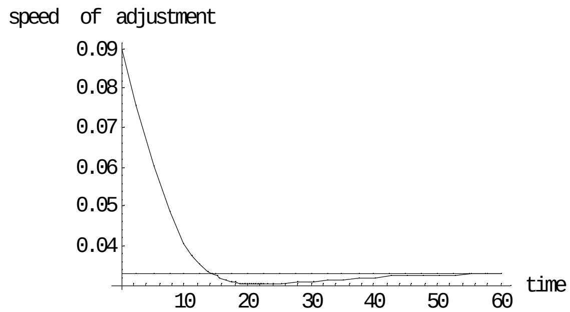
Figure 4

Increasing α_F from 1 to 2 and *n* from .0128 to .025

a) Transition Paths

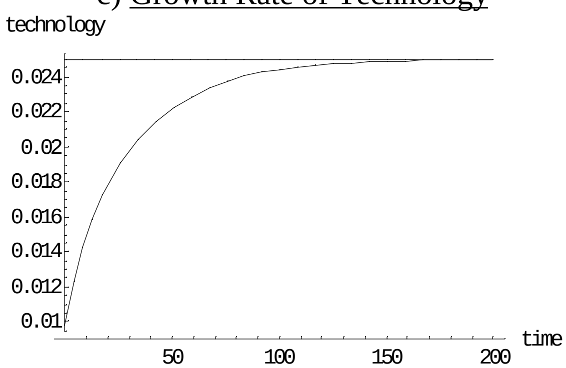


b) Aggregate Speed of Adjustment



Time profile of growth rates

c) Growth Rate of Technology



d) Growth Rate of Capital

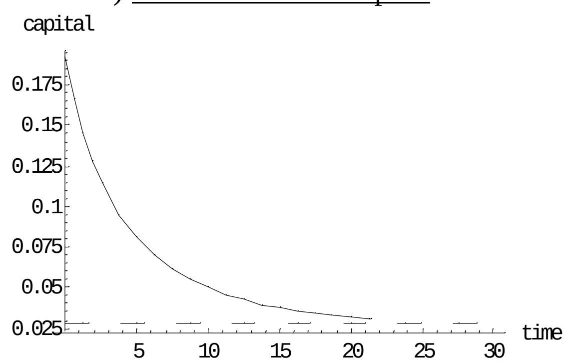
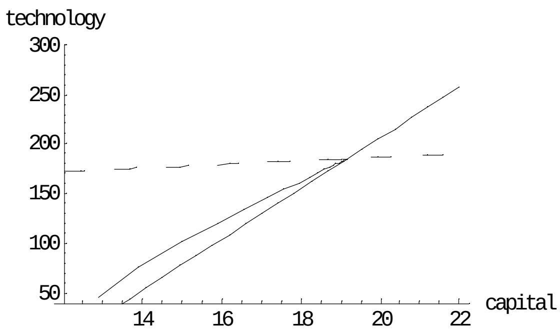


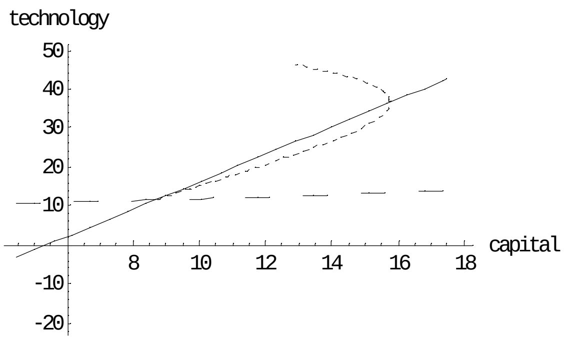
Figure 5

Comparison Of Adjustments To Changes From $\alpha_J = 1$

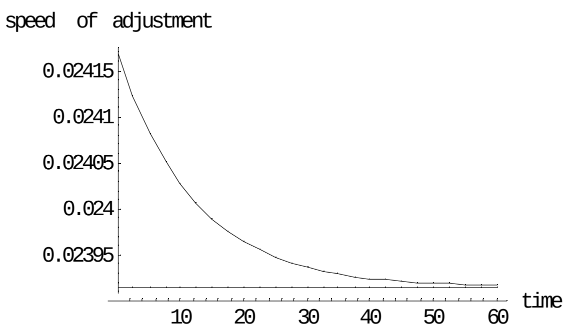
a) Transition Path $\alpha_J = 2$



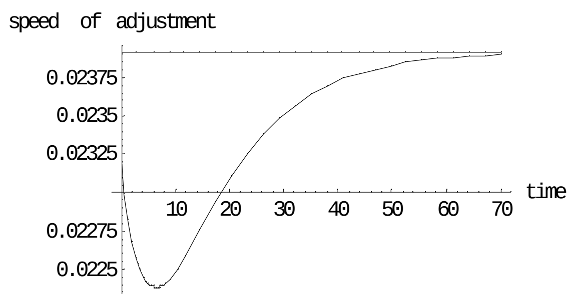
e) Transition Path $\alpha_J = 0.5$



b) Aggregate Speed of Adjustment $\alpha_J = 2$

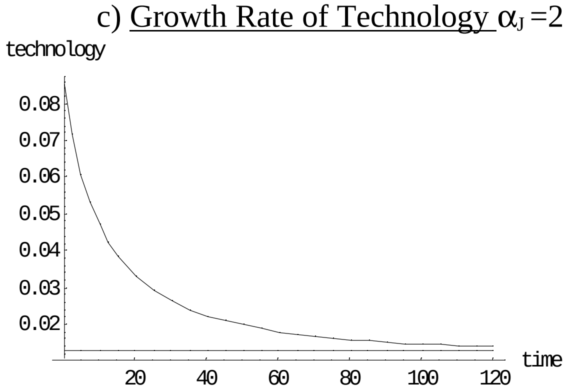


f) Aggregate Speed of Adjustment $\alpha_J = 0.5$

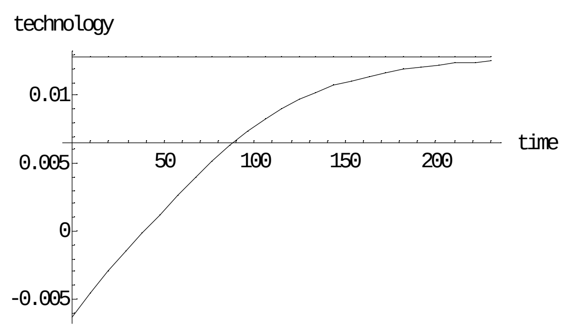


Time profile of growth rates for changes from $\alpha_J = 1$

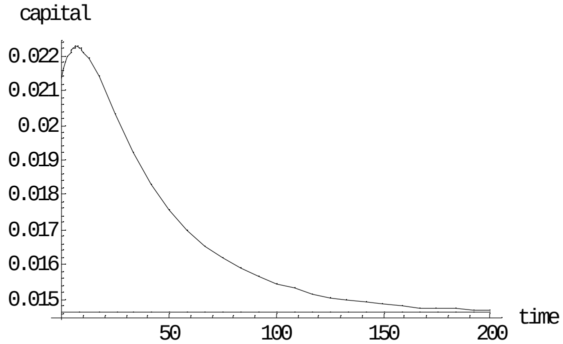
c) Growth Rate of Technology $\alpha_J = 2$



g) Growth Rate of Technology $\alpha_J = 0.5$



d) Growth Rate of Capital $\alpha_J = 2$



h) Growth Rate of Capital $\alpha_J = 0.5$

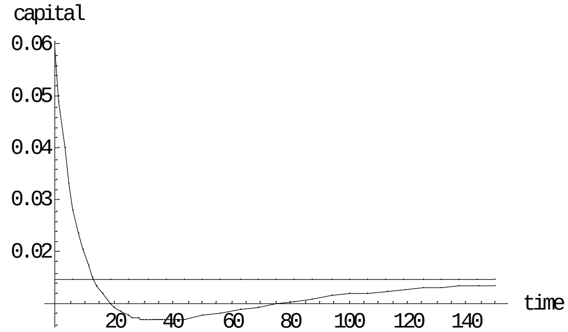
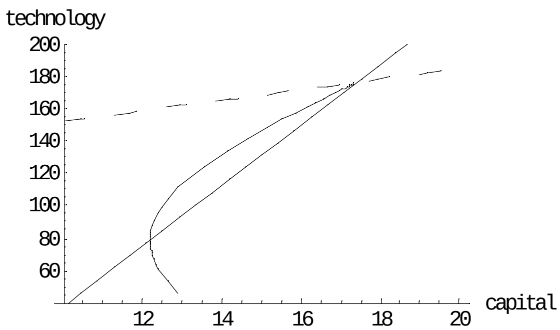


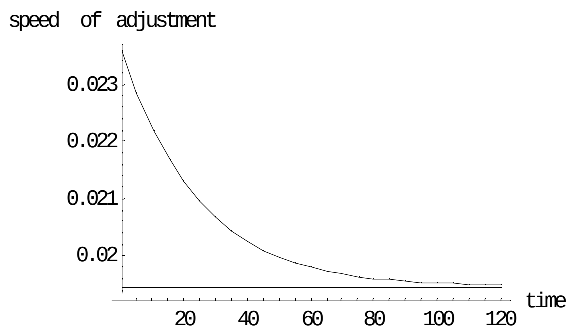
Figure 6

Increasing α_J from 1 to 2 and γ from 1.5 to 5

a) Transition Paths

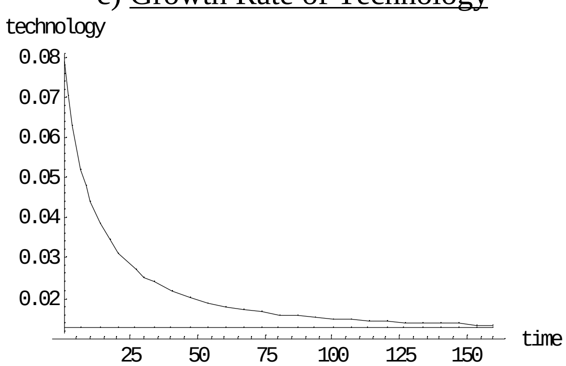


b) Aggregate Speed of Adjustment



Time profile of growth rates

c) Growth Rate of Technology



d) Growth Rate of Capital

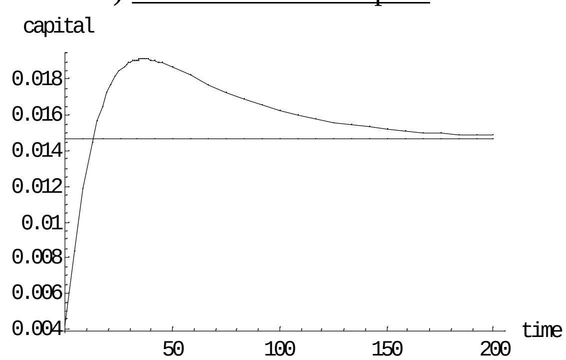
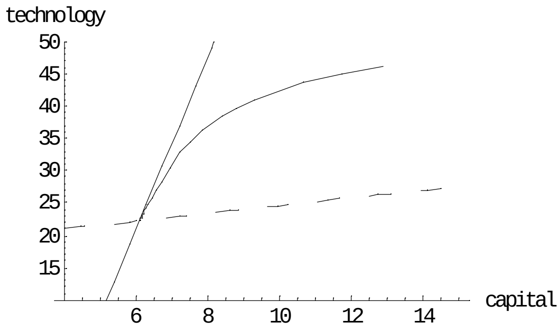


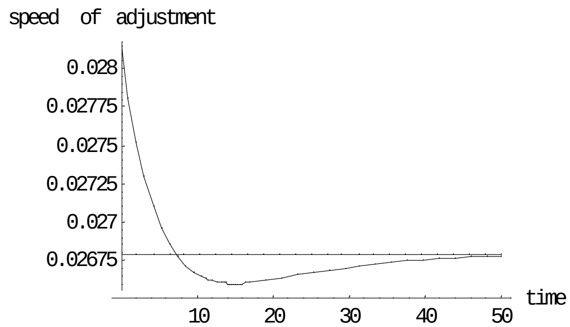
Figure 7

Decreasing Returns in both sectors

a) Transition Paths

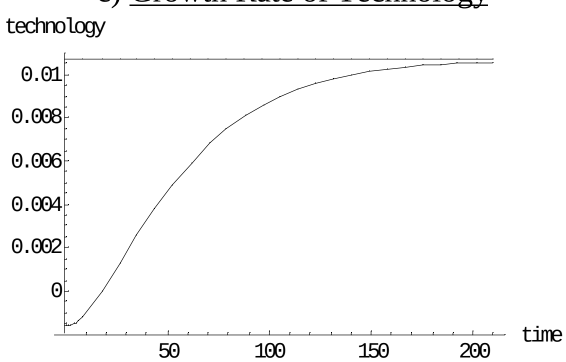


b) Aggregate Speed of Adjustment



Time profile of growth rates

c) Growth Rate of Technology



d) Growth Rate of Capital

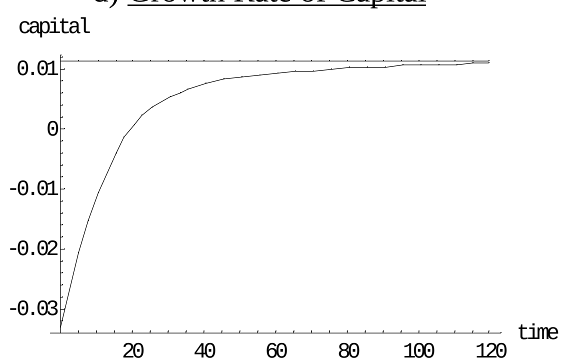
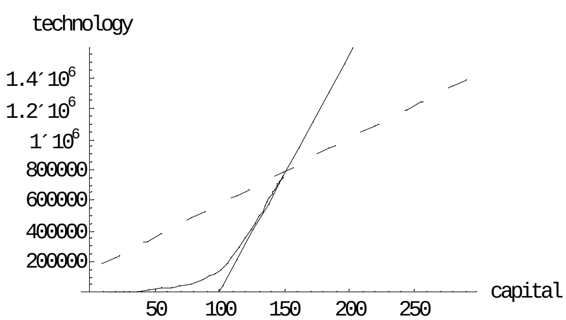


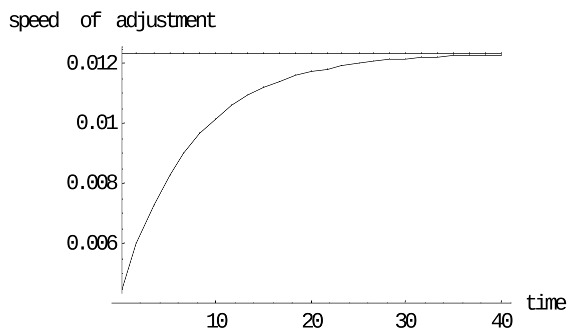
Figure 8

Increasing Returns in both sectors

a) Transition Paths

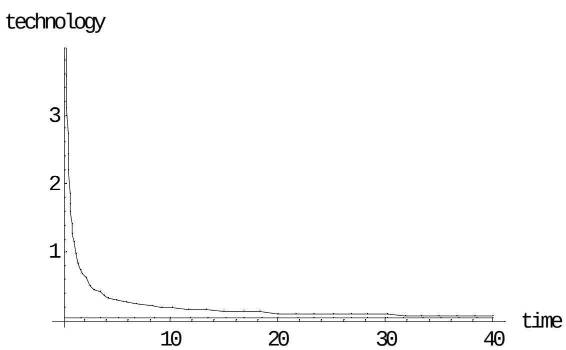


b) Aggregate Speed of Adjustment



Time profile of growth rates

c) Growth Rate of Technology



d) Growth Rate of Capital