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The Tempered Aspirations solution for bargaining problems with a reference point

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ABSTRACT

Gupta and Livne (1988) modified Nash's (1950) original bargaining problem through the introduction of a reference point restricted to lie in the bargaining set. Additionally, they characterized a solution concept for this augmented bargaining problem. We propose and axiomatically characterize a new solution concept for bargaining problems with a reference point: the Tempered Aspirations solution. In Kalai and Smorodinsky (1975), aspirations are given by the so called *ideal* or *utopia* point. In our setting, however, the salience of the reference point mutes or tempers the negotiators' aspirations. Thus, our solution is defined to be the maximal feasible point on the line segment joining the modified aspirations and disagreement vectors. The Tempered Aspirations solution can be understood as a "dual" version of the Gupta–Livne solution or, alternatively, as a version of Chun and Thomson's (1992) Proportional solution in which the *claims* point is endogenous. We also conduct an extensive axiomatic analysis comparing the Gupta–Livne to our Tempered Aspirations solution.

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1. Introduction

In Nash's (1950) bargaining framework, the outcome of a negotiation is a function of the bargaining set, S, and the disagreement point $d \in S$. The first is the set of achievable utility profiles whilst the second represents the utilities the parties receive in the event a bargain is not struck. It has been noted by a number of writers that other aspects of the environment influence the outcome of the bargain; for example, the precedent set by a similar bargain from an earlier period. The practice of pattern bargaining employed by some trade unions and negotiations in modified rebuy contexts in Business to Business settings are other examples. The agreement struck with one employer or business in the industry becomes the reference point for agreements with other employers. To accommodate this, Gupta and Livne (1988) enriched Nash's framework through the introduction of a reference point $r \in S$.

In their work, r is interpreted to be an intermediate agreement which facilitates conflict resolution. They also propose and characterize a solution to the bargaining problem that depends on S, d and r. Their solution chooses the maximal point in S along the segment joining r and the aspirations of agents measured from the disagreement point d. We propose a new and different solution concept: the *Tempered Aspirations solution*. We suggest an outcome that is dictated by the aspirations bargainers draw from the reference, not the disagreement point. Specifically, bargainer i aspires to obtain r_i plus anything left over after satisfying other bargainers' claims. In our solution, the reference point assists in the formation of the players' aspirations. The conflict point no doubt defines the ideal (or utopia) point that each party would like to attain. The salience of the reference point, however, mutes or tempers the negotiators' aspirations. Having defined every bargainer's aspiration, the Tempered Aspirations solution is defined to be the maximal feasible point on the line joining the aspirations and disagreement vectors.

The structure of the paper is as follows. In Section 2, we describe the model and our new solution concept. In Section 3, we show that the proposed Tempered Aspirations solution can be characterized using Weak Pareto Optimality, Symmetry, Scale Invariance, Restricted Monotonicity with respect to the Reference Point, Irrelevance of Trivial Reference Points, and Continuity with Respect to the Bargaining Set. The first three axioms are adaptations of the typical requirements found in Nash (1950). The fourth axiom is a natural variation of Roth's (1979) well-known restricted monotonicity. The fifth axiom requires the reference point to change bargainers' expectations for it not to be trivial. The last axiom addresses technical issues. In Section 4, we study how our proposed solution concept behaves with respect to a number

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of different properties examined in the literature. In Section 5, we compare our Tempered Aspirations solution to its main alternative, the solution concept proposed by Gupta and Livne (1988). In this process, we also reexamine the latter solution concept in more detail. We conclude in the final section by proposing directions for future research.

2. The model

Let $\mathcal{B}$ be an infinite set of potential *bargainers* and let $\mathcal{N}$ be the family of all non-empty finite subsets of $\mathcal{B}$. Fix $N \in \mathcal{N}$. Let $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^N$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^N$. For every $M, N \in \mathcal{N}$ such that $M \subseteq N$, and every $x \in \mathbb{R}^N$, let x_M denote the projection of x onto $\mathbb{R}^M$. If N has more than one member, for every $x \in \mathbb{R}^N$ and every $i \in N$, define $x_{-i} = x_{N\setminus\{i\}}$ and let the unit vector $e_i = (1, \mathbf{0}_{-i}) \in \mathbb{R}^N$. Also, if $t > 0$, let $\frac{x}{t}$ abbreviate $\frac{1}{t}x$. Vector inequalities are treated as follows. Given $x, y \in \mathbb{R}^N$ we write $x \geq y$ if $x_i \geq y_i \; \forall i \in N$, $x > y$ if $x \geq y$ but $x \neq y$, and $x \gg y$ if $x_i > y_i \; \forall i \in N$. Let $Y \subseteq \mathbb{R}^N$. Y is said to be *convex* if for every $y_1, y_2 \in Y$ and every $\lambda \in [0, 1]$, $\lambda y_1 + (1 - \lambda)y_2 \in Y$. Y is called *comprehensive* if for every $x \in \mathbb{R}^N$ the fact that there is a $y \in Y$ such that $y \geq x$, implies $x \in Y$. The *convex and comprehensive hull* of Y, $cch(Y)$, is the smallest of all convex and comprehensive sets containing Y.

A *bargaining set* for N is any non-empty, convex, comprehensive and closed set $S \subseteq \mathbb{R}^N$ that is *bounded above* in the sense that there is a $p \in \mathbb{R}^N_{++}$ and a $w \in \mathbb{R}$ such that $\sum_{i\in N} p_i x_i \leq w$ for all $x \in S$. The bargaining set represents all the utility vectors that can be achieved by the players in N, bargaining among themselves.

Convexity presumes that randomization over the outcomes is possible. The comprehensiveness condition reflects the possibility of free disposal of utility. For every bargaining set S, define its *Pareto-optimal set* as $PO(S) = \{y \in S \mid x > y \text{ implies } x \notin S\}$. Similarly, its *weakly Pareto-optimal set* is defined as $WPO(S) = \{y \in S \mid x \gg y \text{ implies } x \notin S\}$.

A *bargaining problem* for N (Nash, 1950) is a pair (S, d) such that S is a bargaining set for N, $d \in S$, and there exists an $x \in S$ satisfying $x \gg d$. The point d is called the *disagreement point* and represents the utility obtained by the bargainers if no agreement is reached. For every bargaining set S and every $x \in S$, let the *aspirations vector* $a(S, x)$ be defined by $a_i(S, x) = \max\{t \in \mathbb{R} \mid (t, x_{-i}) \in S\}$ for every $i \in N$. Notice that the aspirations vector is well defined as S is closed and bounded above. The *ideal point* of the problem (S, d) represents bargainers' expectations before coming to the negotiation table and is defined by $a(S, d)$. Denote the family of all bargaining problems for N by Σ_0^N. A *solution concept on* Σ_0^N is a function ψ that associates with each $(S, d) \in \Sigma_0^N$ a unique outcome $\psi(S, d) \in S$. For example, the *Kalai–Smorodinsky solution* (Kalai and Smorodinsky, 1975) is defined for every $(S, d) \in \Sigma_0^N$ as $KS(S, d) = \lambda^* a(S, d) + (1 - \lambda^*)d$ where $\lambda^* = \max\{\lambda \in [0, 1] \mid \lambda a(S, d) + (1 - \lambda)d \in S\}$.

A *bargaining problem with a reference point* for N (Gupta and Livne, 1988) is a triple (S, d, r) where (S, d) is a bargaining problem for N and the *reference point* $r \in S \setminus WPO(S)$ satisfies $r \geq d$. We label the family of all bargaining problems with a reference point for N by Σ^N. A *solution concept on* Σ^N is a function ϕ that associates with each triple $(S, d, r) \in \Sigma^N$ a unique outcome $\phi(S, d, r) \in S$.

We now define our proposed solution concept and its main alternative in the literature.

Definition 2.1. The *Tempered Aspirations solution* is defined for every $(S, d, r) \in \Sigma^N$ as

$$TA(S, d, r) = \lambda^* a(S, r) + (1 - \lambda^*)d$$

where $\lambda^* = \max\{\lambda \in [0, 1] \mid \lambda a(S, r) + (1 - \lambda)d \in S\}$.

If a bargaining problem is translated so that the disagreement point is at the origin, our proposed solution is the only point along the frontier of S proportional to the aspirations vector $a(S, r)$.

Definition 2.2. The *Gupta–Livne solution* (Gupta and Livne, 1988) is defined for every $(S, d, r) \in \Sigma^N$ as

$$GL(S, d, r) = \lambda^* a(S, d) + (1 - \lambda^*)r$$

where $\lambda^* = \max\{\lambda \in [0, 1] \mid \lambda a(S, d) + (1 - \lambda)r \in S\}$.

The Gupta–Livne solution is "dual" to the Tempered Aspirations solution in the sense that it exchanges the roles played by the reference and disagreement points. In the Gupta–Livne framework, the disagreement point d has no role to play as a threat in the bargain. It serves only to form the aspirations of the players through the construction of the ideal aspiration point. Instead, we use the reference point r to set bargainers' aspirations, and d as a reference vector from which proportional payoffs are measured. Both solution concepts are illustrated in Fig. 1.

3. Characterization of the Tempered Aspirations solution

3.1. The axioms

The first three axioms, due to Nash (1950), are standard in the bargaining literature. They have been modified to account for the presence of a reference point. We assume for the moment that $N \in \mathcal{N}$ is fixed. In what follows, the axioms are written for a generic solution ϕ on Σ^N.

Weak Pareto-Optimality (WPO): For every $(S, d, r) \in \Sigma^N$, $\phi(S, d, r) \in WPO(S)$.

Let $\Pi(N)$ be the set of permutations of the set N. For every $\pi \in \Pi(N)$ and every $x \in \mathbb{R}^N$, define $\pi(x) \in \mathbb{R}^N$ as the vector such that for every $i \in N$, $(\pi(x))_{\pi(i)} = x_i$. For every $X \subseteq \mathbb{R}^N$ define $\pi(X) = \{\pi(x) \mid x \in X\}$. A problem $(S, d, r) \in \Sigma^N$ is said to be *symmetric* if, for every $\pi \in \Pi(N)$, $\pi(S) = S$, $\pi(d) = d$, and $\pi(r) = r$.

Symmetry (SYM): For every $(S, d, r) \in \Sigma^N$, if (S, d, r) is symmetric then, for every $i, j \in N$, $\phi_i(S, d, r) = \phi_j(S, d, r)$.

Let $\mathcal{L}$ be the family of vectors of functions $L = (L_i)_{i\in N}$ such that for every $i \in N$, there exist $m_i \in \mathbb{R}_{++}$ and $b_i \in \mathbb{R}$ satisfying, for every $t \in \mathbb{R}$, $L_i(t) = m_i t + b_i$.

Scale Invariance (SC.INV): For every $(S, d, r) \in \Sigma^N$ and every $L \in \mathcal{L}$, $\phi(L(S), L(d), L(r)) = L(\phi(S, d, r))$.

The following axiom is the logical counterpart to *restricted monotonicity* when a reference point is introduced. In its original form, the axiom requires that expanding the bargaining set without altering the bargainers' ideal point must not hurt any party's payoff. We argue that the introduction of a reference point necessarily changes expectations. Therefore, in our version of monotonicity, we substitute the ideal point, $a(S, d)$, by the vector of aspirations, $a(S, r)$.

r-Restricted S-Monotonicity (r-REST.S-MON): For every (S, d, r) and $(S', d', r') \in \Sigma^N$, $(d, r) = (d', r')$, $S \subseteq S'$ and $a(S, r) = a(S', r')$, imply $\phi(S, d, r) \leq \phi(S', d', r')$.

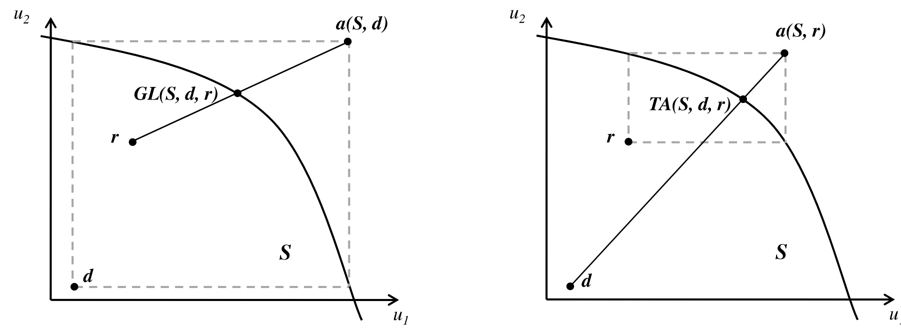
The following axiom states that whenever introducing a reference point does not change bargainers' initial aspirations, given by $a(S, d)$, the reference point might as well be replaced by the disagreement point.

Irrelevance of Trivial Reference Points (ITR): For every $(S, d, r) \in \Sigma^N$, $a(S, r) = a(S, d)$ implies $\phi(S, d, r) = \phi(S, d, d)$.

In what follows, convergence of sets is evaluated using the Hausdorff topology.

S-Continuity (S-CONT): For every sequence $\{(S_k, d, r)\}_k \subset \Sigma^N$ such that $\lim_{k\to\infty} S_k = S$ and $(S, d, r) \in \Sigma^N$, $\lim_{k\to\infty} \phi(S_k, d, r) = \phi(S, d, r)$.

We now show that the axioms above uniquely characterize the Tempered Aspirations solution.



Gupta-Livne solution   Tempered Aspirations solution

Fig. 1. Solution concepts for bargaining problems with a reference point.

Proposition 3.1. *A solution ϕ on Σ^N satisfies WPO, SYM, SC.INV, ITR, r-REST.S-MON, and S-CONT if and only if, for every $(S, d, r) \in \Sigma^N$, $\phi(S, d, r) = TA(S, d, r)$.*

Proof. It is easy to verify that the Tempered Aspirations solution satisfies the six axioms. Conversely, let ϕ be any solution satisfying them. Choose any $(S, d, r) \in \Sigma^N$. Thanks to SC.INV, there is no generality lost by assuming that $d = \mathbf{0}$ and $a(S, r) = \mathbf{1}$. Notice then that, for every $i, j \in N$, $TA_i(S, d, r) = TA_j(S, d, r)$. Define $\check{r} = \min\{r_i \mid i \in N\}$ and $V = \{\check{r}\mathbf{1} + (1-\check{r})e_i \mid i \in N\}$. Define next $S'' = cch(V \cup \{TA(S, d, r)\})$. By WPO and SYM, $\phi(S'', d, \check{r}\mathbf{1}) = TA(S, d, r)$. Now define $S' = \{x \in S \mid x \leq a(S, r) = \mathbf{1}\}$. If $TA(S, d, r) \in PO(S')$, as $S'' \subseteq S'$ and $a(S'', \check{r}\mathbf{1}) = a(S', \check{r}\mathbf{1})$, use r-REST.S-MON to conclude that $\phi(S'', d, \check{r}\mathbf{1}) = \phi(S', d, \check{r}\mathbf{1}) = TA(S, d, r)$. Otherwise, if $TA(S, d, r) \in WPO(S') \setminus PO(S')$, approximate S' with sets S'_k such that, for every $k \in \mathbb{N}$, $S'' \subseteq S'_k$, $(S'_k, d, \check{r}\mathbf{1}) \in \Sigma^N$, and $TA(S, d, r) \in PO(S'_k)$. Use r-REST.S-MON to see that for every $k \geq 1$, $\phi(S'_k, d, \check{r}\mathbf{1}) = TA(S, d, r)$. Hence, $\phi(S', d, \check{r}\mathbf{1}) = TA(S, d, r)$ by S-CONT. Observe that by definition of S', $a(S', r) = a(S', d) = a(S', \check{r}\mathbf{1}) = \mathbf{1}$. Thus, ITR implies $\phi(S', d, r) = \phi(S', d, d) = \phi(S', d, \check{r}\mathbf{1}) = TA(S, d, r)$. Finally, invoke again r-REST.S-MON and a continuity argument if necessary to yield $\phi(S, d, r) = TA(S, d, r)$, as desired. □

Notice that, if N has only two bargainers, the Tempered Aspirations solution is Pareto-optimal and S-CONT is not needed to obtain the result.

3.2. Independence of the axioms

A solution that assigns the disagreement point to every $(S, d, r) \in \Sigma^N$ shows that WPO is logically independent from the remaining axioms. Given a fixed $k \in N$, all axioms except SYM are satisfied by the k-dictatorial solution, defined for every $(S, d, r) \in \Sigma^N$ as $(a_k(S, d), d_{-k})$. The Egalitarian solution (Kalai, 1977), defined for every $(S, d, r) \in \Sigma^N$ as $d + \lambda^* \cdot \mathbf{1}$ where $\lambda^* = \max\{\lambda \in \mathbb{R}_+ \mid d + \lambda \cdot \mathbf{1} \in S\}$, shows the independence of SC.INV. The Nash bargaining solution (Nash, 1950), defined for every $(S, d, r) \in \Sigma^N$ as $\arg\max_{x \in S} \prod_{i \in N}(x_i - d_i)$, satisfies all axioms except r-REST.S-MON. The solution proposed by Herrero and Villar (2010), defined for every $(S, d, r) \in \Sigma^N$ as $\lambda^* a(S, r) + (1 - \lambda^*) r$, where $\lambda^* = \max\{\lambda \in [0, 1] \mid \lambda a(S, r) + (1 - \lambda) r \in S\}$, shows that ITR is not implied by the other five axioms. The independence of S-CONT is still an open question.

4. Other properties of the Tempered Aspirations solution

4.1. Monotonicity properties

The following properties describe how the Tempered Aspirations solution is affected if one of the bargainers improves her position, be it with respect to the disagreement or the reference point. Up to now, such properties have only been defined for standard bargaining problems (Thomson, 1987) and *bargaining problems with claims* (Bossert, 1992, 1993). We adapt these axioms to bargaining problems with a reference point.

The first property states that improving bargainer i's disagreement payoff (d_i) should not hurt her. The second property states that the remaining bargainers should not benefit from such an improvement. More formally:

d-Monotonicity (*d-MON*): For every $(S, d, r) \in \Sigma^N$, every $i \in N$, and every $t \in \mathbb{R}_{++}$, $(S, d + te_i, r) \in \Sigma^N$ implies $\phi_i(S, d + te_i, r) \geq \phi_i(S, d, r)$.

Strong d-Monotonicity (*ST.d-MON*): For every $(S, d, r) \in \Sigma^N$, every $i \in N$, and every $t \in \mathbb{R}_{++}$, $(S, d + te_i, r) \in \Sigma^N$ implies, for every $j \in N \setminus \{i\}$, $\phi_j(S, d + te_i, r) \leq \phi_j(S, d, r)$.

The reason for the names of these properties is that a continuous and weakly Pareto-optimal solution concept satisfying ST.d-MON must also comply with d-MON. If, additionally, $n := |N| = 2$, the two properties are equivalent (Thomson, 1987). Thomson also shows that the Nash and Kalai–Smorodinsky solutions are d-monotonic, but not strongly d-monotonic if $n \geq 3$. We show that the Tempered Aspirations solution satisfies both properties.

Proposition 4.1. *The Tempered Aspirations solution satisfies d-MON and ST.d-MON.*

Proof. The Tempered Aspirations solution satisfies WPO and S-CONT, so it is enough to show the second part. Let $(S, d, r) \in \Sigma^N$, $i \in N$, and $t \in \mathbb{R}_{++}$ be such that $(S, d + te_i, r) \in \Sigma^N$. Without loss of generality, assume $d = \mathbf{0}$. Let $\lambda, \mu \in (0, 1]$ be such that $TA(S, \mathbf{0}, r) = \lambda a(S, r)$ and $TA(S, te_i, r) = \mu a(S, r) + (1 - \mu) te_i$. For a contradiction, suppose there is a bargainer $k \in N \setminus \{i\}$ such that $\lambda a_k(S, r) < \mu a_k(S, r)$, so $\lambda < \mu$. But then, for every $j \in N$ we have $TA_j(S, \mathbf{0}, r) < TA_j(S, te_i, r)$, contradicting the fact that $TA(S, \mathbf{0}, r)$ is weakly Pareto-optimal. □

Analogously, we define properties that study how bargainers are affected once the reference point is altered.

r-Monotonicity (r-MON): For every $(S, d, r) \in \Sigma^N$, every $i \in N$, and every $t \in \mathbb{R}_{++}$, $(S, d, r + te_i) \in \Sigma^N$ implies $\phi_i(S, d, r) \geq \phi_i(S, d, r + te_i)$.

Strong r-Monotonicity (ST.r-MON): For every $(S, d, r) \in \Sigma^N$, every $i \in N$, and every $t \in \mathbb{R}_{++}$, $(S, d, r + te_i) \in \Sigma^N$ implies, for every $j \in N \setminus \{i\}$, $\phi_j(S, d, r + te_i) \leq \phi_j(S, d, r)$.

Only the first property is satisfied by the Tempered Aspirations solution.

Proposition 4.2. *The Tempered Aspirations solution satisfies r-MON.*

Proof. Let $(S, d, r) \in \Sigma^N$, $i \in N$, and $t \in \mathbb{R}_{++}$ be such that $(S, d, r + te_i) \in \Sigma^N$. Without loss of generality, let $d = \mathbf{0}$. Let $\lambda, \mu \in (0, 1]$ be such that $TA(S, \mathbf{0}, r) = \lambda a(S, r)$ and $TA(S, \mathbf{0}, r + te_i) = \mu a(S, r + te_i)$. By contradiction, suppose that $\lambda a_i(S, r) >$

$\mu a_i(S, r + te_i)$. As $a_i(S, r) = a_i(S, r + te_i)$ and, for every $j \in N \setminus \{i\}$, $a_j(S, r) \geq a_j(S, r+te_i)$, then $\lambda > \mu$ and, for every $j \in N$, $\lambda a_j(S, r) > \mu a_j(S, r + te_i)$. This contradicts the fact that $TA(S, \mathbf{0}, r + te_i)$ is weakly Pareto-optimal. □

Proposition 4.2 immediately implies that the Tempered Aspirations solution is also strongly r-monotonic if $n = 2$. We adapt an example by Thomson (1987) to show that this fact is not true if $n > 2$.

Example 4.3. The Tempered Aspirations solution does not satisfy ST.r-MON if $n \geq 3$. If $S = cch(\{(0, 2, 2), (2, 1, 2)\})$, $d = r = (0, 0, 0)$, and $r' = (1, 0, 0)$, then $TA(S, d, r) = (1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$ and $TA(S, d, r') = (1\frac{3}{5}, 1\frac{1}{5}, 1\frac{3}{5})$. The third bargainer is better off after the first coordinate of r was increased by one, contradicting ST.r-MON.

4.2. Uncertainty properties

Bargainers might be uncertain about tomorrow's value of any of the elements of the triple (S, d, r). The next family of properties examines whether our solution concept makes it desirable to write contingent contracts today, compared to waiting until such uncertainty is resolved. This type of analysis has previously been performed with respect to the bargaining set (e.g. Perles and Maschler (1981), Myerson (1981) and Bossert et al. (1996)) and the disagreement point (e.g. Chun and Thomson (1990a,b,c)). To the best of our knowledge, our work is the first one to study uncertain reference points.

The results in this subsection build on the similarities between the Tempered Aspirations solution and Chun and Thomson's (1992) *Proportional solution*. This last concept is defined for a model in which the "reference" point is required to lie outside the bargaining set.

A *bargaining problem with claims* for N is a triple (S, d, c) where (S, d) is a bargaining problem for N and the *claims point* $c \in \mathbb{R}^N$ satisfies $c \geq d$, $c \notin S$ and, for every $i \in N$, $c_i \leq \max\{x_i \mid x \in S\}$ whenever this maximum exists. We label the family of all bargaining problems with claims for N by Σ^N_C. Define the *Proportional solution* for every $(S, d, c) \in \Sigma^N_C$ as $P(S, d, c) = \lambda^* c + (1-\lambda^*)d$ where $\lambda^* = \max\{\lambda \in [0, 1] \mid \lambda c + (1-\lambda)d \in S\}$. Notice that when the aspirations vector $a(S, r)$ plays the role of the claims point, the Tempered Aspirations and the Proportional solutions coincide. That is, for every $(S, d, r) \in \Sigma^N$, $(S, d, a(S, r)) \in \Sigma^N_C$ and $TA(S, d, r) = P(S, d, a(S, r))$.

Chun and Thomson (1992) show that the Proportional solution is *concave with respect to the bargaining set*, i.e., if problems (S, d, c) and (S', d, c) will respectively occur with probabilities α and $1 - \alpha$, then the outcome of committing to a solution today, $P(\alpha S + (1 - \alpha)S', d, c)$, dominates the expected outcome of solving the problem tomorrow, $\alpha P(S, d, c) + (1 - \alpha)P(S', d, c)$. Hence, restricting S' to leave the aspirations vector unchanged leads to another property satisfied by the Tempered Aspirations solution. Notice this restriction was also used in the definition of r-REST.S-MON.

r-Restricted S-Concavity (*r-REST.S-CAV*): For every (S, d, r), $(S', d', r') \in \Sigma^N$ and every $\alpha \in [0, 1]$, $(d, r) = (d', r')$ and $a(S, r) = a(S', r')$ imply $\phi(\alpha S + (1 - \alpha)S', d, r) \geq \alpha\phi(S, d, r) + (1 - \alpha)\phi(S', d', r')$.

Regarding uncertainty with respect to d, Chun and Thomson (1992) show that the proportional solution is *quasi-concave with respect to the disagreement point*. As modifying the disagreement point does not alter the aspirations vector, the Tempered Aspirations solution also satisfies the corresponding property.

d-Quasi-Concavity (*d-QCAV*): For every (S, d, r), $(S', d', r') \in \Sigma^N$, every $i \in N$, and every $\alpha \in [0, 1]$, $(S, r) = (S', r')$ implies $\phi_i(S, \alpha d + (1 - \alpha)d', r) \geq \min\{\phi_i(S, d, r), \phi_i(S', d', r')\}$.

We now define an analogous property, applicable when the reference point is uncertain.

r-Quasi-Concavity (*r-QCAV*): For every (S, d, r), $(S', d', r') \in \Sigma^N$, every $i \in N$, and every $\alpha \in [0, 1]$, $(S, d) = (S', d')$ implies $\phi_i(S, d, \alpha r + (1 - \alpha)r') \geq \min\{\phi_i(S, d, r), \phi_i(S', d', r')\}$.

Example 4.4. The Tempered Aspirations solution does not satisfy r-QCAV. For example, if $S = cch(\{(8, 0), (2, 6)\})$, $d = (0, 0)$, $r = (0, 1)$, $r' = (4, 3)$, and $\alpha = \frac{1}{2}$, then $TA(S, d, r) = (4\frac{4}{13}, 3\frac{9}{13})$, $TA(S, d, r') = (4\frac{4}{9}, 3\frac{5}{9})$, and $TA(S, d, \frac{1}{2}r + \frac{1}{2}r') = (4, 4)$. Bargaining with respect to $\frac{1}{2}r + \frac{1}{2}r'$ leaves the first bargainer worse off than at r and r'.

4.3. Variable population properties

Here we examine how $TA(S, d, r)$ changes when a subset of bargainers leaves the table. The cases to consider depend on the payoffs obtained by those who quit. If they leave and just take their disagreement payoff, then it is expected that the remaining players remain at least as well as before. Solution concepts satisfying this requirement are said to be *population monotonic* (Thomson, 1983). We also define a similar property for the case in which those who leave take with them their reference payoff. The formal definitions follow.

Up to now, the set of bargainers N has remained fixed. In order to work with varying numbers of bargainers, we need the following definitions. Let $\Sigma = \bigcup_{N \in \mathcal{N}} \Sigma^N$. A *solution on* Σ is a function ϕ that associates with each triple $(S, d, r) \in \Sigma$ a unique outcome $\phi(S, d, r) \in S$. The remaining axioms in this subsection are written for a generic solution ϕ on Σ.

Population d-Monotonicity (*POP.d-MON*): For every $M, N \in \mathcal{N}$ such that $M \subseteq N$, every $(S', d', r') \in \Sigma^N$, and every $(S, d, r) \in \Sigma^M$, $(d, r) = (d'_M, r'_M)$ and $S = \{x \in \mathbb{R}^M \mid (x, d'_{N\setminus M}) \in S'\}$ imply $\phi_M(S', d', r') \leq \phi(S, d, r)$.

Population r-Monotonicity (*POP.r-MON*): For every $M, N \in \mathcal{N}$ such that $M \subseteq N$, every $(S', d', r') \in \Sigma^N$, and every $(S, d, r) \in \Sigma^M$, $(d, r) = (d'_M, r'_M)$ and $S = \{x \in \mathbb{R}^M \mid (x, r'_{N\setminus M}) \in S'\}$ imply $\phi_M(S', d', r') \leq \phi(S, d, r)$.

The next example and proposition show that while our solution does not satisfy POP.d-MON, it does comply with POP.r-MON.

Example 4.5. Let $S' = cch(\{(4, 0, 8), (0, 0, 12), (0, 4, 8)\})$, $d' = (0, 0, 0)$ and $r' = (\frac{1}{2}, 2, 8\frac{1}{2})$. Then $a(S', r') = (1\frac{1}{2}, 3, 9\frac{1}{2})$ and $TA(S', d', r') = (1\frac{2}{7}, 2\frac{4}{7}, 8\frac{1}{7})$. If the third bargainer leaves the negotiations empty-handed, the new bargaining set is $S = cch(\{(4, 0), (0, 4)\})$ and the projections of d' and r' are $d = (0, 0)$ and $r = (\frac{1}{2}, 2)$. Then $a(S, r) = (2, 3\frac{1}{2})$ and $TA(S, d, r) = (1\frac{5}{11}, 2\frac{6}{11})$, making the second bargainer worse off. Therefore, the Tempered Aspirations solution is not population d-monotonic.

Proposition 4.6. *The Tempered Aspirations solution satisfies POP. r-MON.*

Proof. Let $M, N \in \mathcal{N}$ be such that $M \subseteq N$, $(S', d', r') \in \Sigma^N$ and $(S, d, r) \in \Sigma^M$ be such that $(d, r) = (d'_M, r'_M)$, and $S = \{x \in \mathbb{R}^M \mid (x, r'_{N\setminus M}) \in S'\}$. Notice that, for every $i \in M$, $\max\{t \mid (t, r'_{N\setminus\{i\}}) \in S'\} = \max\{t \mid (t, r_{M\setminus\{i\}}) \in S\}$, so $a_M(S', r') = a(S, r)$. Let $l \subset \mathbb{R}^M$ denote the line determined by d and $a(S, r)$. Then $TA_M(S', d', r') \in l$ because collinearity is preserved when projecting on $\mathbb{R}^M$. As $TA(S, d, r)$ also lies along l and is weakly Pareto-optimal, it must be the case that $TA_M(S', d', r') \leq TA(S, d, r)$. □

Another possibility is that bargainers leave taking with them their share according to a solution concept ϕ. Then, it is expected that the remaining bargainers are indifferent between renegotiating among themselves and just accepting their previous

ϕ-share. If so, ϕ is said to be *consistent*. This condition was originally proposed by Lensberg (1988), calling it *multilateral stability*.

Consistency (CONS): For every $M, N \in \mathcal{N}$ such that $M \subseteq N$, every $(S', d', r') \in \Sigma^N$, and every $(S, d, r) \in \Sigma^M$, $(d, r) = (d'_M, r'_M)$ and $S = \{x \in \mathbb{R}^M \mid (x, \phi_{N\setminus M}(S', d', r')) \in S'\}$ imply $\phi_M(S', d', r') = \phi(S, d, r)$.

The Tempered Aspirations solution is not consistent, as the following example shows.

Example 4.7. Let S' be defined as in Example 4.5 and $d' = r' = (0, 0, 0)$. Then $a(S', r') = (4, 4, 12)$ and $TA(S', d', r') = (2, 2, 6)$. If the first bargainer leaves the negotiations with a payoff of 2 units, then the new bargaining set is $S = cch(\{(2, 8), (0, 10)\})$ and the projections of d' and r' are $d = r = (0, 0)$. Then $a(S, r) = (2, 10)$ and $TA(S, d, r) = (1\frac{2}{3}, 8\frac{1}{3})$, different from the original payoffs received by the second and third bargainers. Thus, the Tempered Aspirations solution is not consistent.

4.4. Domination axioms

The last group of axioms studies whether using our concept guarantees that agents improve with respect to the disagreement/reference point payoffs.

d-Domination (*d-DOM*): For every $(S, d, r) \in \Sigma^N$, $\phi(S, d, r) \geq d$.

r-Domination (*r-DOM*): For every $(S, d, r) \in \Sigma^N$, $\phi(S, d, r) \geq r$.

It is clear that the Tempered Aspirations solution satisfies d-DOM, but Example 4.5 also shows that, if $n \geq 3$, it is possible that a bargainer receives less than her reference point payoff.[1] An extension of the Tempered Aspirations solution satisfying r-DOM can be constructed, using the method designed by Bossert (1993) to extend the Claims Egalitarian solution.[2]

5. Comparison with the Gupta–Livne solution

5.1. Axiomatic analysis of the Gupta–Livne solution

The *Gupta–Livne solution* (defined in Section 2) is the main solution concept in the bargaining literature with a reference point. In this section, we compare it with our Tempered Aspirations solution. As a byproduct of this comparison, we obtain a number of new results for Gupta and Livne's concept. As we will see, the duality between the two solutions influences the type of axioms satisfied by each of them.

Gupta and Livne characterize their solution using the already familiar WPO, SYM, and SC.INV, plus the following three axioms.

Relevant Domain (RD): For every $(S, d, r) \in \Sigma^N$, $\phi(S, d, r) = \phi(cch(\{x \in S \mid x \geq d\}), d, r)$.

The RD property states that the outcome of the negotiation is only affected by points that weakly Pareto-dominate the disagreement point. This property is also satisfied by the Tempered Aspirations solution, although it is not used in our characterization.

d-Restricted S-Monotonicity (*d-REST.S-MON*): For every (S, d, r), $(S', d', r') \in \Sigma^N$, $(d, r) = (d', r')$, $S \subseteq S'$, and $a(S, d) = a(S', d')$ imply $\phi(S, d, r) \leq \phi(S', d', r')$.

Originally proposed for standard bargaining problems by Roth (1979), the d-REST.S-MON axiom can be seen as dual to r-REST.S-MON. As the bargaining set S grows, the corresponding aspirations must remain fixed in order to preserve monotonicity.

[1] We thank Hervé Moulin for pointing out this possibility.

[2] We thank William Thomson for this observation.

Limited d-Sensitivity (*LIM.d-SENS*): For every $(S, d, r), (S', d', r') \in \Sigma^N$, $(S, r) = (S', r')$ and $a(S, d) = a(S', d')$ imply $\phi(S, d, r) = \phi(S', d', r')$.

This axiom was originally labeled *limited sensitivity to changes in the disagreement point*. It says that if the disagreement point changes without altering the corresponding aspirations, then the outcome of the negotiation is the same. Our solution concept satisfies its dual version, LIM.r-SENS. However, our characterization only uses ITR, a property clearly implied by LIM.r-SENS.

Analogous arguments to Propositions 4.1 and 4.2 work to show that the Gupta–Livne solution satisfies r-MON, ST.r-MON and d-MON. Nevertheless, if $n \geq 3$, ST.d-MON is violated.

Example 5.1. If $n \geq 3$, the Gupta–Livne solution is not strongly d-monotonic. Let S be defined as in Example 4.3, $d = (0, 0, 0)$ and $r = d' = (1, 0, 0)$. Then $GL(S, d, r) = (1\frac{3}{5}, 1\frac{1}{5}, 1\frac{1}{5})$ and $GL(S, d', r') = (1\frac{3}{4}, 1\frac{1}{8}, 1\frac{1}{2})$. The third bargainer is better off after the first coordinate of d was increased by one, contradicting ST.d-MON.

In terms of properties dealing with uncertainty, it is straightforward to define the axioms d-REST.S-CAV and r-QCAV (analogous to r-REST.S-CAV and d-QCAV) and then show that the Gupta–Livne solution satisfies them. However, the example below shows that the Gupta–Livne solution does not satisfy d-QCAV.

Example 5.2. Let S be defined as in Example 4.4, $r = (4, 3)$, $d = (0, 0)$, $d' = (4, 2)$, and $\alpha = \frac{1}{2}$, then $GL(S, d, r) = (4\frac{4}{7}, 3\frac{3}{7})$ and $GL(S, d', r) = (4\frac{2}{3}, 3\frac{1}{3})$, but $GL(S, \frac{1}{2}d + \frac{1}{2}d', r) = (4\frac{1}{2}, 3\frac{1}{2})$. Bargaining with respect to $\frac{1}{2}d + \frac{1}{2}d'$ leaves the first bargainer worse off than at d and d', so the Gupta–Livne solution violates d-QCAV.

Analogous arguments to those shown in Proposition 4.6 ensure that the Gupta–Livne solution does satisfy POP.d-MON. However, Example 4.7 shows that it is not consistent. The following example shows that Gupta and Livne's concept does not comply with POP.r-MON either.

Example 5.3. Let $S' = cch(\{(0, 6, 6), (6, 0, 6), (12, 0, 0)\})$, $d' = (0, 0, 0)$ and $r' = (0, 0, 6)$. Then $a(S', d') = (12, 6, 6)$ and $GL(S', d', r') = (4, 2, 6)$. If the third bargainer leaves with her reference point payoff of 6, the new bargaining set is $S = cch(\{(6, 0), (0, 6)\})$. The projections of d' and r' are $d = (0, 0)$ and $r = (0, 0)$. Then $GL(S, d, r) = (3, 3)$, making the first bargainer worse off. Although in this example $r' \in WPO(S')$, moving r marginally toward the origin does not change the essence of the argument. Therefore, the Gupta–Livne solution is not population r-monotonic.

Table 1 compares the two solution concepts. Observe that the number of properties satisfied by our concept but not by Gupta–Livne is equal to the number of properties for which the opposite holds. It is interesting to note that this informal counting analysis did not result in lopsided results.

5.2. The linear frontier case

Given a set of bargainers $N \in \mathcal{N}$, we define a *linear bargaining problem with a reference point* as a triple $(S, d, r) \in \Sigma^N$ such that $S = \{x \in \mathbb{R}^N \mid \sum_{i\in N} p_i x_i \leq w\}$ for some $(p, w) \in \mathbb{R}^N_{++} \times \mathbb{R}$. We denote the family of linear bargaining problems with a reference point as Θ^N. As $\Theta^N \subseteq \Sigma^N$, all definitions made in previous sections remain valid.

Throughout this subsection, we normalize the bargaining set to be $\bar{S} = \{x \in \mathbb{R}^n \mid \sum_{i\in N} x_i \leq 1\}$ and d to be at the origin. SC.INV prevents any loss of generality. This new problem is equivalent to

Table 1
Axioms satisfied by the Tempered Aspirations (TA) and the Gupta–Livne (GL) solutions.

Axiom	Acronym	TA	GL
Weak Pareto-optimality	WPO	✓ $*$	✓ $*$
Symmetry	SYM	✓ $*$	✓ $*$
Scale invariance	SC.INV	✓ $*$	✓ $*$
S-continuity	S-CONT	✓ $*$	✓ $*$
r-restricted S-monotonicity	r-REST.S-MON	✓ $*$	×
d-restricted S-monotonicity	d-REST.S-MON	×	✓ $*$
Irrelevance of trivial reference points	ITRP	✓ $*$	×
Limited r-sensitivity	LIM.r-SENS	✓	×
Limited d-sensitivity	LIM.d-SENS	×	✓ $*$
Restricted domain	RD	✓	✓ $*$
d-monotonicity	d-MON	✓	✓
Strong d-monotonicity	ST.d-MON	✓	× ($n \geq 3$)
r-monotonicity	r-MON	✓	✓
Strong r-monotonicity	ST.r-MON	× ($n \geq 3$)	✓
r-restricted S-concavity	r-REST.S-CAV	✓	×
d-restricted S-concavity	d-REST.S-CAV	×	✓
r-quasi-concavity	r-QCAV	×	✓
d-quasi-concavity	d-QCAV	✓	×
Population r-monotonicity	POP.r-MON	✓	×
Population d-monotonicity	POP.d-MON	×	✓
Consistency	CONS	×	×
r-domination	r-DOM	× ($n \geq 3$)	✓
d-domination	d-DOM	✓	✓

$*$ denotes axioms used to characterize the corresponding solution concept.

dividing an estate of size one among the bargainers in N. Besides the Tempered Aspirations and Gupta–Livne solutions, two other possible distributions are equal division, represented by the vector $\frac{\mathbf{1}}{n}$, and proportional division according to r. To abbreviate notation, given a reference point $r \in \mathbb{R}^N$, let $R := \sum_{i\in N} r_i$. Then proportional division is represented by the vector $\frac{r}{R}$.

Restricting the bargaining set to have a linear frontier enables us to further compare our solution with Gupta–Livne. Moulin (1987) shows that any solution concept satisfying WPO, SYM, and SC.INV, is *decentralizable* and *additive*[3] if and only if it can be expressed as a linear combination of the vectors representing equal and proportional division. The Tempered Aspirations and the Gupta–Livne solution satisfy both properties, so the four vectors are aligned.

Proposition 5.4. *For every $(S, d, r) \in \Theta^N$ such that $r > d$ and $TA(S, d, r) \neq GL(S, d, r)$, $TA(S, d, r)$ is closer to equal division than $GL(S, d, r)$; and $GL(S, d, r)$ is closer to proportional division than $TA(S, d, r)$.*

Proof. After normalizing the problem (S, d, r), assume that $n > 1$ and r is asymmetric, otherwise $TA(S, d, r) = GL(S, d, r)$ and there is nothing to prove. Expressing the solution concepts as linear combinations of the equal and the proportional division vectors yields

$$TA(\bar{S}, \mathbf{0}, r) = \frac{R}{R + n(1-R)}\left[\frac{r}{R}\right] + \frac{n(1-R)}{R + n(1-R)}\left[\frac{\mathbf{1}}{n}\right]$$

and

$$GL(\bar{S}, \mathbf{0}, r) = \frac{(n-1)R}{n-R}\left[\frac{r}{R}\right] + \frac{n(1-R)}{n-R}\left[\frac{\mathbf{1}}{n}\right].$$

It only remains to show that $\frac{R}{R+n(1-R)} \leq \frac{(n-1)R}{n-R}$. As $n - 1 \geq 1$, we know that $n-R = n(1-R)+(n-1)R \leq n(n-1)(1-R)+(n-1)R = (n-1)[R + n(1-R)]$. Multiplying by R and rearranging achieves the desired result. □

The procedure above also shows that when $n = 2$, both coefficients are equal to $\frac{R}{2-R}$. Therefore we have the corollary below. Notice that a geometric argument based on the symmetry along the Pareto-frontier is also possible.

Corollary 5.5. *The Tempered Aspirations and the Gupta–Livne solutions coincide in linear problems with two bargainers.*

The next example illustrates the fact that, even assuming a linear frontier, the Tempered Aspirations and the Gupta–Livne solutions do not coincide if $n \geq 3$.

Example 5.6. Consider three bargainers deciding about how to divide \$12, so that the bargaining set is described by $S = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 \leq 12\}$. The disagreement point is $d = (0, 0, 0)$ and the reference point is $r = (2, 5, 1)$. Equal division is $(4, 4, 4)$ while proportional division according to r yields $(3, 7\frac{1}{2}, 1\frac{1}{2})$. Furthermore, $GL(S, d, r) = (3\frac{3}{7}, 6, 2\frac{4}{7})$ and $TA(S, d, r) = (3\frac{3}{5}, 5\frac{2}{5}, 3)$. Notice that the four solutions are collinear, with our Tempered Aspirations solution being closest to equal division of the estate.

6. Concluding remarks

It is important to note that we are not arguing in this paper that the Tempered Aspirations solution concept is superior to the solution concept of Gupta and Livne (1988) or the reverse. In fact, to the contrary, we wish to emphasize the point that there is no one solution for all seasons. What we do claim is that the context of the bargain will affect the manner in which the reference point influences the negotiated outcome. The axiomatic approach taken here is useful to understand their differences and similarities, therefore making it easier to choose an appropriate concept in any given situation.

Regarding future research, we believe that one fruitful avenue is an experimental comparison of the Tempered Aspirations and the Gupta–Livne solutions. It would be useful to determine the situational and contextual boundary conditions under which each of the solution concepts would be significantly better predictors. A second direction for future work is to exploit the similarities with Chun and Thomson's (1992) Proportional solution and study the implications of making the claims point depend on the shape of the bargaining set. Finally, Balakrishnan and Eliashberg (1995) also study aspirations, but from an alternating offers perspective. It would be interesting then to find a bargaining process leading to our solution concept, enabling us to compare both approaches.

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[3] Let ϕ be a solution concept on Θ^N satisfying WPO, SYM, and SC.INV. ϕ is *decentralizable* if, for every normalized linear problem $(\bar{S}, \mathbf{0}, r) \in \Theta^N$ and every $i \in N$, $\phi_i(S, d, r)$ is a function of r_i and R. This property is equivalent to *non-advantageous reallocation* (Moulin, 1985), an axiom limiting the strategic possibilities of multiple-player coalitions. ϕ is *additive* if, for every normalized linear problem $(\bar{S}, \mathbf{0}, r) \in \Theta^N$, every $i \in N$, and every $\alpha_1, \alpha_2 \geq 1$, $\phi_i((\alpha_1 + \alpha_2)\bar{S}, \mathbf{0}, r) = \phi_i(\alpha_1\bar{S}, \mathbf{0}, r) + \phi_i(\alpha_2\bar{S}, \mathbf{0}, r)$.

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