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Theory and Methodology

A maximin procedure for the optimal insertion timing of ad executions

P.V. (Sundar) Balakrishnan [a,*], Nicholas G. Hall [b]

[a] *Business Administration Program, University of Washington, Bothell, WA 98021, USA*

[b] *Department of Management Sciences, College of Business, The Ohio State University, Columbus, OH 43210-1399, USA*

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Abstract

It has been recommended that dominant brands should not suffer from long periods with little or no advertising. For such brands in a product category which is not subject to seasonality, it is important not to let the minimum effectiveness level fall too low. Toward this end, we present an analytical model for determining the optimal insertion timing pattern of a long run ad campaign which consists of a number of varying executions. We develop a computational procedure to calculate the time between the insertions within a pulse in order to maximize the minimum effectiveness level at any point in time. The focus of the procedure is to provide conceptual insights and on benchmark solutions, which can reduce the computational burden of simulations, rather than on providing an exact solution. We also provide a theoretical justification for the two common economic axioms of nonsatiation and diminishing marginal returns, in the context of an advertising model with exponential forgetting.

Keywords: Maximin strategy; Optimal insertion timing; Algorithm; Dominant brand

1. Introduction and literature review

One of the major research issues in advertising, that continues to be the focus of attention of both practitioners and scholars, is the scheduling of advertisements (Mahajan and Muller, 1986), since it was first brought to the notice of academics by Zielske (1959). It is only lately that academics have started to pay attention to the important issue of utilizing *varying executions* in an ad campaign. Consequently, most of the analytical models developed have tended to focus on investigating different aspects of the problem of repeating the same ad. However, one of the major problems faced by advertising planners is: given a pool of ads, how should one best space the timing between the different exposures (Poltrack, 1983)? In this paper, we present an analytical model that might be employed for determining the optimal timing between insertions of a long-run ad campaign consisting of a number of varying executions.

We now briefly review the literature to establish the factors that are relevant to the advertisement timing decision. We begin with the motivation, followed by discussions on the choice of an appropriate timing objective, the theory of consumer learning and forgetting, and the current timing models.

* Corresponding author.

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1.1. Motivation

In the seminal Zielske (1959) study, the question posed was: "Should an advertising schedule be concentrated in an intensive 'burst' or should it be spread over a longer period?". The experimental stream (Zielske and Henry, 1980; Zielske, 1986; Strong, 1974), which has provided much of the motivation for this research, has concerned itself with obtaining primary data on the recall of advertising by the target audience, when exposed to both television and mail advertising at varying frequencies and intervals. In the original Zielske (1959) study, two similar groups of the target audience were mailed 13 different advertisements from the same newspaper campaign. One group received the advertisements at weekly intervals, thus after 13 weeks the 'annual budget' allotted for advertising, for this group, was exhausted. The budget for the other group, on the other hand, was spread over the entire year by placing the advertisements at four-week intervals. This resulted in the now well-known patterns of advertising recall for the two different schedules. An econometric model, resulting from the extension of the tracking study to the domain of television advertising, was employed to generate and compare recall patterns for different ways of scheduling the same number of rating points (Zielske and Henry, 1980). An examination of these different recall patterns leads directly to the question "Which of the options for scheduling advertisements is best?".

To summarize, if the spacing between the exposures is very far apart, then there will be forgetting of the ads which would destroy their cumulative impact. On the other hand, if the exposures are tightly spaced together, the effective time span of the campaign is lessened. Consequently, the ideal media plan, according to Rossiter and Percy (1987, p.583), is one that "...swamps the competition in terms of learned communication effects and employs a relatively large and varied pool of ads to prevent wearout [i.e.,] a continuous, even schedule, which would even ignore seasonal sales trends". However, most advertisers, they acknowledge, "...cannot afford to simply out-advertise the competition", especially for long periods. Therefore, naturally enough, the answer to the above question, given the trade-off to be made in the choice of the best exposure pattern, is that it depends upon the scheduling objective of the decision-maker.

1.2. Scheduling objectives

The optimal choice of the exposure timing pattern and implicitly the related objective function depends on a number of factors. Some of the relevant factors mentioned in the literature are the consumer's ability to recall ad messages; nature of the product; seasonality of product consumption; intensity of competitive advertising; and, purchase cycle (Strong, 1977; Poltrack, 1983; Rossiter and Percy, 1987). For instance, as Zielske (1959) notes, if the objective is "to make a maximum number of people at least temporarily remember the advertising, then: an intensive burst of thirteen weekly exposures would be preferable...". This concentration of exposures during the early part of the budget period, i.e., a blitz scheduling strategy (Mahajan and Muller, 1986), however, would result in leaving a large portion of the time frame with little or no recall, by the customers, of the advertising. On the other hand, Zielske (1959) conjectures, for products which require continual remembrance of the ads in order to maintain their sales, a blitz scheduling strategy would be suboptimal. A better objective would be "to obtain a maximum average weekly number of consumers remembering the advertising". For this, he recommends spreading the budgeted allowance of rating points over the planning period, i.e., a pulsing strategy, as preferable to an intensive burst. This criterion of average effect (i.e., the consequence of the ad: recall, awareness, etc.) per time period, has become quite widely accepted.

One of the major problems with this criterion is the problem associated with any 'average' measure. That is, the same number of rating points could be scheduled in many different ways, which would all have the same average effect (Poltrack, 1983, p. 289; Zielske and Henry, 1980). In other words, the average measure is deceiving, since it evens out the peaks and valleys. That is, if a firm were to employ such a criterion, it could conceivably end up with a schedule which suffered from a lot of variability, as the recall pattern would exhibit extreme levels of maximum and minimum effect.

If a particular brand was, for instance, relatively weak or new, and seeking to establish itself by 'outshouting' the dominant brands, then such a scheduling pattern would make sense. Similarly, this schedule would be effective if the market for the category was highly seasonal, such as with skis or Christmas candles. However, for a dominant brand, Poltrack (1983, p. 290) notes, it is less critical to outshout the competition than it is to "... avoid any long period of absence from the air, because that would give the challenging brands exclusive exposure to the dominant brand's customers".

Additionally, such a highly variable recall pattern schedule could considerably exacerbate this problem for dominant brands in an unseasonal product category with a relatively short purchase cycle, especially if the response function were S-shaped. For at the convex part of the response curve, the marginal effort and expense involved in increasing the advertising effect from where it hits its minimum value could be considerable. In such situations, the firm might address the problem by imposing a requirement, in the form of a constraint, to have the minimum advertising effect exceed a certain value.

A problem with this approach, however, is that the media-planner may have no idea as to what might be a reasonable value of the minimum effect constraint. An alternative to this approach is to make the minimum effect level a part of the firm's objective function. By doing so, this problem of having a high and consistent effect (Katz, 1980) can be addressed by attempting to push this criterion of minimum advertising effect to the maximum possible level. Such an approach is modeled explicitly by us. In this manner we can design an improved scheduling strategy, especially for those products requiring continual remembrance to maintain their sales, by ensuring that the advertising effect is not compromised at any point in time.

1.3. Consumer learning and forgetting

Only lately have marketing academics begun investigating issues relating to the use of varying ads (Pechmann and Stewart, 1990, p. 324). However, practitioners have long recognized that there may be differences in the execution of ads which in turn acknowledges that different information may be put out by each such commercial. For instance, a campaign for a product, such as Miller beer, may typically consist of a pool of ads which are tested *a priori* for attitude favorability and then ordered and rotated.

The notion that different ad executions provide different information is consistent with the view espoused by Stewart (1964), who notes that advertising should be viewed as a channel of communication whose basic function is to transmit messages to consumers. Of the series of measurable effects that advertising could be visualized as having, one of these is "to inform consumers about product attributes such as features, usage, and price". As Little and Lodish (1969) note, it is also important to recognize that each of these executions may provide differing amounts of information, i.e., a "larger ad may convey more information". For instance, in a pool of ads, one ad may be oriented toward providing information on price, another may stress product warranty, a third may emphasize ease of use, and so on.

There are multiple measures of effect that could be employed ranging from the measurable such as recall, awareness and sales to unobservable constructs such as exposure level (Lodish, 1971). In order to incorporate this aspect explicitly, without loss of generality, we characterize each advertisement execution in terms of its effectiveness (i.e., relative strength) which then can be related back to the appropriate communication surrogate measures such as its information content, presentation time, reach and size.

The behavioral literature has examined this question of scheduling, at the individual level, in terms of learning and memory theory with the focus on understanding and explaining the underlying mental processes that determine how individuals respond to a message repeated over time (Katz, 1980). There are two basic premises of this research stream. First, effect over time will increase as more information is retained with each additional exposure. Second, in a critical reexamination of research on forgetting curves, Bogartz (1990) concludes that the amount of information retained at any time is independent of the initial level of learning. This is supported by Zielske and Henry's (1980) data from studies of television commercials that "did not show any relationship between the number of rating points and forgetting". Therefore, in learning models, these different pieces of information with which the consumer is presented are assumed to increase the 'exposure level' of the individual in an additive fashion (Lodish, 1971). Most forgetting models, on the other hand, assume that the forgetting phenomenon is best represented using an exponential decay pattern (Bogartz, 1990; Blattberg and Jeuland, 1981).

1.4. Timing models

The analytical stream has attempted to develop models that help select optimal media schedules. These can be categorized as optimization models, simulation models or heuristics (Rust, 1986). The use of simulations or heuristics arose due to the complexity of the scheduling task. For, as Aaker and Myers (1982) note, even for firms operating with a small budget, the number of possible media schedules to choose from can be extremely large. Consequently, evaluating the large numbers of possible schedules in terms of the firm's effect criteria (e.g., sales, recall, awareness, etc.) is computationally exhausting. Therefore, any analytical model that develops a procedure for reducing the number of alternative insertion schemes to be examined will be helpful (Zielske and Henry, 1980). That is, by the provision of bench-marks to guide the simulation, the computational burden can be reduced.

Among the optimization models, there has been extended debate on the superiority of the pulsing strategy. In important recent papers that generalize prior results, Hahn and Hyun (1991) and Park and Hahn (1991) prove that a pulsing policy is optimal in the long run under fairly reasonable conditions, such as when the ratio of the pulsing cost to the fixed cost of advertising is relatively small. Their models do not, however, address the issue of how the pulses should be timed.

In addition to the theoretical superiority of pulsing, there is also the pragmatic problem of employing other strategies such as 'constant spending' (Simon 1982), also referred to as 'continuous advertising' (Sethi, 1971) or 'even policy' (Mahajan and Muller, 1986). All of this means that, at least as far as radio and television are concerned, there is really only one advertising strategy which is implementable, namely the pulsing policy. This is not as strange as it may seem, especially given the practical difficulty for firms of employing the even policy, unless one had access to, say, a 24 hour all-advertisements cable network. The only difference between the various terminologies then reduces to the question of timing of the successive exposures.

In this paper, our objective then becomes to develop a model for advertisement scheduling, which will incorporate the following factors: firstly, a budget constraint, which explicitly recognizes the number of different ads which can be run during the planning period; secondly, the development and production of the different executions which vary in their relative effectiveness; and thirdly, the learning and forgetting of information by the consumers. The model then provides as the output the timing of the various ad executions so as to maximize the minimum total effect at any point in time. In the next section, we present the analytical modeling procedure and the corresponding theoretical results. Section 3 presents the associated comparative statics, and Section 4 concludes the paper.

2. Analytical model

The focus of the paper is on determining the insertion timing pattern between the varying executions that are to be repeated periodically. In other words, we would like to develop a computational procedure to calculate the time between insertions of the varying executions within a pulse. This timing pattern is repeated periodically over the long run. We assume, similar to Hahn and Hyun (1991), that the time between the pulses is a constant of period T.

The analytical model presented in this section is an attempt to define an optimal timing sequence procedure for n ad executions. The underlying model of the consumer adopted here is based on the work of Blattberg and Jeuland (1981). As in their case, we let q denote the probability of an individual being exposed to an ad. Also, the exposure process is Bernoulli, and the population is homogeneous with respect to, q, the probability of reach.

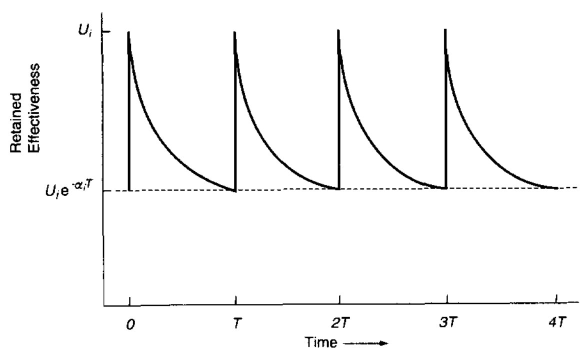
Similar to MEDIAC (Little and Lodish, 1969), each ad insertion, which can be characterized by its effectiveness level (i.e., the relative strength of an ad), increases the total amount of effect (the consequence of the ad) which then gradually decays over time. However, we do not simply consider the effectiveness level of one advertisement i to be Y_i, which then decays asymptotically to zero. Instead, we follow Blattberg and Jeuland (1981) (see also Sethi, 1983) in looking at the long run 'steady state' of a system, where the 'aggregate effectiveness function', has a sawtoothed representation between two asymptotes. The lower asymptote is the total amount of effectiveness retained immediately prior to the next ad insertion. The upper asymptote is the total amount of effectiveness retained immediately following the last advertisement insertion. For an individual advertisement i, we let U_i denote the value of the upper asymptote, and $U_i\,e^{-\alpha_i T}$ denote the value of the lower asymptote. That is,

$$U_i = \frac{qY_i}{1-(1-q)e^{-\alpha_i T}},$$

where q is as before, and α_i (>0), for $i = 1,\ldots,n$, is the exponential decay factor for the different executions (Sethi, 1983; Blattberg and Jeuland, 1981). The total effectiveness retained over time for a single advertisement is as shown in Fig. 1.

Note that since the total response function is merely the sum of the response functions to individual advertisements, it follows immediately that the problem of maximizing the average effectiveness is a trivial one. If $t_1,\ldots,t_n$ denote the times within a pulsing cycle of length T at which the executions are inserted, with $t_1 = 0$, then any value of $t_1,\ldots,t_n$ will give the same value for the average total response. Moreover, it can be shown that maximizing the maximum value at any point in time for the total response function is also trivially solvable by inserting all advertisements simultaneously.

Our objective here is to find the optimal timing policy which will maximize the minimum total effectiveness retained as a result of n advertisements of the type illustrated in Fig. 1. It is important to note that all our results are independent of the order of the advertisements within the pulse. We



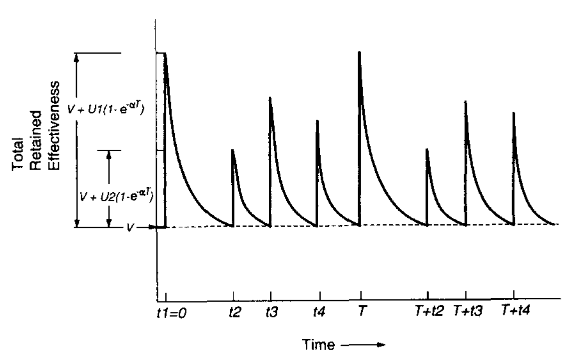


Fig. 1. Effectiveness of a single advertisement.

Fig. 2. Total effectiveness of four advertisements.

therefore assume an arbitrary order. Fig. 2 gives an example of the total effectiveness function in which $n = 4$. In Fig. 2, each advertisement is used once in a common cycle of duration T.

Let V_i denote the total amount of effectiveness retained *immediately prior* to the insertion of the ith advertisement, $i = 1, \dots, n$. We begin by establishing conditions on an optimal timing policy.

Lemma 1 (Necessary condition for optimality). *In an optimal policy,*

$$V_1 = V_2 = \cdots = V_n. \tag{1}$$

Proof of Lemma 1. We will show that if (1) is not satisfied, then the solution cannot be optimal. Let $k = \operatorname{argmax}_{1 \le i \le n} V_i$ and consider the effect, on the minimum total amount of information, of the following change:

$$t_k \leftarrow t_k + \Delta, \quad \text{where } 0 < \Delta < t_{k+1} - t_k \text{ and } t_{n+1} = T.$$

The new values for the total effectiveness retained immediately prior to each advertisement are

$$V_k^* = V_k + \sum_{t_i < t_k} U_i\left[e^{-\alpha_i(t_k - t_i + \Delta)} - e^{-\alpha_i(t_k - t_i)}\right] + \sum_{t_i > t_k} U_i\left[e^{-\alpha_i(T + t_k + \Delta - t_i)} - e^{-\alpha_i(T + t_k - t_i)}\right],$$

and

$$V_i^* = \begin{cases} V_i + U_k\left[e^{-\alpha_i(t_k - t_i)} - e^{-\alpha_i(t_k - t_i + \Delta)}\right] & \text{if } t_i < t_k,\ 1 \le i \le n, \\ V_i + U_k\left[e^{-\alpha_i(T + t_k - t_i)} - e^{-\alpha_i(T + t_k - t_i + \Delta)}\right] & \text{if } t_i > t_k,\ 1 \le i \le n. \end{cases}$$

Now for $\Delta > 0$ and small, $\min_{1 \le i \le n} V_i^* > \min_{1 \le i \le n} V_i$, therefore the original solution cannot be optimal. □

While Lemma 1 shows that condition (1) is necessary for optimality, it does not prove that it is sufficient. However, sufficiency will be established by the uniqueness of the optimal value which we prove in Theorem 1 below. Since we now have a characterization of an optimal timing policy, we will attempt to find closed form solutions for that policy. We make the simplifying assumption of identical decay rates throughout the analysis which follows. Thus, let $\alpha = \alpha_1 = \cdots = \alpha_n$ represent the decay parameter. Let $V = V_1 = \cdots = V_n$, and define $t_1, \dots, t_n$ as above, with $t_1 = 0$. The following result will be a useful step in defining an optimal policy.

Lemma 2. *In an optimal policy,*

$$t_{i+1} - t_i = \frac{1}{\alpha} \ln\left[\frac{V + U_i - U_i\, e^{-\alpha T}}{V}\right]. \tag{2}$$

Proof. From condition (1), we know that $V_i = V_{i+1}$, which gives

$$\begin{aligned}
& U_1\, e^{-\alpha t_i} + U_2\, e^{-\alpha(t_i - t_2)} + \cdots + U_i\, e^{-\alpha T} + U_{i+1}\, e^{-\alpha(T + t_i - t_{i+1})} + U_{i+2}\, e^{-\alpha(T + t_i - t_{i+2})} \\
&\quad + \cdots + U_n\, e^{-\alpha(T + t_i - t_n)} \\
&\quad = U_1\, e^{-\alpha t_{i+1}} + U_2\, e^{-\alpha(t_{i+1} - t_2)} + \cdots + U_i\, e^{-\alpha(t_{i+1} - t_i)} + U_{i+1}\, e^{-\alpha T} + U_{i+2}\, e^{-\alpha(T + t_{i+1} - t_{i+2})} \\
&\qquad + \cdots + U_n\, e^{-\alpha(T + t_{i+1} - t_n)} \\
&\quad \Rightarrow V_{i+1} - U_i\, e^{-\alpha(t_{i+1} - t_i)} - U_{i+1}\, e^{-\alpha T} = e^{-\alpha(t_{i+1} - t_i)}\left[V_i - U_i\, e^{-\alpha T} - U_{i+1}\, e^{-\alpha(T + t_i - t_{i+1})}\right] \\
&\quad \Rightarrow V - U_i\, e^{-\alpha(t_{i+1} - t_i)} - U_{i+1}\, e^{-\alpha T} = e^{-\alpha(t_{i+1} - t_i)} V - e^{-\alpha T} U_i\, e^{-\alpha(t_{i+1} - t_i)} - U_{i+1}\, e^{-\alpha T} \\
&\quad \Rightarrow \left(V + U_i - U_i\, e^{-\alpha T}\right) e^{-\alpha(t_{i+1} - t_i)} = V
\end{aligned}$$

$$\Rightarrow \mathrm{e}^{-\alpha(t_{i+1}-t_i)} = \left(\frac{V}{V+U_i-U_i\,\mathrm{e}^{-\alpha T}}\right)$$

$$\Rightarrow t_{i+1}-t_i = \frac{1}{\alpha}\ln\left(\frac{V+U_i-U_i\,\mathrm{e}^{-\alpha T}}{V}\right). \qquad \square$$

A few comments must be made about Lemma 2. This result will give us a way to compute the optimal values of $t_2, \ldots, t_n$ using $t_1 = 0$, if we can find the optimal value of V (which we can think of as the objective function). So the optimal values for the decision variables are defined implicitly in (2), in terms of the (unknown) optimal objective value. While a closed form expression for the value of V will not be obtainable, we show how that value can always be found very easily.

Theorem 1 (Existence of a unique optimal value).
(a) *The optimal value of V is a solution to the polynomial*

$$\mathrm{e}^{\alpha T}V = U_1 + U_2\left(\frac{V+U_1-U_1\,\mathrm{e}^{-\alpha T}}{V}\right) + U_3\frac{(V+U_1-U_1\,\mathrm{e}^{-\alpha T})(V+U_2-U_2\,\mathrm{e}^{-\alpha T})}{V^2} + \cdots + U_n\prod_{i=1}^{n-1}\frac{V+U_i-U_i\,\mathrm{e}^{-\alpha T}}{V}. \tag{3}$$

(b) *The polynomial* (3) *has exactly one solution which is both positive and real. Consequently, that solution is V.*

Proof. (a): From condition (1),

$$V = V_1 \Rightarrow V = U_1\,\mathrm{e}^{-\alpha T} + U_2\,\mathrm{e}^{-\alpha(T-t_2)} + U_3\,\mathrm{e}^{-\alpha(T-t_3)} + \cdots + U_n\,\mathrm{e}^{-\alpha(T-t_n)}$$

$$\Rightarrow V = \mathrm{e}^{-\alpha T}\left[U_1 + U_2\,\mathrm{e}^{\alpha t_2} + U_3\,\mathrm{e}^{\alpha t_3} + \cdots + U_n\,\mathrm{e}^{\alpha t_n}\right]$$

$$\Rightarrow \mathrm{e}^{\alpha T}V = U_1 + U_2\left(\frac{V+U_1-U_1\,\mathrm{e}^{-\alpha T}}{V}\right) + U_3\frac{(V+U_1-U_1\,\mathrm{e}^{-\alpha T})(V+U_2-U_2\,\mathrm{e}^{-\alpha T})}{V^2} + \cdots + U_n\prod_{i=1}^{n-1}\frac{V+U_i-U_i\,\mathrm{e}^{-\alpha T}}{V},$$

using (2).

(b): The polynomial of part (a) is of the form

$$f(V) = C_0V^n + C_1V^{n-1} + \cdots + C_{n-1}V + C_n = 0$$

where $C_0 > 0$, and $C_j < 0$ for $j = 1, \ldots, n$. Clearly if $n = 1$, there is at most one real, positive solution. Assume, for induction, that there is at most one real, positive solution for $n = 1, \ldots r$, but more such solutions for $n = r + 1$. However, the derivative $(\mathrm{d}f(V)/\mathrm{d}V) = nC_0V^{n-1} + (n-1)C_1V^{n-2} + \cdots + C_{n-1}$ inherits the same signs of coefficients as $f(V)$ and therefore, by the assumption, $(\mathrm{d}f(V)/\mathrm{d}V) = 0$ has at most one real positive solution. Thus $f(V)$ can have at most one turning point in the interval $0 \le V < \infty$, over which it is continuously differentiable. Since $f(0) < 0$ and $f(+\infty) > 0$, the assumption that there are two or more real, positive solutions is contradicted. Indeed, the function $f(V)$ looks as shown in Fig. 3.

Further, since $f(V)$ is continuous, it is clear that it must have exactly one real, positive solution. This establishes that the polynomial (3) always has exactly one real, positive solution, namely the optimal solution value. $\square$

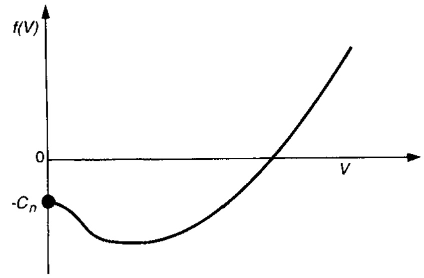


Fig. 3. The function $f(V)$.

Theorem 1 suggests the following simple algorithm, based on the idea of binary search (Aho, Hopcroft and Ullman, 1974), which will always find the optimal value of V. Substitution of this value into (2), using $t_1 = 0$, will then provide the optimal values of $t_2, \dots, t_n$.

Algorithm Adtime

Step 0. (Input and Initialization)

Input $U_1, \dots, U_n$, T and α. [Note that the order of $U_1, \dots, U_n$ is arbitrary]

Input ε [ε is a prespecified tolerance]

LL $\leftarrow 0$ [LL represents the lower limit of the interval being searched]

UL $\leftarrow \sum_{i=1}^{n} U_i$ [UL represents the upper limit of the interval being searched]

Step 1. (Test Step)

$$V \leftarrow \frac{\mathrm{LL} + \mathrm{UL}}{2}$$

$$f(V) \leftarrow \mathrm{e}^{\alpha T} V - U_1 - U_2\left(\frac{V + U_1 - U_1\,\mathrm{e}^{-\alpha T}}{V}\right) - U_3 \frac{\left(V + U_1 - U_1\,\mathrm{e}^{-\alpha T}\right)\left(V + U_2 - U_2\,\mathrm{e}^{-\alpha T}\right)}{V^2}$$

$$- \cdots - U_n \prod_{i=1}^{n-1} \frac{V + U_i - U_i\,\mathrm{e}^{-\alpha T}}{V^{n-1}}$$

If $f(V) \leq 0$, go to Step 2

If $f(V) > 0$, go to Step 3

Step 2. (Search Upper Half of Interval)

LL $\leftarrow V$

If $|f(V)| < \varepsilon$, go to Step 4; otherwise, go to Step 1.

Step 3. (Search Lower Half of Interval)

UL $\leftarrow V$

If $|f(V)| < \varepsilon$, go to Step 4; otherwise, go to Step 1.

Step 4. (Compute Optimal Advertising Times)

$t_1 \leftarrow 0$

for $i = 2, \dots, n$ *do*

$$t_{i+1} \leftarrow t_i + \frac{1}{\alpha} \ln\left[\frac{V + U_i - U_i\,\mathrm{e}^{-\alpha T}}{V}\right]$$

Terminate.

A few comments about Algorithm Adtime seem appropriate. The number of steps in which Adtime will terminate is a function of the prespecified accuracy requirement, ε. Nevertheless, this number of steps is typically small, since the only computations needed are elementary operations incorporated into a binary search procedure. Thus, although no closed form solution is known to the equation $f(V)=0$, we have provided an easily implementable computer-based procedure for finding that solution, and consequently values for advertising times which are optimal or extremely close to optimal, depending on the value of, ε, the tolerance.

3. Comparative analysis and managerial implications

In this section, we further assess the properties of the proposed insertion timing model. First, the more general model is considered with respect to the specifics of a few special cases with a view to better understanding the managerial implications of these results. The timing patterns of the proposed model are contrasted with those of alternate timing policies and finally the sensitivity analysis to changes in individual executions is provided.

3.1. Special cases

We begin with two potentially useful special cases, for which closed form solutions are obtainable. First, consider the special case where there are only two different advertisements ($n=2$) which need to be inserted in each cycle. Without loss of generality, we let $t_1=0$ and need to find the optimal value of t_2. Using expression (2), we find

$$t_2=\frac{1}{\alpha}\ln\left[\frac{V+U_1-U_1\,\mathrm{e}^{-\alpha T}}{V}\right], \tag{4}$$

and we need to find the value of V in order to find t_2. Evaluating the total effectiveness retained immediately prior to advertisement 2,

$$V=U_2\,\mathrm{e}^{-\alpha T}+U_1\,\mathrm{e}^{-\alpha t_2},$$

then, using (4),

$$\begin{aligned}
V&=U_2\,\mathrm{e}^{-\alpha T}+\left[\frac{U_1V}{V+U_1-U_1\,\mathrm{e}^{-\alpha T}}\right]\\
&\Rightarrow V^2+U_1V-U_1V\,\mathrm{e}^{-\alpha T}=VU_2\,\mathrm{e}^{-\alpha T}+U_1U_2\,\mathrm{e}^{-\alpha T}-U_1U_2^{-2\alpha T}+U_1V\\
&\Rightarrow V^2-(U_1+U_2)V\,\mathrm{e}^{-\alpha T}+U_1U_2(\mathrm{e}^{-2\alpha T}-\mathrm{e}^{-\alpha T})=0\\
&\Rightarrow V=\tfrac{1}{2}\left\{(U_1+U_2)\,\mathrm{e}^{-\alpha T}\pm\left[(U_1+U_2)^2\,\mathrm{e}^{-2\alpha T}-4U_1U_2(\mathrm{e}^{-2\alpha T}-\mathrm{e}^{-\alpha T})\right]^{1/2}\right\}\\
&\Rightarrow V=\tfrac{1}{2}\left\{(U_1+U_2)\,\mathrm{e}^{-\alpha T}+\left[(U_1-U_2)^2\,\mathrm{e}^{-2\alpha T}+4U_1U_2(\mathrm{e}^{-\alpha T})\right]^{1/2}\right\},
\end{aligned} \tag{5}$$

taking the positive square root, since only the larger of the two solutions is positive, which we know from Theorem 1. The optimal timing of two advertisements under exponential decay is therefore easy to find, since we first use expression (5) to find V, and then use expression (4) to find t_2.

Therefore, returning to the more general situation of n advertisements, $n \geq 2$, and considering the special case $U_1 = U_2 = \cdots = U_n$, it is clear that an optimal policy in this case is $t_1 = 0$, $t_2 = T/n$, $t_3 = 2T/n, \dots, t_n = (n-1)T/n$.

3.2. Comparison with blitz policy

We next examine how great is the increase in the minimum value of total effectiveness retained which is achieved by such a policy, relative to a naive blitz policy of inserting all advertisements simultaneously. The blitz or simultaneous advertising policy has a minimum value of $V = (U_1 + U_2 + \cdots + U_n)e^{-\alpha T} = nU\,e^{-\alpha T}$, where $U = U_1 = \cdots = U_n$. For the optimal timing policy, we find

$$\begin{aligned} V &= U_n\,e^{-\alpha T/n} + U_{n-1}\,e^{-2\alpha T/n} + \cdots + U_1\,e^{-\alpha T} \\ &= U\left[Q + Q^2 + \cdots + Q^n\right], \quad \text{where } Q = e^{-\alpha T/n} < 1, \\ &= QU\left[1 + Q + \cdots + Q^{n-1}\right] = \frac{QU[1-Q^n]}{[1-Q]}. \end{aligned}$$

Therefore the ratio of the optimal value of V to that provided by the simultaneous advertising policy is

$$\frac{QU(1-Q^n)}{(1-Q)} \Big/ nUQ^n = \frac{Q(1-Q^n)}{(1-Q)nQ^n} = \frac{(Q^{-n}-1)}{n(Q^{-1}-1)} = \frac{(e^{\alpha T}-1)}{n(e^{\alpha T/n}-1)}$$

We can get some feeling for how big this improvement is by trying different values of α and T. In particular, if $\alpha T = 1$, this ratio becomes

$$\frac{e-1}{n(e^{1/n}-1)}$$

and as $n \to \infty$ the limit of this expression is $e - 1$. This means that the percentage increase in the minimum value over time of the total amount of effectiveness retained when the number of advertisements is large becomes $100(e-2) \approx 71.8\%$, a very considerable improvement.

3.3. Comparison with equal interval policy

Another comparison could be made with the alternative policy of equal spacing of the insertions (i.e., the 'equal interval' policy). It is interesting to note that with an equal interval policy the minimum value of total effectiveness becomes dependent upon the *order* of insertions of the ads. This is unlike the optimal timing policy proposed here, wherein the insertion times are independent of the ordering of the ads. This, of course, implies that magnitude of improvement (in the minimum value) will vary depending upon the specific ordering of the ads desired by the campaign manager. (This is relatively easy to prove and left to the reader.) Thus, it is only in the optimal maximin policy does the campaign manger not have to worry about the impact of *a priori* sequencing of the ads.

Numerical example. We illustrate the relative merits of the alternative scheduling strategies by means of a numerical example. The different policies are compared with respect to their performance on the minimum exposure level. Consider a case wherein 3 different executions are to be inserted within, without loss of generality, a common unit cycle (i.e., $T = 1$). The rate of forgetting $\alpha = 0.2$ and is constant across all executions. Also, the relative effectiveness of the three executions, are set at 10, 20 and 40, in terms of U_i, the value of their upper asymptote.

For the above values, algorithm Adtime suggests that the optimal value of V, i.e., the maximum value of the minimum total effectiveness, is 60.786. To achieve this optimal value, the three ads are inserted at times $t_1 = 0$, $t_2 = 0.147$, and $t_3 = 0.437$. On the other hand, a blitz scheduling strategy of inserting all three ads at the same time would result in a value of minimum total effectiveness of 57.311. The equal interval policy of spacing the ads would, in this case, result in a quite acceptable performance of $V = 60.211$. In addition, we considered three randomly generated timing schemes. The first random timing policy of scheduling the ads at $t_1 = 0.18$, $t_2 = 0.20$, and $t_3 = 0.84$, resulted in the value of $V = 59.109$. A second random timing policy of scheduling the ads at $t_1 = 0.07$, $t_2 = 0.54$, and $t_3 = 0.67$, resulted in the value of $V = 59.09$. The third random timing policy of scheduling the ads at $t_1 = 0.29$, $t_2 = 0.73$, and $t_3 = 0.91$, resulted in the value of $V = 59.483$. The relative performance of each of these five different scheduling policies resulted in a performance that ranged from 94.28% to less than 98% of the value obtained by the optimal policy suggested by the proposed algorithm.

Essentially, in a highly competitive environment wherein a lot effort and energy is devoted to the creative side of the advertisement business, we are able to show that media planners who rely purely on simulations for scheduling and timing of insertions will not be typically able to squeeze out all of the inefficiencies. At the same time, the above numerical example should also be encouraging to the ad industry, as it seems to suggest that the amount of inefficiencies in the timing of insertions is fairly low.

3.4. Sensitivity analysis

We now investigate the effect of changes in the effectiveness of individual executions on the achievable maximum value for the minimum total amount of effectiveness retained. While the lack of a closed form expression for V does present a problem, however, we can still perform a sensitivity analysis through the use of implicit differentiation. Cross-multiplying in (3), we let

$$f(V) = e^{\alpha T} V^n - \sum_{i=1}^{n} U_i V^{n-1} - (1 - e^{-\alpha T}) \sum_{i=1}^{n} \sum_{j>i}^{n} U_i U_j V^{n-2} - (1 - e^{-\alpha T})^2 \sum_{t=1}^{n} \sum_{j>t}^{n} \sum_{k>j}^{n} U_t U_j U_k V^{n-3} - \cdots,$$

then setting

$$\begin{aligned}\frac{\partial f(V)}{\partial U_i} &= n\, e^{\alpha T} V^{n-1} \frac{\partial V}{\partial U_i} - \sum_{\substack{j=1\\ j\neq i}}^{n} U_j (n-1) V^{n-2} \frac{\partial V}{\partial U_i} - V^{n-1} \\ &\quad - U_i (n-1) V^{n-2} \frac{\partial V}{\partial U_i} - (1 - e^{-\alpha T}) \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{k>j\\ k\neq i}}^{n} U_j U_k (n-2) V^{n-3} \frac{\partial V}{\partial U_i} - (1 - e^{-\alpha T}) \sum_{\substack{j=1\\ j\neq i}}^{n} U_j V^{n-2} \\ &\quad - (1 - e^{-\alpha T}) U_i \sum_{\substack{j=1\\ j>i}}^{n} U_j (n-2) V^{n-3} \frac{\partial V}{\partial U_i} \\ &\quad - (1 - e^{-\alpha T})^2 \sum_{\substack{t=1\\ t\neq i}}^{n} \sum_{\substack{j>t\\ j\neq i}}^{n} \sum_{\substack{k>i\\ k\neq i,t}}^{n} U_t U_j U_k (n-3) V^{n-4} \frac{\partial V}{\partial U_i} - (1 - e^{-\alpha T})^2 \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{k>j\\ k\neq i}}^{n} U_j U_k V^{n-3}\end{aligned}$$

$$-(1-e^{-\alpha T})^2 U_i \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{k>j\\ k\neq i}}^{n} U_j U_k (n-3) V^{n-4} \frac{\partial V}{\partial U_i} - \cdots = 0$$

$$\Rightarrow \frac{\partial V}{\partial U_i} = \frac{V^{n-1} + (1-e^{-\alpha T}) \sum_{\substack{j=1\\ j\neq i}}^{n} U_j V^{n-2} + (1-e^{-\alpha T})^2 \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{k>j\\ k\neq i}}^{n} U_j U_k V^{n-3} + \cdots}{n\, e^{\alpha T} V^{n-1} - (n-1) \sum_{j=1}^{n} U_j V^{n-2} - (n-2)(1-e^{-\alpha T}) \sum_{j=1}^{n} \sum_{k>j}^{n} U_j U_k V^{n-3} - \cdots} \tag{6}$$

Now the sign of the denominator in (6) must be positive, since V is a solution of (3), and in (6) the first (positive) term in the denominator of $f(V)$ is multiplied by a factor of $n\, e^{\alpha T} > n$, compared to (3), and all the other (negative) terms by a smaller factor. Thus $(\partial V/\partial U_i) > 0$, $i = 1, \ldots, n$, as we would expect. Examining the (all positive) terms in the numerator of (6), we see that every individual term, and thus the total, is smaller the larger is the value of U_i. In a sense we have 'diminishing marginal returns' as we increase the effectiveness of the superior executions which may have been longer, better designed, etc. Specifically, it is always worthwhile to increase a U_i-value if no cost is incurred by doing so, but the marginal rate of increase in V which occurs as a result, will itself decrease as U_i increases. In other words, the managerial implication of the above finding is that the greatest return will always be obtained by first improving the least effective execution rather than on refining the most effective execution.

4. Conclusions

In this paper we have provided an innovative model for the insertion timing of varying advertisement executions, incorporating the use of an objective function which maximizes minimum effectiveness. This objective is particularly appropriate when it is not desirable to have a portion of the time frame with little or no recall of the product by the consumers, for example with dominant brands not subject to seasonality. For such situations would, even if occurring only rarely, render the market share of a company or a product vulnerable to attack.

While the assumption of exponential decay prevents closed form solutions from being found, we are nevertheless able to provide a very quick computer-based algorithm for finding the optimal insertion timing. Such procedures can be used to provide benchmark solutions rather than an exact answer. Also, we are able to show theoretically that investment in improved effectiveness of individual advertisements should be directed to those executions with relatively low effectiveness. Moreover, such investment always produces a positive return. We are thus able to provide a theoretical justification for two common economic axioms, namely nonsatiation and diminishing marginal returns, in the context of advertising.

It is also instructive to note at this point that the independence of ad sequencing on minimum effectiveness in the maximin strategy is relatively robust. That is, even relaxing the exponential decay assumption of forgetting with, perhaps, a less realistic but more tractable assumption of linear delay decay leads to essentially the same conclusion. Please see the Appendix for the derivation of the results. Additionally, given that the model developed and proposed here is of a very general nature it is possible to employ it for timing the insertion of executions across different media (e.g., newspaper, magazine, radio and television).

The limitations of the current model lead us to suggest several interesting extensions or variations for future research. One of these would be to permit the individual executions to have different cycle times. This assumption, however, would present considerable computational difficulties, and the existence of an efficient, or even reasonably tractable, algorithm is very unlikely. Another related analytical extension would be to determine the optimal pulsing time. The emphasis in the model of this paper is on finding a 'steady state' policy which could be used over an infinite horizon, but a corresponding finite horizon model would also be of considerable interest. In that connection, an enumerative approach using dynamic programming would seem to have the most promise. Finally, instead of the maximin objective function, it would be interesting to examine the difference in the insertion timing pattern, if any, required to minimize the difference between the upper and lower values of the total effectiveness function.

Additionally, among the most profitable avenues for future research would be incorporate the competitive dynamics of the marketplace. A important question in this regard would be determine the best response to a dominant competitor employing a maximin strategy. Additionally, more myopic policies are sometimes of interest to managers who may be currently involved in promotional activities emphasizing short-term returns. These promotional activities combined with seasonality would perhaps demand very different strategies. Toward this end, we encourage other researchers to conduct both laboratory studies as well as to report the results of actual applications. This should provide more empirical data to help develop richer analytical models that will be of greater use in developing advertising decision support systems.

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Appendix

Linear decay of information

In this Appendix, we derive the optimal timing sequence of n advertisement executions within a cycle of known duration, T. Each advertisement, i, disseminates an effect level Y_i immediately upon release. Under the linear decay scenario, the effect level retained declines linearly from Y_i to 0 over the next T units of time. The entire cycle repeats over an infinite horizon.

Our objective in this analysis is to find the optimal time to place the n executions within a common cycle of duration, T, in order to maximize the minimum total effect retained at any point in time. Clearly, if all executions are inserted at time 0, then at time T, the total effect retained is zero. But, an alternative policy may ensure that the total effect retained is never zero.

The central idea of our approach is to prevent the total effect retained from falling to a very low level, by not permitting the points of low effect for different advertisements to occur simultaneously. A policy of this type will never schedule two advertisements simultaneously. Thus, let t_i denote the time at which advertisement i is executed within the cycle of length T. Since the total effect retained depends only on the relative timing of advertisements to each other within the cycle, we can fix any one of the advertisements. Thus, we let $t_1 = 0$, and we have $0 < t_i < T, i = 2, \dots, n$, as decision variables. There are n advertisements in every cycle of length T, and we let Z_i denote the total effect retained immediately prior to the insertion of the ith execution, $i = 1, \dots, n$. We begin to formalize our search for an optimal timing policy with the following lemma.

Lemma A.1. *In an optimal policy*, $Z_1 = \cdots = Z_n$.

Proof. If the lemma does not hold, then $Z_i < Z_j$ for some $1 \le i \ne j \le n$. Now consider the impact, on the minimum total effect, of the following change:

$$t_i = t_i + \Delta,$$

where $0 < \Delta < t_{i+1} - t_i$ and $T_{n+1} = T$. The new values for the total effect retained immediately prior to each advertisement are

$$Z_j^* = Z_j - \Delta \sum_{i \ne j}^{n} Y_i,$$

and $Z_k^* = Z_k + \Delta Y_j$, $k = 1,\dots,n$; $k \ne j$. Now for $\Delta > 0$ and small, $\min_{1 \le i \le n} Z_i^* > \min_{1 \le i \le n} Z_i$, therefore the original solution cannot be optimal. □

To establish that the condition of Lemma A.1 is not only necessary for optimality, but also sufficient, it will be enough to show that the minimum value for the total effect retained, by any policy satisfying the lemma, is unique.

Lemma A.2. $Z_A = Z_1 = \cdots = Z_n$ *and* $Z_B = Z_1 = \cdots = Z_n$ *be the total effect retained immediately prior to the respective advertisements, under two different timing policies, both of which satisfy Lemma* A.1. *Then* $Z_A = Z_B$.

Proof. Consider the area under the graph representing the total effect retained. Since the height of the graph is simply the sum of the heights for individual advertisements, we know that the area must be $T\Sigma_{i=1}^{n}(\frac{1}{2}Y_i)$. This area consists of two parts, firstly a rectangle of area TZ_A or TZ_B, and secondly n triangles of areas $\frac{1}{2}TY_1^2\Sigma_{i=1}^{n}Y_i,\dots,\frac{1}{2}TY_n^2\Sigma_{i=1}^{n}Y_i$. Thus

$$TZ_A + T\sum_{i=1}^{n}\left(\tfrac{1}{2}Y_i^2\right)\sum_{j=1}^{n}Y_j = TZ_B + T\sum_{i=1}^{n}\left(\tfrac{1}{2}Y_i^2\right)\sum_{j=1}^{n}Y_j,$$

and therefore $Z_A = Z_B$. □

We can now use these two results to define an optimal timing policy.

Theorem A.1. (a) *The timing policy which maximizes the minimum total effect retained at any point in the cycle is given by* $t_1 = 0$, *and* $t_i = T\Sigma_{j=1}^{i-1}Y_j/(\Sigma_{j=1}^{n}Y_j)$, $i = 2,\dots,n$.
(b) *The maximal value of the minimum effect retained using the policy of* (a) *is* $\Sigma_{j=2}^{n}\Sigma_{k=1}^{j-1}Y_kY_j/(\Sigma_{k=1}^{n}Y_k)$.

Proof. (a): By induction on i. For $i = 1$, note that Lemma A.1 implies $Z_1 = Z_2$, which gives

$$\sum_{j=2}^{n} t_jY_j/T = Y_1(T - t_2)/T + \sum_{j=3}^{n}(t_j - t_2)Y_j/T \Rightarrow t_2 = TY_1/\left(\sum_{j=1}^{n}Y_j\right).$$

For the induction hypothesis, we assume $t_i = T\Sigma_{k=1}^{i-1}Y_k/(\Sigma_{k=1}^{n}Y_k)$, for $i = 1,\dots,r$, where $1 \le r \le n-1$, and show that this holds for $i = r+1$. Equivalently, we must show $t_{i+1} - t_i = TY_i/(\Sigma_{k=1}^{n}Y_k)$.

From Lemma A.1, $Z_i = Z_{i+1}$, which gives

$$\sum_{k=i+1}^{n}(t_k - t_i)Y_k/T + \sum_{k=1}^{i-1}(T + t_k - t_i)Y_k/T = \sum_{k=i+2}^{n}(t_k - t_{i+1})Y_k/T + \sum_{k=1}^{i}(T + t_k - t_{i+1})Y_k/T$$

$$\Rightarrow t_{i+1}Y_{i+1} - t_i\sum_{k=i+1}^{n}Y_k - t_i\sum_{k=1}^{i-1}Y_k = TY_i - t_{i+1}\sum_{k=i+2}^{n}Y_k + t_iY_i - t_{i+1}\sum_{k=1}^{i}Y_k$$

$$\Rightarrow t_{i+1} - t_i = TY_i/\left(\sum_{k=1}^{n}Y_k\right),$$

as required.

(b): From Lemma A.1, the minimum total effect retained can be found from

$$Z_1 = \sum_{j=1}^{n} t_j Y_j / T = T \sum_{j=2}^{n} \left\{ \sum_{k=1}^{j-1} Y_k \Bigg/ \left(\sum_{k=1}^{n} Y_k \right) \right\} Y_j / T = \sum_{j=2}^{n} \sum_{k=1}^{j-1} Y_k Y_j \Bigg/ \left(\sum_{k=1}^{n} Y_k \right). \qquad \square$$

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