Lesson Week 10

Response of LTI Systems (Transfer Functions, Partial Fraction Expansion, and Convolution), LTI System Characteristics (Stability and Invertibility)
where $h(t)$ is an impulse response, is called the system function or transfer function and it completely characterizes the input/output relationship of an LTI system. We can use it to determine time responses of LTI systems.

**Transfer Functions**

We can use Laplace Transforms to solve differential equations for systems (assuming the system is initially at rest for one-sided systems) of the form:

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{m} b_k \frac{d^k}{dt^k} x(t)$$

Taking the Laplace Transform of both sides of this equation and using the Differentiation Property, we get:

$$Y(s) \sum_{k=0}^{n} a_k s^k = X(s) \sum_{k=0}^{m} b_k s^k$$

From this, we can define the transfer function $H(s)$ as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k}$$
Partial Fraction Expansion

Instead of taking contour integrals to invert Laplace Transforms, we will use Partial Fraction Expansion. We review it here. Given a Laplace Transform,

\[ F(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)}, \quad m < n \]

We write its Partial Fraction Expansion as:

\[ F(s) = \frac{N(s)}{(s-p_1)(s-p_2) \cdots (s-p_n)} = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_n}{s-p_n} \]

where

\[ k_j = (s-p_j) F(s)\big|_{s=p_j} \quad \text{is the residue of the pole at } p_j. \]

Thus

\[ f(t) = \sum_{j=1}^{n} k_j e^{p_j t} u(t) \]

because the Inverse Laplace Transform of

\[ \frac{k_j}{s+p_j} \quad \text{is } k_j e^{p_j t} \]
Example 1. Find $y(t)$ where the transfer function $H(s)$ and the input $x(t)$ are given. Use Partial Fraction Expansion to find the output $y(t)$:

$$H(s) = \frac{3s + 1}{s^2 + 6s + 5}, \quad x(t) = e^{-3t}u(t)$$

$$e^{-3t}u(t) \xrightarrow{L} \frac{1}{s + 3}$$

$$Y(s) = \frac{3s + 1}{(s + 3)(5 + s)(s + 1)} = \frac{A}{s + 3} + \frac{B}{s + 5} + \frac{C}{s + 1}$$

$$A = \frac{-9 + 1}{2(-2)} = \frac{-8}{-4} = 2$$

$$B = \frac{-15 + 1}{(-2)(-4)} = \frac{-14}{8} = -\frac{7}{4}$$

$$C = \frac{-3 + 1}{(2)(4)} = \frac{-2}{8} = -\frac{1}{4}$$

$$y(t) = \left(2e^{-3t} - \frac{7}{4}e^{-5t} - \frac{1}{4}e^{-t}\right)u(t)$$
Example 2. Find the transfer function $H(s)$ for the differential equation. Assume zero initial conditions.

$$y'(t) + 2y(t) = 3x'(t).$$

$$sY(s) + 2Y(s) = 3[X(s)]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s}{s + 2} = 3 \left( \frac{s}{s + 2} \right)$$

$$= 3 \left[ \frac{s + 2 - 2}{s + 2} \right] = 3 \left[ 1 - \frac{2}{s + 2} \right]$$

$$h(t) = 3 [\delta(t) - 2e^{-2t}]u(t)$$
Example 3  Now let the input to the system be $x(t) = 5u(t)$. Find $y(t)$.

\[ 5u(t) \xrightarrow{L} \frac{5}{s} \]

\[ Y(s) = X(s)H(s) = \frac{5}{s} \left( \frac{3s}{s + 2} \right) = \frac{15}{s + 2} \]

\[ y(t) = 15e^{-2t}u(t) \]
**Convolution**

As we saw for the Fourier Transform

\[ x(t) * h(t) \leftrightarrow X(s)H(s) \]

This is useful for studying LTI systems.

In fact, we can completely characterize an LTI system from:

1. The system differential equation
2. or the system transfer function \( H(s) \)
3. or the system impulse response \( h(t) \).
Example 4  Find the step response $s(t)$ to

$$h(t) = e^{-t}u(t)$$

Hint:  $u(t) \xrightarrow{L} \frac{1}{s}$

$$H(s) = \frac{1}{s + 1}$$

$$S(s) = \frac{1}{s} H(s) = \frac{1}{s(s + 1)} = \frac{A}{s} + \frac{B}{s + 1}$$

$A = 1 \quad B = -1$

$$S(t) = u(t) - e^{-t}u(t) = h(t) * u(t)$$
Example 5. Find the output of an LTI system with impulse response $h(t) = e^{bt}u(t)$ to an input $x(t) = e^{at}u(t)$, where $a \neq b$.

\[
Y(s) = \frac{1}{(s-a)(s-b)} = \frac{\left(\frac{1}{a-b}\right)}{s-a} + \frac{\left(\frac{1}{b-a}\right)}{s+b}
\]

\[
y(t) = \left(\frac{1}{a-b}e^{at} + \frac{1}{b-a}e^{bt}\right)u(t)
\]

$\text{Re}(s) > a \Rightarrow \text{Re}(s) > \text{Max}(a,b)$

$\text{Re}(s) > b$
We saw that a condition for bounded-input bounded-output stability was:

\[ \int_{-\infty}^{\infty} |h(t)| \, dt < \infty \]

Let's look at stability from a system function standpoint. Given a Laplace Transform \( H(s) \), we expand \( H(s) \) with Partial Fraction Expansion:

\[
H(s) = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \cdots + \frac{k_n}{s - p_n}
\]

The corresponding impulse response is:

\[
h(t) = k_1 e^{p_1 t} u(t) + k_2 e^{p_2 t} u(t) + \cdots + k_n e^{p_n t} u(t)
\]

What happens to \( h(t) \) as \( t \to \infty \)? For a system to be stable, its impulse response must not blow up as \( t \to \infty \).
If \( \text{Re}\{p_i\}, \forall i \), then \( h(t) \) decays to 0 as \( t \to \infty \) and the system is stable. (just evaluate \( \int_{-\infty}^{\infty} |h(t)| \, dt \).)

Therefore, the system is BIBO stable if and only if all poles of \( H(s) \) are in the left half plane of the \( s \)-plane.

If you study CONTROL THEORY, you will learn more about this. Using feedback, you can build systems to steer the poles into the left half plane and thus stabilize the system. Here is an example of such a system.
Example 6

1. You are given a system with impulse response
   \[ h(t) = e^t u(t) \]

Is the system stable?

\[ e^t u(t) \xrightarrow{L} \frac{1}{s + 1} \]

No \( t \to \infty \) \( h(t) \to \infty \)
2. You now hook up the system up into a "Feedback" system as shown. Find the new impulse response or transfer function. Find the range on the parameter A such that the system is stable.

\[ H'(s) = \frac{Y(s)}{X(s)} \]

\[ w(t) = x(t) - Ay(t) \]

\[ W(s) = X(s) - AY(s) \]

\[ y(t) = w(t) \ast h(t) \]

\[ Y(s) = W(s) H(s) \]

\[ \frac{Y(s)}{H(s)} = X(s) - AY(s) \]

\[ Y(s) \left[ \frac{1}{H(s)} + A \right] = X(s) \]

\[ H'(s) = \frac{Y(s)}{X(s)} = \frac{H(s)}{1 + AH(s)} \]

Plugging in \( H(s) \) & choose A

\[ H'(s) = \frac{\left( \frac{1}{s-1} \right)}{\left( 1 + \frac{A}{s-1} \right)} = \frac{1}{s-1+\frac{A}{s}} \]

if \( A > 1 \) (pole in LHP), then the system is BIBO stable
**Invertibility**
You can find the inverse of a system using Laplace Transforms. This is because:

\[ h(t) * h_I(t) = \delta(t) \]

Take the Laplace Transform of both sides of this equation:

\[ H(s) H_I(s) = 1 \]

Therefore, the Laplace Transform of the inverse system is simply

\[ H_I(s) = \frac{1}{H(s)} \]
Example 7  Find the inverse system of

\[
H(s) = \frac{s + a}{s + b}, \quad \text{Re}\{s\} > -b
\]

\[
H_I(s) = \frac{s + b}{s + a} = \frac{s}{s + a} + \frac{b}{s + a} = \frac{s + a}{s + a} + \frac{b - a}{s + a}
\]

\[
\begin{align*}
\mathcal{L}^{-1}\{s\} & \quad \frac{d}{dt}\{e^{-at}u(t)\} \\
\mathcal{L}^{-1}\{s\} & \quad L^{-1}\{e^{-at}u(t)\}
\end{align*}
\]

\[
h_I(t) = -ae^{-at}u(t) + \delta(t) + be^{-at}u(t)
\]

\[
= \delta(t) + (b - a)e^{-at}u(t)
\]
In this section, we cover the Bilateral Laplace Transform. We will see that for the Bilateral Laplace Transform we must specify Region of Convergence (ROC) because multiple signals have the same Laplace Transform. The Bilateral Laplace Transform is defined as:

\[ X_b(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \]

Example 1  Find the Bilateral Laplace Transform of \( e^{4t}u(t) \).

Example 2  Find the Bilateral Laplace Transform of \(-e^{4t}u(-t)\) and plot its ROC.

Notice that the Laplace Transforms for Examples 1 and 2 were the same, except that they had different Regions of Convergence. Therefore, we must always specify the ROC when working with the Bilateral Laplace Transform so that we can determine whether the time signal is left-sided or right-sided.
We state without proof here that the ROC of the Laplace Transform of the sum of multiple time functions is the INTERSECTION of the individuals ROCs. Use this fact to solve the next two Examples.

**Example 3**  Find the Bilateral Laplace Transform of \[e^{-3t}u(t) - e^{-t}u(-t)\] and plot its ROC.

**Example 4**  Find the Bilateral Laplace Transform of \[e^{-t}u(t) - e^{-3t}u(-t)\] and plot its ROC.
Inverse Bilateral Laplace Transform

To compute the Inverse Bilateral Laplace Transform, we'll again use Partial Fraction Expansion. We will ignore the case of multiple poles. Write the Bilateral Laplace Transform \( H(s) \) as:

\[
H(s) = \frac{B(s)}{A(s)} = b_N + \sum_{k=1}^{N} \frac{r_k}{s + s_k}
\]

Here, \( s_k \) are the poles, \( r_k \) are the residues, and \( b_N \) is nonzero if the order of the numerator \( B(s) \) is greater than the order of the denominator \( A(s) \).

**Example 5**  Find the Inverse Bilateral Laplace Transform of:

\[
X(s) = \frac{2s + 6}{s^2 + 6s + 8}, \quad \text{Re}\{s\} > -2
\]

To start, write \( X(s) \) as:

\[
\frac{2s + 6}{s^2 + 6s + 8} = \frac{A}{s + 2} + \frac{B}{s + 4}
\]
Example 6  Find the Inverse Bilateral Laplace Transform of

\[ X(s) = \frac{4s + 2}{s^2 + 3s + 2}, \quad \text{Re}\{s\} < -2 \]

Example 7  Find the Bilateral Inverse Laplace Transform of

\[ X(s) = \frac{2s + 1}{s^2 + s}, \quad -1 < \text{Re}\{s\} < 0 \]

Remember, you can always check your results of taking the inverse transform by taking the transform and comparing.