Scale, Congestion and Growth

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1. Introduction

The scope and effectiveness of public expenditures has been a major policy issue as governments throughout the world design stimulus packages, as in Japan, and face difficult choices to meet the EMU criteria in Europe. Empirical research investigating the impact of public investment on the productivity of private capital and output was stimulated by Aschauer (1989a, 1989b), who found the effects to be strongly positive. Aschauer's results have generated debate, and the consensus view is that public investment has a significant positive impact on the productivity of private capital.[1]

The theoretical analysis of the productivity of public investment has focused on its impact on the accumulation of private capital and output in the economy. Typically this is done by introducing government expenditure as an argument in the production function to reflect an externality in production. Aschauer and Greenwood (1985) and Barro (1989) are early examples to follow this approach in a neoclassical Ramsey framework. These models have the characteristic that the economy converges either to a stationary state (if population is fixed) or to a balanced growth path along which the economy grows at the exogenous growth rate of population. Fiscal policy therefore has no long-run effect on the equilibrium growth rate. By contrast, new growth theory attributes a prominent role to fiscal policy as a determinant of long-run growth. In particular, productive government expenditures have been shown to have important effects on long-run growth rates, and in Barro's (1990) well known AK model the growth-maximizing share of productive government expenditure is shown to be welfare-maximizing.[2]

The concept of public expenditures is, however, diverse. Starting with Borcherding and Deacon (1972) and Bergstrom and Goodman (1973) there is a vast empirical literature that examines the “publicness” of such expenditures. In general, public goods vary extensively in terms of their level of services, spanning the range between the polar cases of pure private and pure public goods; see, Oakland (1972). But the treatment of public expenditures in growth models is limited. Barro (1990)

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[1]A documentation of the relevant empirical literature is provided in the survey by Gramlich (1994).
[2]The coincidence between welfare maximization and growth maximization obtained in the Barro model is not robust, however. see [Futagami, Morita, and Shibata, 1993] for an example where it ceases to hold.

and Futagami, Morita, and Shibata (1993) treat government expenditure as a pure public good, and in so doing, they fail to take account of the congestion typically associated with public goods. However, as Barro and Sala-i-Martin (1995) have argued, almost all public services are characterized by some degree of congestion, so that the pure public good should be viewed as only a benchmark.[3] These considerations suggest that congestion is an important consideration in assessing the effect of public expenditures on economic growth.

Attempts to analyze the effects of congestion on economic growth have been limited in various dimensions. Drawing on the public goods literature, Edwards (1990) proposes a number of specifications for the congestion function, though the empirical evidence supporting these alternative formulations is mixed. In the context of a growth model, two notions that are both applicable and relevant are those of *relative* congestion and *aggregate* (*absolute*) *congestion*.[4] The former specifies the level of services derived by an individual from the provision of a public good in terms of the usage of his individual capital stock relative to the aggregate capital stock. An example of this is the service provided by highway usage. Unless an individual drives his car, he derives no service from a publicly provided highway, and in general the services he derives depends upon his own usage relative to that of others in the economy, as total usage contributes to congestion. Aggregate congestion (sometimes called *crowding*) indicates how aggregate usage of the service alone influences the services received by an individual. Police protection may serve as an example of this. In principle, people always enjoy this service, independent of their own actions, though the amount of service they may actually derive varies inversely with aggregate activity and the demands this places on the limited resources devoted to this public service.

Existing applications of congestion to growth theory introduce it into AK models; see Barro and Sala-i-Martin (1992, 1995), Glomm and Ravikumar (1994, 1997) and Turnovsky (1996, 1997). However, the AK model features two limitations that restrict its flexibility to examine the relationship between congestion and growth. First, sustained ongoing growth is possible only as long as the

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[3]Thompson (1976) has argued that even national defense, sometimes cited as the purest of public goods, is not congestion-free.

[4]Edwards (1990) also considers local congestion and clubs, aspects that we do not address here.

production functions satisfy the restrictive knife-edge condition of constant returns to scale in the accumulated factors [Solow 1994]. Because of this, previous papers either restrict themselves to the special case of relative congestion, or alternatively assume a very specific degree of aggregate congestion. The second limitation is that the equilibrium growth rates generated by these models are, in general, sensitive to the scale of the economy [Jones 1995].[5] When public services contain any element of non-rivalry, scale effects become crucial determinants of growth rates, unless one is willing to make the assumption that the externality is exactly proportional to each individual [Lucas 1988]. Empirical evidence does not support the presence of these scale effects.[6]

Since congestion is inherently related to the size, (scale) of the economy, it is important to consider the effects of congestion on growth within a model that imposes less restrictive assumptions on the technology. One model which is particularly convenient for this purpose is the non-scale endogenous growth model recently developed by Jones (1995), Young (1998), and Eicher and Turnovsky (1997). This model has important advantages of flexibility. First, it allows for constant equilibrium growth in the presence of any arbitrary degree of returns to scale (consistent with stability), not just if the restrictive knife-edge conditions of the AK model hold. Second, it allows us to introduce population growth, and more general specifications of externalities. Third, the flexibility of the non-scale model permits us to introduce both relative and absolute congestion simultaneously. And most importantly, non-scale models are appealing from an empirical perspective, since they support the evidence suggesting that distortionary taxes have no effects on long-run growth rates; see Stokey and Rebelo (1995) and Jones (1995).

The non-scale model allows us to distinguish precisely how different types of congestion and scale affect growth, the equilibrium growth path, and its transitional dynamics. Indeed, we find that the two measures of congestion impact the economy in profoundly different ways. First, aggregate congestion reduces the effective productivity of the aggregate capital stock, whereas relative congestion reduces the effective productivity of the aggregate supply of labor. Second, numerical

[5]The scale could be measured by the size of the population, for example.
[6]See e.g. Jones (1995) and Backus, Kehoe and Kehoe (1992).

simulations suggest that the equilibrium capital-output ratio is more sensitive to relative congestion than it is to aggregate congestion. Consequently, the equilibrium consumption-output ratio declines with the former, but increases with the latter. These contrasts extend to the transitional dynamics, which reveal that absolute congestion has a stabilizing effect on the economy, whereas relative congestion is shown to be irrelevant in ensuring stability. Finally, we find that relative congestion retards the transitional speed of adjustment; aggregate congestion has precisely the opposite effect.

Both forms of congestion and any deviation of government expenditure from its optimum generate externalities. We show that these can be fully corrected, both along short-run transitional adjustment paths, and in steady state, by a simple time-invariant income tax. This contrasts with other dynamic models, where the replication of the first-best optimum requires a time-varying tax rate along the transitional path; see Turnovsky (1997). Our results are similar to the implications of Oakland (1972) whose empirical work finds that whether or not taxes can fully defray the congestion costs depends on the cost curves and the congestion functions. We highlight how the degree of "publicness" (relative and aggregate congestion) is a crucial determinant of optimal fiscal policy.

The rest of the paper is organized as follows. In Sections 2 and 3, we discuss the decentralized economy. Section 2 focuses on the specification of congestion and its impact on the individual's production, while Section 3 characterizes the implications for the aggregate equilibrium and growth rate. Section 4 discusses optimal tax policy and the impact of congestion on its design to replicate the first-best outcome of a central planner. Section 5 supplements our analytical results with some numerical calculations, thereby highlighting some important differences between the two forms of congestion. Section 6 reviews our main findings, while details of stability and transitional dynamics are provided in the Appendix.

2 The Decentralized Economy

2.1 Production and Technology

Consider an economy comprised of N individual units (households). Each individual i produces output, Y_i, using his inelastically supplied labor input, L_i, his capital stock, K_i, together with public services that are provided by the government, G_s, in accordance with the Cobb Douglas production function:

$$Y_i = a' L_i^{1-s_K} K_i^{s_K} G_s^{s_G} = a K_i^{s_K} G_s^{s_G} \tag{1}$$

where $a \equiv a' L_i^{1-s_K}$. The production function has constant returns to scale in the two private inputs, L_i and K_i and increasing returns in all three factors. At this point we impose no restrictions on the productive elasticities, s_K, s_G, other than $1 \geq s_K \geq 0$, $s_G \geq 0$, so that all three factors have positive marginal products. Population grows at the exogenous rate

$$\dot{N}/N = n \tag{2}$$

The services derived by the individual from government expenditure are represented by:

$$G_s = G(K_i/K)^{q_R} K^{-q_A} \quad q_R > 0, \; q_A > 0 \tag{3}$$

where $K \equiv NK_i$ denotes the aggregate capital stock and G denotes the aggregate flow of government expenditure. Equation (3) incorporates the potential for the public good to be associated with alternative types and degrees of congestion, and is a convenient specification in that it is consistent with sustained, endogenous growth and population growth. This representation of public services specified in (3) can be classified into 3 categories.[7]

The first category occurs if $q_A = q_R = 0$, so that $G_s = G$, in which case the public good is available equally to each individual, independent of the usage of others. Government services are both non-rival and non-excludable, and therefore constitute a pure public good in the original Samuelson (1954) sense. Few examples of such pure public goods exist, so that this case should be treated primarily as a benchmark. As either q_R or q_A deviate from zero, the non-excludable public

[7]Some specifications of congestion express (3) in terms of individual and aggregate output, rather than capital. This does not lead to any substantive changes in the analysis.

service loses its non-rival nature, and as this occurs, the level of public services enjoyed by the individual is tied to his individual and/or the aggregate usage of capital.

The second category encompasses public goods that allow for pure *relative* congestion, parameterized by $q_R > 0,\ q_A = 0$ in equation (3). With relative congestion the agent can maintain a fixed level of government services, G_s, from a constant level of government expenditure, *G*, if and only if the usage of his own individual capital stock increases in proportion to the usage of the aggregate capital stock. Congestion increases if aggregate usage increases relative to individual usage. We have already noted that highways may be subject to this form of congestion; only if individuals use their cars in proportion to the aggregate usage does their level of highway services remain unchanged. A special case of interest arises if $q_R = 1$. This case may be referred to as proportional (relative) congestion. The public good is like a private good in that since $K_i/K = 1/N$ the individual receives his proportional share of services, $G_s = G/N$. The case $0 < q_R < 1$ can be interpreted as describing partial relative congestion, in the sense that, given K_i, *G* can increase at a slower rate than does *K* and still provide a fixed level of public services to the firm. Alternatively, the case $q_A > 1$ describes the extreme situation in which the congestion of the public good exceeds the growth rate of the economy. This case is unlikely at the aggregate level, but may well be plausible for local public goods; see Edwards (1990).

The third category of public goods are subject to pure *aggregate* congestion, parameterized by $q_A > 0,\ q_R = 0$. In this case congestion is directly proportional to the aggregate level of capital in the economy. An example might be local police services or education financed by local property taxes. Note that in these cases, congestion increases with the absolute size of the economy. This, however, is associated with scale effects, and for this reason has been excluded from most previous endogenous growth models, which instead concentrate on relative congestion.[8] In this model we allow the government provided good to be subject to both absolute and relative congestion.

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[8]A special subset of the pure aggregate congestion case corresponds to the simplest endogenous growth models with positive externalities, with $q_R = 0,\ q_A < 0$. In these models, sustained growth becomes feasible through positive (and free) spillovers from the aggregate level of investment (Arrow 1962), physical capital (Romer 1986) or human capital (Lucas 1988). These models are not exactly analogous to our framework, however, since the spillovers are purely accidental and not subject to tax financing.

Combining (1) and (3), enables the individual's production function to be written in the form:

$$Y_i = aK_i^{s_K + q_R s_G} G^{s_G} K^{-s_G(q_R + q_A)} \tag{4}$$

From this expression it is seen that the net (private) marginal physical product of individual capital depends upon $s_K + q_R s_G$. The productivity of individual capital thus comprises two components. The first is the usual elasticity of private capital, s_K. Second, to the extent that there is relative congestion, from the perspective of the individual, increasing his capital stock will increase the level of government services he derives. Aggregate capital, to the extent that it raises both aggregate and relative congestion, has an adverse effect on individual output. In this respect these play the role of negative externalities in Romer's (1986) model.

Note that although equation (4) is interpreted as reflecting elements of both relative and absolute congestion, it is observationally equivalent to two alternative scenarios in which congestion is assumed to be purely relative and absolute, respectively. Specifically, modifying (1) and (3) to:

$$Y_i = a' L_i^{1-s_K} K_i^{s_K + q_R s_G} G_s^{s_G} \tag{1'}$$

$$G_s = G K^{-(q_R + q_A)} \tag{3'}$$

again produces (4). And the same applies if these equations are modified to:

$$Y_i = a' L_i^{1-s_K} K_i^{s_K - q_A s_G} G_s^{s_G} \tag{1''}$$

$$G_s = G\left(\frac{K_i}{K}\right)^{-(q_R + q_A)} \tag{3''}$$

Thus the individual production function with constant returns to scale in labor and private capital and with both relative and aggregate congestion [equations (1) and (3)] is observationally equivalent to one in which the productive elasticity of individual capital is increased to $s_K + q_R s_G$, yielding *increasing* returns to scale in the private factors of production, while congestion is of the purely *aggregate* form with an elasticity equal to $(q_R + q_A)$. It is also equivalent to a production function in which the productivity of individual capital is reduced to $s_K - q_A s_G$, yielding *decreasing* returns to scale in the private factors, while congestion is now purely *relative*, again being equal to $(q_R + q_A)$. Thus a tradeoff exists between private returns to scale, relative congestion, and absolute congestion.

Summing (4) over the *N* individuals in the economy yields the aggregate production function:

$$Y = aK^{s_K - q_A s_G} N^{1 - s_K - q_R s_G} G^{s_G} \tag{5}$$

To assure that capital and labor are productive in the aggregate economy, we impose the conditions:

$$s_K > q_A s_G; \;\; 1 > s_K + q_R s_G \tag{6}$$

In Section 3 below, we shall discuss the properties of (5) under general assumptions about returns to scale. For constant population *N* $(n = 0)$ the production function (5) will yield endogenous growth if and only if it has constant returns to scale in *K* and *G*, i.e. if and only if:

$$s_K + s_G(1 - q_A) = 1 \tag{7}$$

If, following Barro (1990) and others, we assume that the production function is constant returns to scale in capital and in government services, $s_K + s_G = 1$, then (7) requires the complete absence of aggregate congestion. Alternatively, the presence of aggregate congestion is consistent with endogenous growth if and only if the returns to scale satisfy: $s_K + s_G = 1 + s_G q_A > 1$. In this case, there should be sufficiently increasing returns to scale in capital and government, so as to exactly offset the negative externality caused by congestion. Indeed, this restriction is imposed by Glomm and Ravikumar (1994, 1997) in their analysis of absolute congestion.[9] By contrast, (7) imposes no restriction on the degree of relative congestion, and for this reason, that is the form of congestion primarily adopted in endogenous growth models.[10]

[9]The analogous condition for endogenous growth in the Romer (1986) model, where output $Y_i = aK_i^{s_K} K^h$, is $s_K + h = 1$

[10]We should point out that the restriction (7) applies to aggregate congestion associated with productive government expenditure. Aggregate congestion implies no such restriction in the case of utility-enhancing, government consumption expenditure; see Turnovsky (1996).

The functional form we have adopted incorporates all three specifications of public services adopted by Barro and Sala-i-Martin (1992). One version of their model treats public services as a Samuelson (1954) public good and assumes $q_A = q_R = 0$, as in our first category. Since they are concerned with endogenous growth, they impose the restriction $s_K + s_G = 1$ [in accordance with (7)], and thus in our notation, individual output (4) is described by:

$$Y_i = aK_i^{1-s_G} G^{s_G} \tag{8a}$$

A second version of their model assumes that government services rival but not excludable. Barro and Sala-i-Martin express this by writing the production function in the form:

$$Y_i = aK_i \left(\frac{G}{K}\right)^{s_G} \tag{8b}$$

This can be obtained from our general production (4) by imposing the restrictions:

$$q_R + q_A = 1; \quad s_K + s_G q_R = 1$$

In the absence of aggregate congestion $(q_A = 0)$, these restrictions reduce to proportional relative congestion, together with constant returns to scale in capital and government services ($s_K + s_G = 1$). But it is also consistent with absolute congestion ($q_A = 1$), together with linearity in the individual capital stock ($s_K = 1$). Equation (8b) asserts that individual production satisfies constant returns to private capital as long as the government maintains a fixed ratio of expenditure to total capital. Since the good is rival but non-excludable, lump sum taxation leads to over investment and excessive use of public services. Consequently, the decentralized growth rate exceeds that of the planned economy. We show below that this result depends on the *type* of congestion assumed, and on the fact that taxes influence the growth rate in the AK model.

The final version they consider is one in which the public good is rival but excludable, so that $G_s = G/N$, which they express by writing the production function as:

$$Y_i = aK_i^{1-s_G}\left(\frac{G}{N}\right)^{s_G} \tag{8c}$$

This function can be obtained by substituting the aggregation condition $K = NK_i$ into (8b). In this case production is subject to diminishing returns with respect to private capital, for given *G*, but constant returns to scale with respect to K_i and *G* together.

2.2 Individual Optimization

The representative agent in the economy chooses individual consumption and the rate of capital accumulation to maximize his intertemporal utility function:

$$\Omega \equiv \int_0^\infty \frac{C_i^g}{g} e^{-rt} dt; \quad r > 0 \tag{9a}$$

where r denotes the constant rate of time preference. The constant elasticity utility function is chosen for convenience and implies a constant elasticity of substitution equal to $1/(1-g)$. In the decentralized economy the optimization is subject to the individual accumulation equation:

$$\dot{K}_i = (1-t_y)Y_i - (1+t_c)C_i - T_i - nK_i \tag{9b}$$

where t_y and t_c represent the income and consumption tax respectively, and T_i denotes a lump sum rebate received by the agent. Individual output is generated by (4), and in performing his optimization, each agent takes government expenditures, G, and the aggregate capital, *K*, as given.

The following first order conditions are obtained:

$$C_i^{g-1} = l(1+t_c) \tag{10a}$$

$$(1-t_y)(s_K + q_R s_G)\frac{Y_i}{K_i} - n = r - \frac{\dot{l}}{l} \tag{10b}$$

Equation (10a) equates marginal utility to the shadow value of an additional unit of capital, l, while equation (10b) equates the net after-tax rate of return on capital to the rate of return on consumption.

The former is given by the left hand side of (10b) and takes account of the fact that the presence of relative congestion enhances the return to private investment. This is due to the fact that more private investment increases the volume of government services received by the individual. Note that the absolute congestion parameter does not affect the marginal product of private capital, although it affects output. This leads the agent to neglect the negative effects of absolute congestion, since the focus of his maximization is only his individual capital stock. This does not imply, however, that aggregate congestion is irrelevant for growth and welfare, as we will show below.

3. Equilibrium Behavior of the Aggregate Economy

To derive the behavior of the aggregate economy we must introduce the government and consider the aggregate rate of capital accumulation. Since we wish to focus on a growing economy, we assume that the government sets its aggregate expenditure level, G as a constant fraction, g, of aggregate output, Y, namely:

$$G = gY \tag{11}$$

so that an expansion in government expenditure is parameterized by an increase in the output share, g. We assume that the government finances its expenditure in accordance with a balanced budget, which aggregating over the N individuals can be expressed as:

$$t_y NY_i + t_c NC_i + NT_i = gNY_i \tag{12}$$

or in terms of the aggregate quantities: $Y \equiv NY_i$, $C \equiv NC_i$, $T \equiv NT_i$

$$t_y Y + t_c C + T = gY \tag{12'}$$

To complete the macroeconomic equilibrium, we must consider the aggregate accumulation of capital, $K \equiv NK_i$. To do this, we first note:

$$\dot{K} = N\dot{K}_i + nK$$

Multiplying the individual accumulation equation (9b) by *N*, and combining with the government budget constraint (12), aggregate capital in the economy is accumulated in accordance with the product market equilibrium condition:

$$\dot{K} = (1-g)Y - C \tag{13}$$

3.1 Aggregate Production, Congestion, and Returns to Scale

Combining (11) with (5), enables the aggregate production function to be written in the form:

$$Y = \left(ag^{s_G}\right)^{1/(1-s_G)} K^{b_K} N^{b_N} \equiv aK^{b_K} N^{b_N} \tag{14a}$$

where

$$b_K \equiv \frac{s_K - q_A s_G}{1 - s_G}; \quad b_N \equiv \frac{1 - s_K - q_R s_G}{1 - s_G} \tag{14b}$$

From the restrictions in (6) we know that the numerators in (14b) are positive. In order to ensure that the productive elasticities, b_K, b_N, remain positive in the aggregate, we assume $s_G < 1$. An important observation from (14) is that at the aggregate level absolute congestion manifests itself in a reduction in the productive elasticity of aggregate *capital*, while relative congestion is similarly reflected in the productive elasticity of aggregate *labor*.

A central point of our analysis concerns the relationship between congestion and returns to scale. From (14) we immediately see that the returns to scale of the aggregate production function in terms of the primary factors *K* and *N* are given by:

$$b_K + b_N = \frac{1 - s_G(q_R + q_A)}{1 - s_G} \tag{15}$$

Thus aggregate production exhibits increasing or decreasing returns to scale according to whether

$$s_G(1-q_A-q_R) \gtrless 0 \tag{16}$$

Constant returns to scale therefore prevails in two cases. The first is if $s_G = 0$, so that government expenditure is unproductive, in which case the model essentially reverts back to a Solow type model with non-productive taxation. More interesting, constant returns to scale will apply with productive government expenditure $(s_G > 0)$, if and only if the elasticities of relative and absolute congestion sum to unity, i.e. $q_R + q_A = 1$. In this case, the individual production function (4) is of the form

$$Y_i = aK_i^{s_K + q_R s_G}(G/K)^{s_G}$$

This relationship highlights that even among models with constant returns to scale, the previous literature examined only a restricted set of examples of congestion and growth.

3.2 Balanced Growth

Along the balanced growth path, aggregate output, capital, consumption, and government expenditure all grow at the identical, constant rate. This common growth rate is determined by taking differentials of the aggregate production function (14a), and is given by

$$f = \left(\frac{b_N}{1-b_K}\right)n = \frac{1-s_K-s_G q_R}{1-s_K-s_G(1-q_A)}n \tag{17a}$$

where in the Appendix we show that stability implies $b_K < 1$. Together with the restriction $s_G < 1$, this condition assures positive long-run growth. From (17a) we find that per capita growth is:

$$f-n = \left(\frac{b_N+b_K-1}{1-b_K}\right)n = \frac{s_G(1-q_R-q_A)}{1-s_K-s_G(1-q_A)}n \tag{17a'}$$

Equation (17a') clarifies that positive per capita growth rate is attained if and only if there are increasing returns to scale in *K* and *N*, and that this occurs if and only if the sum of the elasticities with respect to aggregate and relative congestion sum to less than unity. Clearly moderate degrees of both relative and aggregate congestion are compatible with positive per capita growth.

Taking the differentials of the government services function, (3), and using (17a), we see that the growth rate of government services, $\hat{G}_s$, is given by:

$$\hat{G}_s = f(1-q_A) - q_R n = \frac{(1-s_K)(1-q_A-q_R)}{1-s_K-s_G(1-q_A)} n \leq f \qquad (17b)$$

Congestion causes the growth rate of government services to be less than the equilibrium growth rate of the economy. The interesting point about (17b) is that the growth rate of services will be positive or negative according to whether $(1-q_A-q_R) \gtrless 0$, that is, according to whether the aggregate production function has increasing or decreasing returns to scale. Under constant returns to scale, the economy and the size of the government grow at the rate of population and government services remain constant. Per capita government services therefore fall. Per capita government services will remain constant if and only if the congestion is sufficiently weak so that the aggregate production function exhibits sufficiently strong increasing returns.[11]

It is important to note that the steady-state growth rate has the characteristics of the non-scale growth model in that it is independent of the absolute size (scale) of the economy. Instead, the equilibrium growth rate is determined by the following exogenous parameters: (i) the rate of population growth, and (ii) the coefficients of the production technology, in this case the production elasticities (s_K , s_G) and the respective degrees of congestion (q_A, q_R). It is straightforward to show than an increase in either form of congestion will reduce the equilibrium growth rate. Note also that neither tax rates, nor the share of government in output, *g*, enters the expression in (17a). This is characteristic of non-scale models [Jones 1995] and is generally consistent with the empirical evidence. We have already noted the empirical studies of Stokey and Rebelo (1995) and Jones (1995) that suggest that taxes do not seem to have a long-term impact on growth rates. Kormendi and Meguire (1985) found no significant relation between average growth rates of GDP and average growth and the share of government consumption in GDP, although other evidence on government expenditure and growth is more mixed; see Grier and Tullock (1987).

[11]The condition for constant per capita level of government services may be written as: $q_A + q_R = (s_G/(1-s_K))(1-q_A)$.

3.3 Transitional Dynamics and Stability

To characterize the transitional dynamics around the balanced growth equilibrium (17a) it is convenient to transform the system in terms of the stationary variables:

$$k \equiv \frac{K}{N^{b_N/(1-b_K)}}\,;\;\; c \equiv \frac{C}{N^{b_N/(1-b_K)}} \tag{18}$$

The stationary variables, *k* and *c*, can be characterized as being "scale-adjusted" per capita quantities and in the case that the aggregate production function has constant returns to scale they reduce to standard per capita quantities.

In the Appendix we show that the equilibrium dynamics of the economy can be represented by the pair of equations:[12]

$$\dot{k} = k\left\{a(1-g)k^{b_K-1} - \frac{c}{k} - \frac{b_N}{1-b_K}n\right\} \tag{19a}$$

$$\dot{c} = c\left\{\frac{(1-t_y)(s_K+q_R s_G)ak^{b_K-1}}{1-g} - \frac{r}{1-g} - \left(\frac{g}{1-g} + \frac{b_N}{1-b_K}\right)n\right\} \tag{19b}$$

There we show that (19) will be locally saddle path stable if and only if $b_K < 1$. Recalling (14b), this can be expressed in the form:

$$s_K + s_G(1-q_A) < 1 \tag{20}$$

which thus restricts the elasticities of private capital, public services and aggregate congestion. Several key features of this stability condition merit comment. First, stability is independent of the level of relative congestion. To the extent that previous models introduced only relative congestion, this would give the misleading impression that congestion itself does not matter for stability. Second, previous models with fiscal expenditures have shown that excessively large shares of public services

[12]Equation (19a) describes goods market equilibrium, while (19b) specifies the intertemporal efficiency condition, both expressed in terms of the scale-adjusted per capita quantities *k*, *c*.

in output may lead to instability [Barro and Sala-i-Martin, 1992]. Equation (20) highlights that aggregate congestion has a *stabilizing* impact on the economy. Thus, if the production function is constant returns to scale in capital and public services, then stability is assured as long as the latter is subject to some aggregate congestion. Indeed it is possible for increasing returns to scale to be associated with stability, provided the degree of aggregate congestion is sufficiently large.

4. Optimal Tax Policy and Congestion

Since individuals ignore the consequences of their own actions on the aggregate economy, congestion generates an externality, the effect of which can be corrected by the introduction of appropriate distortionary taxes. As a benchmark, we consider the first-best outcome of the centrally planned economy and determine the tax structure that will enable this to be replicated.

4.1 Centrally Planned Economy

The central planner possesses complete information and chooses all quantities directly, taking into account the congestion caused by all agents. The formal optimization is to maximize per capita utility in the economy:

$$\Omega^* = \int_0^\infty U(C/N)e^{-rt}dt \qquad (21a)$$

subject to the aggregate accumulation constraint

$$\dot{K} = (1-g)Y - C = (1-g)(ag^{s_G})^{1/(1-s_G)}K^{b_K}N^{b_N} - C \qquad (21b)$$

The first order conditions are similar to those in the decentralized economy, namely:

$$(C^*)^{g-1}N^{-g} = l^* \qquad (10a')$$

$$(1-g)b_K\left(\frac{Y}{K}\right)^* = r - \frac{\dot{l}^*}{l^*} \qquad (10b')$$

where * denotes the optima in the centrally planned economy. There are two main qualitative differences between the decentralized and planned economies. First, the return to capital, given by the left hand side of (10b'), now depends on the planner's share of output claimed for public services, while in the decentralized case it was the tax rate that influenced the net return on capital. Second, the productive elasticity determining the planner's accumulation of capital is the *social* productivity of capital, b_K, rather than the private productivity, $(s_K + q_R s_G)$, relevant for the representative agent. This reflects the differences in perception regarding congestion.

As in the decentralized economy, we introduce the normalization (18), thereby expressing the dynamics in terms of stationary variables. Following the procedure laid out in the Appendix, we can summarize the dynamics of the aggregate system by the pair of equations:

$$\dot{k} = k^* \left\{ a(1-g)k^{b_K - 1} - \frac{c^*}{k^*} - \frac{b_N}{1-b_K} n \right\} \tag{19a'}$$

$$\dot{c} = c^* \left\{ \frac{(1-g)a(k^*)^{b_K - 1}}{1-g} - \frac{r}{1-g} - \left(\frac{g}{1-g} + \frac{b_N}{1-b_K} \right) n \right\} \tag{19b'}$$

The parallels between these equations and (19a), (19b) are apparent. Linearizing these two equations around their steady-state equilibrium, we find that $b_K < 1$ continues to be necessary and sufficient for saddlepath stability, just as it was in the decentralized economy.

Given that the long-run growth rate is determined entirely by characteristics of the aggregate production function [which remains invariant between the two economies] it follows that the long-run equilibrium growth rate continues to be determined by (17). The equilibrium growth rate in both the centralized and decentralized economies are determined by the same structural elasticities that comprise b_K and b_N and are independent of government policy. Accordingly, issues of growth-maximizing size of government and its relationship to welfare maximization, discussed in the context of the Barro (1990) AK model, are irrelevant in the present non-scale equilibrium.

By contrast, the welfare-maximizing size of government remains a crucial issue. To maximize welfare, the planner maximizes the Hamiltonian with respect to the share of output claimed by the planner, g. As in Barro's (1990) model, we find that the welfare-maximizing share of output claimed to finance public services equals elasticity of public services in production:

$$\hat{g} = s_G \tag{22}$$

The reasoning is the same as in Barro. On the one hand an increase in g raises the marginal product of capital and that has positive welfare effects. At the same time, however, an increase in g also decreases the amount of output available for consumption. The optimal size of government balances off these two effects and occurs where (22) holds.

4.2 Optimal Tax Structure

Comparing (19) with (19') we see that the decentralized economy will fully replicate the dynamic time path of the centrally planned economy, and its steady-state equilibrium, and therefore attain its welfare level if and only if the income tax is set such that

$$\left(1 - t_y\right)\left(s_K + q_R s_G\right) = (1 - g)b_K \tag{23}$$

Setting the tax in this way ensures that the marginal return to investment as perceived by the representative agent exactly equals the return as viewed by the central planner. Solving (23) for t_y implies the optimal income tax rate:

$$\hat{t}_y = \frac{s_G(q_A + q_R)}{s_K + q_R s_G} + \frac{b_K}{s_K + q_R s_G}(g - s_G) \tag{24}$$

Condition (24) is a very strong result and differs in important respects from previous results in the literature. Specifically, (24) determines a *time-invariant* tax rate that will replicate *both* the transitional path and the steady-state equilibrium of the first-best equilibrium. Previous AK models that lack transitional dynamics, also derive a constant tax rate that will replicate the balanced-growth equilibrium. On the other hand, multi-capital AK models, having transitional dynamics, find that a two-part time-varying tax rate is necessary in order to attain the first best outcome. A constant part is necessary to replicate the steady state; a second, time-varying component is necessary to replicate the changing equilibrium along the transition; see Turnovsky (1997). Here, both objectives are accomplished by the time-invariant tax rate (24).

Equation (24) indicates several important characteristics of the optimal income tax. First, the optimal tax is necessary to correct for two distortions; one related to the non-optimality of the government’s claim on output, $(g-\sigma_G)$, the other to address the externalities, both aggregate and relative, caused by congestion. If the government's claimed output share is set optimally, to the extent that there is congestion, the optimal income tax is positive. Intuitively, this is because the private agent ignores the negative externality caused by congestion and over accumulates capital relative to the optimum. On the other hand, in the absence of congestion, the optimal income tax will in fact be negative (a subsidy) if productive government expenditure is less than its social optimum. This is because the marginal social benefits exceed the marginal costs. Only in the absence of these two effects will the optimal tax on capital income be zero, as in the steady state of the Chamley (1986)-Judd (1985) model. Moreover, a subsidy may continue to be optimal in the presence of congestion, provided *g* is sufficiently below its optimum. This is to reap the benefits of the public services, which are highly productive despite a little congestion.

The responsiveness of the optimal tax on (capital) income to the different degrees of congestion are given by the expressions:

$$\frac{\partial \hat{\tau}_y}{\partial \theta_A}=\frac{\sigma_G(1-g)}{(1-\sigma_G)\left(\sigma_K+\theta_R\sigma_G\right)}>\frac{\sigma_G(1-g)\left(\sigma_K-\theta_A\sigma_G\right)}{(1-\sigma_G)\left(\sigma_K+\theta_R\sigma_G\right)^2}=\frac{\partial \hat{\tau}_y}{\partial \theta_R}>0 \qquad (25)$$

Thus both forms of congestion raise the optimal tax on capital. From (24) we see that an increase in either form of congestion, θ_A or θ_R, increases the corresponding externality. This *raises* the tax on capital necessary for its correction [the first component of (24)]. On the other hand, congestion *decreases* the optimal tax in order to correct for the non-optimality of government expenditure [the second component of (24)]. Overall, the congestion effect dominates and the optimal tax is unambiguously raised. Equation (25) implies further that the optimal tax is more responsive to absolute congestion than to relative congestion. This is because an increase in absolute congestion lowers the marginal social return to capital, β_K, by $-\sigma_G/(1-\sigma_G)$, while an increase in relative congestion raises the marginal private return by only s_G. The former drives a larger wedge between the social and marginal rates of return to capital and therefore requires a larger tax to correct.

The idea that the presence of congestion favors an income tax over lump-sum taxation or a consumption tax has been shown previously by Barro and Sala-i-Martin (1992) and Turnovsky (1996). In the simple AK model, for example, these authors show that in the case where the government good is subject to proportional relative congestion $(q_R = 1)$ it should be fully financed by a tax on income; i.e. $\hat{t}_y = g$. The question naturally arises as to the extent to which this result continues to hold under our more general production conditions and specifications of congestion.

Substituting for b_K into (24), it is straightforward to establish that $\hat{t}_y = g$ if and only if:

$$q_A + q_R(1 - s_G) = s_K \qquad (26)$$

That is, it will be optimal to finance government expenditure fully with an income tax if and only if the absolute and relative degrees of congestion satisfy (26). Combining this condition with the positive productivity condition (6), we see that $\hat{t}_y = g$ is optimal if and only if:

$$1 > q_A + q_R$$

that is, if and only if the aggregate production function has increasing returns to scale leading to positive per capita growth.

This contrasts with the condition when the production function is of the AK form and therefore satisfies the condition (7). Combining this with (26), and assuming $s_G < 1$, yields:

$$q_A + q_R = 1$$

That is, in the AK endogenous growth model, full financing with an income tax will be optimal if and only if the elasticities with respect to aggregate and relative congestion sum to unity. This of course includes the special case considered by Barro and Sala-i-Martin and Turnovsky which assumed the polar case $q_A = 0$, $q_R = 1$.

5. Some Numerical Results

To obtain further insights into the effects of the two types of congestion, we perform some numerical analysis. Our calibrations of the economy allow us to examine the magnitudes of the tradeoffs between the two congestion parameters, q_A, q_R, and the rate of growth, the capital- and consumption-output ratios, the speeds of adjustment, as well as the optimal tax. Tables 1 - 5 report some effects for the model based on the following benchmark parameter values:

Calibration Parameters[13]

Production	$s_K = 0.35$ $s_G = 0.15$ $a = 0.3$
Taste and Population Growth	$r = 0.04$ $g = -1$ $n = 0.015$
Fiscal Policy	$t_y = 0.15$ $g = 0.15$ Tables 2 - 4 $g = 0.25$ Table 5
Congestion	$s_A = 0,\ 0.5,\ 1$ $s_R = 0,\ 0.5,\ 1$

In Tables 2- 4 we assume that the share of government expenditure devoted to production is 0.15, which is consistent with the social optimum and equal to the income tax rate. In Table 5, where we explore how the optimal tax responds to deviations of government expenditure from its optimum g is set higher at 25% of output.

Table 1 summarizes the common steady-state growth rates. These reflect the determinants in (17). In contrast to the private production function, which is subject to constant returns to scale in the private factors, and increasing returns to scale in all three factors (L_i, K_i, G_s), the aggregate

[13]Our congestion measures cover the range from a pure public good to varying degrees of excludability for both types of congestion. The capital elasticity of 0.35 and the implied labor elasticity of 0.65 are standard. The government production elasticity of 0.15 is consistent with the consensus evidence which reports a mild degree of increasing returns to scale in the presence of government expenditures in the individual production function. Given g, the implied intertemporal elasticity of substitution equals 0.5.

production function has increasing, constant, or decreasing returns to scale, according to whether the sum of the congestion elasticities is less than, equal to, or greater than unity. In the case of the pure public good ($q_A = q_R = 1$), we have mildly increasing returns to scale in the aggregate (1.18) in which case the equilibrium growth rate (1.95%) exceeds the population growth rate (1.5%). The table reflects the fact that increasing either form of congestion reduces the equilibrium growth rate, leading to negative per capita growth when the sum of the two elasticities exceeds unity.

Tables 2 and 3 report the steady-state capital-output, and consumption-output ratios. The capital-output ratios increase with both forms of congestion, reflecting the fact that congestion lowers the aggregate productivity in the economy. Table 3 indicates that the capital-output ratio is more responsive to relative congestion than it is to absolute congestion. This is because agents perceive private capital as having a higher productivity when relative congestion increases, since it enhances the services of the public good they receive; see (3). The accumulation of relatively more capital is thereby encouraged, even though in equilibrium the public good will be congested, and the productivity in the economy will decline, just as it does when absolute congestion increases.

The consumption-output ratios in Table 4 are relatively stable around 0.75. Nevertheless, the table does bring out a similar contrast between the two forms of congestion as in Table 2. Higher degrees of absolute congestion raise the consumption-output ratio, while a higher degrees of relative congestion have the opposite effect. The reason can be seen from the following relationship:

$$\frac{C}{Y} = 1 - g - \frac{\dot{K}/K}{Y/K}$$

Comparing Tables 1 and 3, we observe that an increase in the degree of absolute congestion reduces the growth rate by more than it does the output to capital ratio. The fraction of output devoted to investment therefore declines and the consumption-output ratio therefore rises. In contrast, an increase in the relative congestion reduces the output to capital ratio by more than it does the growth rate. The investment-output ratio therefore rises, and the fraction of output available for consumption falls.

Recent growth models have emphasized the speed of convergence; see Barro and Sala-i-Martin (1992), Ortigueira and Santos (1997). Since congestion affects the productivity of capital it therefore also influence the transitional speed of adjustment. In the one-sector model employed here, where the stable manifold is a one-dimensional locus, the transitional adjustment is parameterized uniquely by the magnitude of the stable eigenvalue, as derived in the Appendix, with all variables converging at this common rate. Table 4 reports these convergence speeds for alternative degrees of congestion. The benchmark case of a pure public good implies a convergence rate of around 3.4%, which although slightly high is not implausible. From the present perspective, the striking feature of these results is that the rate of convergence increases with aggregate congestion but decreases with relative congestion. Indeed, $q_R = 1$ implies a rate of convergence of 2.7%, which is entirely consistent with the empirical evidence.

The intuition is most easily understood by turning to equation (18a), which describes the (scale adjusted) rate of capital accumulation in the present economy. Recalling equation (14a) an increase in relative congestion, q_R reduces the productivity of labor b_N and therefore reduces the long-run equilibrium growth rate (17). This implies that less output is necessary to equip the "scale-adjusted" growing labor force, (18), leaving more output available for further capital accumulation, and thus slowing down the convergence process. This is the only effect of relative congestion, insofar as this accumulation equation is concerned. Absolute congestion, on the other hand has two separate effects on convergence. First, it reduces the b_K, and therefore like the reduction in b_N it also reduces the long-run equilibrium growth rate and in this respect has a similar retarding effect on the rate of convergence as does relative congestion. But in addition, by reducing b_K it reduces the productivity of capital, which depresses the level of output available for per capita investment and thereby speeds up the rate of convergence. On balance, we find that this latter effect dominates [at least for all parameter sets we employed] and more absolute congestion is associated with more rapid convergence. Indeed, this result is perfectly consistent with the uniformly stabilizing role absolute congestion was shown to play in the stability condition (20).

Finally, Table 5 summarizes the optimal income tax rate, (24) for the array of congestion parameters.[14] As previously noted, the optimal tax must correct for two distortions, congestion $(q_A, q_R \neq 0)$, as well as the non-optimal government expenditure $(g \neq \hat{g} = s_G)$. Until now we have assumed that $g = \hat{g} = s_G$. In order to include the explore the two parts to the optimal tax, we now set $g = 0.25 > \hat{g} = s_G = 0.15$. The numerical values of the optimal tax rates reflect the analytical properties noted in connection with (24). The optimal tax rate increases with the congestion externality, being more responsive to absolute than to relative congestion. Starting from a pure public good, when this component of the optimal tax should be zero, it increases to 0.3 if $q_R = 1$ and to 0.43 if $q_A = 1$. By contrast, the optimal tax to correct for the non-optimality of government expenditure declines with both forms of congestion. There is therefore a tradeoff between these two externalities insofar as the optimal tax on capital income is concerned.

6 Conclusions

Congestion, returns to scale, and economic growth are intimately related economic variables. This paper has explored the relationship between them in a one-sector non-scale growth model that extends the familiar AK model. We have introduced two notions of congestion and contrasted the way they impinge on the economy. Aggregate congestion reduces the effective productivity of capital, while relative congestion reduces the effective productivity of labor. The effects of these measures of congestion on the equilibrium growth rate have been contrasted, as have their effects on the transitional dynamics. The most striking difference we find is that relative congestion slows down the rate of convergence, whereas aggregate convergence has the opposite effect. To the extent that the standard one-sector neoclassical growth model implies a convergence rate that is too fast, the assumption of relative congestion at least partially corrects this aspect. Finally, we have characterized the optimal tax policy and shown its responsiveness to the distortions caused by congestion, as well as the deviation of government expenditure from its social optimum.

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[14]Corresponding to these tax rates one can calculate corresponding values for the consumption tax or lump-sum tax [which are essentially in the absence of the labor-leisure choice].

The analysis has treated the two congestion elasticities as exogenous parameters, which influence the productivity of the two primary factors of production. As a result, they determine the equilibrium growth rate along with the other productive elasticities, in a manner characteristic of all non-scale growth models [Jones 1995]. Government policies such as expenditures and tax rates have no effect on the equilibrium growth rate. But the assumption that the degree of congestion is exogenous is questionable. The decision to build a four lane highway rather than a six lane highway is a policy decision and the choice will presumably affect the magnitudes of the congestion elasticities q_A and q_R. This decision is presumably made as part of the normal budgetary process of the government. An important extension of this analysis would be to treat the degree of congestion as the outcome of a policy decision made in conjunction with the conventional expenditure and revenue choices of fiscal policy. This would seem a very natural avenue for government policy to influence long-run growth rates in a non-scale economy.

APPENDIX

Transitional Dynamics and Stability

To analyze the transitional dynamics and stability of the economy about its steady-state growth path, we express the system in terms of the stationary variables:

$$k \equiv \frac{K}{N^{b_N/(1-b_K)}}\,;\;\; c \equiv \frac{C}{N^{b_N/(1-b_K)}} \qquad \text{(A.1)}$$

Differentiating (A.1) with respect to *t* yields:

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{b_N}{1-b_K}n; \;\; \frac{\dot{c}}{c} = \frac{\dot{C}}{C} - \frac{b_N}{1-b_K}n \qquad \text{(A.2)}$$

Combining (A.2) with (13) and (14a), we obtain:

$$\dot{k} = k\left\{a(1-g)k^{b_K-1} - \frac{c}{k} - \frac{b_N}{1-b_K}n\right\} \qquad ()$$

In addition, the normalization (18) enables us to rewrite the first order condition (10b) as

$$\frac{\dot{l}}{l} = r + n - (1-t_y)\left(s_K + q_R s_G\right)ak^{b_K-1}$$

Given that, we can totally differentiate (10a) and use the nomalization in (18) to find

$$\dot{c} = c\left\{\frac{(1-t_y)(s_K + q_R s_G)ak^{b_K-1}}{1-g} - \frac{r}{1-g} - \left(\frac{g}{1-g} + \frac{b_N}{1-b_K}\right)n\right\} \qquad \text{(A.3b)}$$

Thus equations (A.3a) and (A.3b) represent the complete, stationary dynamic system of the aggregate economy.

The steady-state values of the transformed variables, $\tilde{k}, \tilde{c}$ are given by:

$$\tilde{k} = \left\{ \frac{r + (1-g)\left[\frac{g}{1-g} + \frac{b_N}{1-b_K}\right]n}{(1-t_y)(s_K + q_R s_G)(a g^{s_G})^{1/(1-s_G)}} \right\}^{\frac{1}{b_K - 1}} \tag{A.4a}$$

$$\tilde{c} = \left(a g^{s_G}\right)^{1/(1-s_G)}(1-g)\tilde{k}^{b_K} - \frac{b_N}{1-b_K} n\tilde{k} \tag{A.4b}$$

From (A.4a) it is clear that the scale-adjusted per capita stock of capital is a decreasing function of the income tax and a positive function of the share of productive government expenditure.

Linearizing (A.3a) and (A.3b) around the steady state (A.4a) and (A.4b), the transitional dynamics of the system may be approximated by:

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} a(1-g)b_K\tilde{k}^{b_K - 1} - \frac{b_N}{1-b_K} n & -1 \\ \frac{\tilde{c}(1-t_y)(s_K + q_R s_G)(b_K - 1)a\tilde{k}^{b_K - 2}}{1-g} & 0 \end{pmatrix} \begin{pmatrix} k - \tilde{k} \\ c - \tilde{c} \end{pmatrix} \tag{A.5}$$

From (A.5) we see that the system is saddlepath stable if and only if $b_K < 1$. In that case, the stable saddlepath in *c*-*k* space is described by:

$$k(t) - \tilde{k} = (k_0 - \tilde{k})e^{mt} \tag{A.6a}$$

$$\frac{c(t) - \tilde{c}}{\tilde{c}} = \frac{1}{m}(1-t_y)(s_K + q_R s_G)(b_K - 1)\tilde{k}^{b_K - 1}\frac{(k(t) - \tilde{k})}{\tilde{k}} \tag{A.6b}$$

where $m < 0$ denotes the stable eigenvalue. The stable locus (A.6b) is attained by an initial jump in the consumption ratio *c*. Equations (A.6) imply that along the stable saddlepath consumption increases with capital, just as in the conventional Ramsey growth model.

The transitional adjustment paths of the growth rates of capital and consumption may also be of interest and linear approximations can be derived by combining (A.2) with (A.6a) and (A.6b) to yield:

$$\frac{\dot{K}}{K} = \frac{b_N}{1-b_K} n + \frac{m}{k}(k - \tilde{k}) \qquad \text{(A.7a)}$$

$$\frac{\dot{C}}{C} = \frac{b_N}{1-b_K} n + \frac{m}{c}(c - \tilde{c}) \qquad \text{(A.7b)}$$

Since consumption and capital and move together (see (A.6b)), the growth rate of both consumption and capital approach their respective steady-state rates in the same direction.

The transitional dynamics and stability of the centrally planned economy can be analyzed identically.

APPENDIX

Transitional Dynamics and Stability

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$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{b_N}{1-b_K}n; \;\; \frac{\dot{c}}{c} = \frac{\dot{C}}{C} - \frac{b_N}{1-b_K}n \tag{A.2}$$

Combining (A.2) with (13) and (14a), we obtain:

$$\dot{k} = k\left\{a(1-g)k^{b_K-1} - \frac{c}{k} - \frac{b_N}{1-b_K}n\right\} \tag{A.3a}$$

In addition, the normalization (18) enables us to rewrite the first order condition (10b) as

$$\frac{\dot{l}}{l} = r + n - (1-t_y)\left(s_K + q_R s_G\right)a k^{b_K-1}$$

Given that, we can totally differentiate (10a) and use the nomalization in (18) to find

$$\dot{c} = c\left\{\frac{(1-t_y)(s_K + q_R s_G)a k^{b_K-1}}{1-g} - \frac{r}{1-g} - \left(\frac{g}{1-g} + \frac{b_N}{1-b_K}\right)n\right\} \tag{A.3b}$$

Thus equations (A.3a) and (A.3b) represent the complete, stationary dynamic system of the aggregate economy.

The steady-state values of the transformed variables, $\tilde{k}, \tilde{c}$ are given by:

$$\tilde{k} = \left\{ \frac{r + (1-g)\left[\frac{g}{1-g} + \frac{b_N}{1-b_K}\right]n}{(1-t_y)(s_K + q_R s_G)(ag^{s_G})^{1/(1-s_G)}} \right\}^{\frac{1}{b_K - 1}} \qquad \text{(A.4a)}$$

$$\tilde{c} = \left(ag^{s_G}\right)^{1/(1-s_G)}(1-g)\tilde{k}^{b_K} - \frac{b_N}{1-b_K}n\tilde{k} \qquad \text{(A.4b)}$$

From (A.4a) it is clear that the scale-adjusted per capita stock of capital is a decreasing function of the income tax and a positive function of the share of productive government expenditure.

Linearizing (A.3a) and (A.3b) around the steady state (A.4a) and (A.4b), the transitional dynamics of the system may be approximated by:

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} a(1-g)b_K\tilde{k}^{b_K-1} - \frac{b_N}{1-b_K}n & -1 \\ \frac{\tilde{c}(1-t_y)(s_K + q_R s_G)(b_K - 1)a\tilde{k}^{b_K-2}}{1-g} & 0 \end{pmatrix} \begin{pmatrix} k - \tilde{k} \\ c - \tilde{c} \end{pmatrix} \qquad \text{(A.5)}$$

From (A.5) we see that the system is saddlepath stable if and only if $b_K < 1$. In that case, the stable saddlepath in *c*-*k* space is described by:

$$k(t) - \tilde{k} = (k_0 - \tilde{k})e^{mt} \qquad \text{(A.6a)}$$

$$\frac{c(t) - \tilde{c}}{\tilde{c}} = \frac{1}{m}(1-t_y)(s_K + q_R s_G)(b_K - 1)\tilde{k}^{b_K-1}\frac{(k(t) - \tilde{k})}{\tilde{k}} \qquad \text{(A.6b)}$$

where $m < 0$ denotes the stable eigenvalue. The stable locus (A.6b) is attained by an initial jump in the consumption ratio *c*. Equations (A.6) imply that along the stable saddlepath consumption increases with capital, just as in the conventional Ramsey growth model.

The transitional adjustment paths of the growth rates of capital and consumption may also be of interest and linear approximations can be derived by combining (A.2) with (A.6a) and (A.6b) to yield:

$$\frac{\dot{K}}{K} = \frac{b_N}{1-b_K} n + \frac{m}{k}(k-\tilde{k}) \qquad \text{(A.7a)}$$

$$\frac{\dot{C}}{C} = \frac{b_N}{1-b_K} n + \frac{m}{c}(c-\tilde{c}) \qquad \text{(A.7b)}$$

Since consumption and capital and move together (see (A.6b)), the growth rate of both consumption and capital approach their respective steady-state rates in the same direction.

The transitional dynamics and stability of the centrally planned economy can be analyzed similarly.

Table 1

Steady-State Growth Rate (in percent)

		q_A		
		0	**.5**	**1**
q_R	**0**	1.950	1.696	1.500
	.5	1.725	1.500	1.327
	1	1.500	1.304	1.154

Table 2

Steady-State capital-output ratio

		q_A		
		0	**.5**	**1**
q_R	**0**	4.65	5.05	5.41
	.5	6.07	6.57	7.01
	1	7.73	8.32	8.84

Table 3

Steady-State consumption-output ratio

		q_A		
		0	**.5**	**1**
q_R	**0**	0.759	0.764	0.769
	.5	0.745	0.751	0.757
	1	0.734	0.741	0.748

Table 4

Transitional Adjustment Speeds (in percent)

		q_A		
		0	**.5**	**1**
q_R	**0**	3.42	3.93	4.48
	.5	3.04	3.46	3.91
	1	2.69	3.05	3.43

Table 5

Optimal Income Tax Rate (in percent)

Correcting for non optimal government share, $g \neq \hat{g}$

and congestion, $q_A, q_R \neq 0$

		q_A								
		0			**.5**			**1**		
		Congestion $q_A, q_R \neq 0$	**Gov't Share** $g \neq \hat{g}$	**Total**	**Congestion** $q_A, q_R \neq 0$	**Gov't Share** $g \neq \hat{g}$	**Total**	**Congestion** $q_A, q_R \neq 0$	**Gov't Share** $g \neq \hat{g}$	**Total**
q_R	**0**	0	11.8	**11.8**	21.4	9.2	**30.6**	42.9	6.7	**49.6**
	.5	17.6	9.7	**27.3**	35.3	7.6	**42.9**	52.9	5.5	**58.4**
	1	30.0	8.2	**38.2**	45.0	6.5	**51.5**	60	5.4	**64.7**

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