I want to post your grades (anonymously) on the web. If you want this, please give me an alias, or give me permission to use your student number:

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Please indicate whether I should or should not put your exam and grade information in a box for you to pick up.
☐ *Put my exam etc in the box.*
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You may use two sides of a single sheet of notes you personally prepared.

Please put your name on each page. There are 6 pages with 7 problems with a total of 16 parts labeled with (a), (b), (c), (d). All those parts count equally. Some are further divided and labeled like "i."

The exam goes from 8:30 until 10:20.

Please make your logic clear. If you just write down numbers it is impossible to tell what you are doing. And if you have a calculator that does all your units conversions, you must still show what you are doing.

If you need to do scratch work, use the back of the pages. If you mess up the space for the answer, put it *on the back of the page with the question on it,* in a box and clearly labeled, and make a note in the space the answer should go.

1. Muons are made in the upper atmosphere by cosmic ray collisions with the nuclei of the air atoms. What energy muons (moving directly down toward the Earth) made at 30,000 m altitude have a 50% chance of reaching the surface before decaying? (Muons have mass 106 MeV/c^2 and a half life of 1.5×10^{-6} s.) Hint: if you find $\gamma\beta >> 1$ you can use $\beta = 1$ in places where it is reasonable.

   The muon half life is a proper time. In the Earth frame this time is dilated by a factor γ. The energy is γmc^2, so we want to find γ. If the muons are to reach the Earth in a dilated half life, they must go $l = 3 \times 10^4$ m in the dilated lifetime. Thus

$$l = \beta c \gamma t_{1/2}$$

   or

$$\gamma\beta = \frac{l}{ct_{1/2}} = \frac{3 \times 10^4}{(3 \times 10^8)(1.5 \times 10^{-6})} = 67$$

   since $\gamma\beta >> 1$ we approximate $\beta = 1$ so $\gamma = 67$ and $E = \gamma mc^2 = (67)(106) = 7.10$ GeV

   Alternately, you could say in the muon's rest frame it sees a contracted length (the altitude, 30,000 m is a propoer length) go by, and the muon lasts 1.5 μS. The atmosphere is going by at a speed of βc so you start with

$$l/\gamma = \beta c t_{1/2}$$

   which gives the same result.

2. A Δ^{++} particle of mass 1239 MeV/c^2 decays into a proton and a positive pion. (Proton mass is 938 MeV/c^2 and pion mass is 140 MeV/c^2).

   (a) What is the combined kinetic energy of the proton and pion in the rest frame of the Δ^{++}?

   Energy is conserved and $E = mc^2$ so initially $E = m_\Delta c^2$ and finally

$$E = E_p + E_\pi = m_p c^2 + m_\pi c^2 + K_p + K_\pi$$

   so

$$m_\Delta c^2 - m_p c^2 - m_\pi c^2 = K_p + K_\pi = K_{\text{tot}} = 1239 - 938 - 140 = 161 \text{ MeV}$$

   (b) What is the total energy of the proton in the Δ^{++} rest frame?

   This one is harder. Energy and momentum are conserved. Intially the Δ is at rest so $P = 0$. Finally, $P = 0$ so

$$P_\pi = -P_p.$$

   Since we are dealing with E, m and P, we can write conservation of energy and square:

$$E_\Delta = m_\Delta c^2 = E_p + E_\pi$$

   this will work best if we rewrite it as

$$m_\Delta c^2 - E_p = E_\pi$$

   and then square:

$$(m_\Delta c^2)^2 - 2m_\Delta c^2 E_p + E_p^2 = E_\pi^2$$

   using the expression for $E^2 = (mc^2)^2 + (Pc)^2$ with $P_p = P_\pi$ we get

$$(m_\Delta c^2)^2 - 2m_\Delta c^2 E_p + (m_p c^2)^2 = (m_\pi c^2)^2$$

which we solve

$$E_p = \frac{m_\Delta^2 + m_p^2 - m_\pi^2}{2m_\Delta}c^2 = \frac{1239^2 + 938^2 - 140^2}{(2)(1239)} = 967 \text{ MeV}$$

This is the total energy of the proton. As a rough check, note the kinetic energy is $967 - 938 = 29$ MeV, which is not nuts considering there is 161 MeV kinetic energy to share between the proton and pion. The pion, being lighter, will need more kinetic energy than the proton to have the same momentum.

3. A distant galaxy is moving away from us at 0.60c.

   (a) Lyman-alpha photons have a wavelength of 121 nm (in the rest frame of the emitting hydrogen.) This radiation comes from transitions from which state to which state in hydrogen? (give a brief reason for your answer)

   Lyman-alpha come from the transition from $n = 2$ to $n = 1$, i.e. the first excited state to the ground state. Since the energies are E_R/n^2 and $E_R = 13.6$ eV, this transition produces a photon of (13.6)(3/4) eV. That is about 10 eV. Since $E = hf = hc/\lambda$ for photons, and $hc = 1240$ eV, a 121 nm photon has about 10 eV too. No other transition is close in energy to this, we don't have to do the math any better.

   (b) What is the wavelength we see for Lyman-alpha light from that galaxy?

   The galaxy is receeding from us, so the Doppler shift will give us a longer wavelength. The factor is $\sqrt{(1+\beta)/(1-\beta)}$, where I chose the signs to make a factor bigger than 1. This factor is

   $$\sqrt{(1.6/0.4)} = \sqrt{4.0} = 2$$

   so we see the light at $2 \times 121 = 242$ nm

   (c) Material is ejected from that galaxy at a speed of 0.90 c (relative to that galaxy) in a direction toward us. How fast do we observe this material to move, and in what direction?

   We have a system moving away from us, with $V = 0.6c$, and a velocity in that system. The x-direction is the direction the distant galaxy is going. The velocity in that system is $U_x' = -0.9c$, and we want U_x in our system.

   $$U_x = \frac{U_x' + V}{1 + \frac{U_x' V}{c^2}} = \frac{-0.9c + 0.6\text{c}}{1 - (0.9)(0.6)} = \frac{-0.3\text{c}}{0.46} = 0.65\text{c}$$

4. A 1.0 MeV photon compton scatters.

   (a) What angle of scattered photon corresponds to the highest energy electron?

   If the photon scatters backwards (at 180° with respect to its initial direction) it will transfer the most momentum to the electron. So scattering at 180° gives the highest energy electrons. If you look at the Compton formula for change in wavelength, you will see the biggest change is at 180°, so this is another way to see the electron gets the most energy for the photon scattering to 180°.

   (b) What is the kinetic energy of that electron?

   The Compton formula gives us the difference in wavelengths. We must put this in terms of energy, using $E = hf = hc/\lambda$ for photons. So

$$\lambda_1 - \lambda_0 = \frac{hc}{E_1} - \frac{hc}{E_2}$$

   which we can put into the Compton formula

$$\frac{hc}{E_1} - \frac{hc}{E_0} = \frac{hc}{m_e \mathrm{c}^2}(1 - \cos 180^\circ)$$

   divide out the hc, use $\cos 180^\circ = -1$, and combine the difference of inverse E_i to get

$$\frac{E_0 - E_1}{E_0 E_1} = \frac{2}{m_e c^2}$$

$$E_0 - E_1 = \frac{2E_0 E_1}{m_e c^2}$$

$$E_1(\frac{2E_0}{m_e c^2} + 1) = E_0$$

$$E_1 = \frac{E_0}{1 + \frac{2E_0}{m_e c^2}} = \frac{1}{1 + \frac{2}{0.511}} = 0.20 \text{ MeV}$$

   and the energy given the electron is $E_0 - E_1 = 1.0 - 0.20 = 0.80$ MeV

5. Recall that for the circular orbits of the Bohr model, the kinetic energy is equal to the magnitude of the binding energy. For a circular orbit the average momentum is 0 and the spread $\Delta p = |p|$.

   (a) Show that the Bohr ground state satisfies the uncertainty principle for position and momentum.

   There are several ways to do this:

   i. You can use Bohr's hypothesis, which says the angular momentum of the ground state is $\hbar$. Since angular momentum is $\vec{L} = \vec{r} \times \vec{p}$ and for a circular orbit, $\vec{r}$ and $\vec{p}$ are always perpendicular, then $r|p| = \hbar$ for the ground state. The orbit is centered at 0 with a spread of r, so we can say $\Delta x = r$, and use $\Delta p = |p|$, so

   $$\Delta x \Delta p = \hbar \geq \hbar/2$$

   which is the undertainty principle.

   ii. You can use numbers. Since $K = -E = 13.6$ eV, and since (this is a non-relativistic problem) $pc = \sqrt{2mc^2K}$ we get $pc = \sqrt{2 \cdot 511 \times 10^3 \cdot 13.6} = 3728$ eV and $r = a_B = 0.05$ nm so $c\Delta p \Delta x = 186$ eV $\cdot$ nm. This is close to $\hbar c = 197$ ev$\cdot$ so $\Delta p \Delta x$ is close to $\hbar$ and greater than $\hbar/2$. Of course the 186 is no more accurate than the 0.05 and should be exactly 197.

   iii. You can use algebra. The kinetic energy is the Ridberg energy $= \alpha^2 mc^2/2$ and the radius is $a_B = \hbar c/(\alpha mc^2)$ where $\alpha = k_e e^2/(\hbar c)$ is the fine structure constant. The momentum is $pc = \sqrt{2mc^2K} = \alpha mc^2$ Multiplying the radius and pc gives $\hbar c$, since the α and mc^2 cancel. So once again, if we equate Δx with r and Δp with p, considering the circular orbit, we find the uncertainty principle is satisfied for the ground state. Note that any charge or mass would still work, since they cancel.

   (b) Using the dependence of the radius and $|p|$ on n, show that $\Delta x \Delta p/n$ is constant.

   All you need is the dependence on n. You know E_n is proportional to $1/n^2$ so, following the connection between E_n and p_n done above, p_n is proportional to $1/n$. The radius is proportional to n^2. The product of r and p, which we equated to $\Delta x \Delta p$ above, is proportional to n, so $\Delta x \Delta p/n$ is constant. (Since you know the value for $n = 1$ you know the constant, but that was not the question.) This illustrates that ground states are near the equality in the uncertainty principle, whereas excited states get bigger and bigger $\Delta x \Delta p$ as the excitation goes up.

6. As you should know, visible light has a wavelength around 500 nm.

   (a) Compare the wavelength of 1 keV electrons to that of visible light. (I.e. give a ratio or something, don't just say one is bigger or smaller than the other.)

   The electron is non-relativistic (1 keV is much less than $mc^2 = 511$ keV) so the momentum is $p = \sqrt{2mK}$ and the wavelength is $\lambda = h/p = hc/\sqrt{2mc^2K} = 1.240/\sqrt{2 \cdot 511 \cdot 1} = 0.039$ nm. Here I converted the usual 1240 eV $\cdot$ nm to 1.240 keV $\cdot$ nm to match the units of the denominator. You could just as well put the mc^2 and 1 keV into eV. You have to do one or the other to match units.

   The 1 keV electron wavelength is $.039/500 = 4/50000 = 8 \times 10^{-5}$ that of the visible light. This is why electron microscopes are needed to look at very small things, and why they work for that job.

   (b) Those 1 keV electrons make x-rays when they hit material.

      i. What is the highest energy x-ray they make?

      Electrons can convert all their kinetic energy into x-ray energy, so the maximum x-ray energy is 1 keV.

      ii. What is the speed of that x-ray?

      X-rays are electromagnetic radiation, so they go c = 3×10^8m/s. An answer of just c is good enough.

      iii. What is the wavelength of that x-ray?

      For photons, $E = hf = hc/\lambda$ so $\lambda = hc/E = 1240/1000 = 1.24$ nm. Since the x-ray energy was specified to 1 significant figure, an answer of 1 nm is acceptable.

7. A particle of mass m is bound in a finite square well (non-rigid box) in which the potential is 0 for $-a/2 < x < a/2$ and is U_0 for $x < -a/2$ and $x > a/2$. There are more than two bound states. In some place the wave function corresponding to bound state n (with energy E_n) is $A_n \sin k_n x + B_n \cos k_n x$, in another place it is $C_n e^{\alpha_n x}$ and in another place it is $D_n e^{-\alpha_n x}$. α_n and k_n are positive, and $x = 0$ at the middle of the well. (One of A_n or B_n might be 0 when the other one isn't.)

   (a) Write down the time independent Schrödinger equation

      i. for the region $-a/2 < x < a/2$
      Here the potential is 0 so the Schrödinger equation is

$$\psi_n'' = -\frac{2m}{\hbar^2} E_n \psi_n$$

      ii. for the region $x < -a/2$ and $x > a/2$
      Here the potential is U_0 so the Schrödinger equation is

$$\psi_n'' = \frac{2m}{\hbar^2}(U_0 - E_n)\psi_n$$

   (b) Show that one of the functions given above is a solution in the appropriate region. (Pick your favorite one.) A complete answer will include your getting a formula for k_n or α_n in terms of natural constants and U_0, E_n, a, m.
   In any of the regions you pick the function and you calculate ψ_n'' from ψ_n. You need to realize that the ones with α go outside the well, and the ones with k go where $-a/2 < x < a/2$. You will find that $\psi_n'' = \alpha_n^2 \psi_n$ or $\psi_n'' = -k_n^2 \psi_n$ depending on your region. Putting these into the Schrödinger equation for the appropriate region will give either

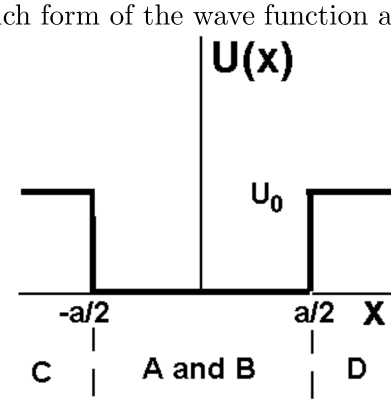
$$\alpha_n^2 = \frac{2m}{\hbar^2}(U_0 - E_n)$$

   or

$$k_n^2 = \frac{2m}{\hbar^2} E_n$$

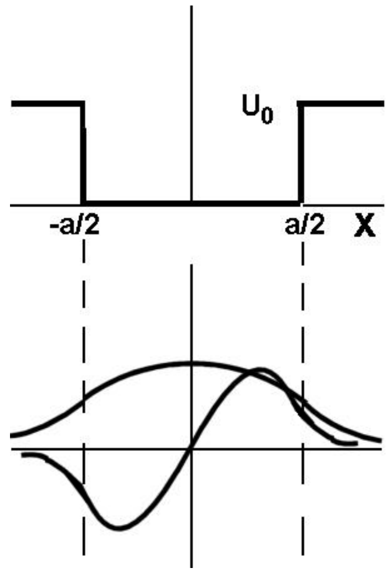
   (c) Sketch

      i. the well and indicate which form of the wave function applies in which place.



      The letters on the figure indicate which solutions belong in the different regions bounded by -a/2 and a/2. C corresponds to $C_n e^{\alpha_n x}$, etc.

ii. Sketch the wavefunctions for the lowest and next to lowest energy. Either sketch them on top of the well, or use a graph with the same x scale. Make a good sketch, or identify features of the wave functions that are not clear in your particular sketch.



The curve that does not cross the axis is the ground state and the one that crosses at $x = 0$ is the first excited state. The ground state is symmetric (even) about $x = 0$ and the first excited state is anti-symmetric (odd). They both have curvature toward the axis inside the well, away from the axis outside, the slopes are equal just before and just after the boundaries of the well (not well illustrated in the sketch), and the curves are supposed to approach 0 for large $|x|$. The first excited state should approach 0 slower than the ground state, since α is smaller. (This is not well illustrated in my sketch.)

(d) What is the binding energy for a bound state with energy E_n? (Be careful - E_n is measured from the bottom of the well, and $U(x)$ is U_0 at large x, unlike the case with atoms, where we took $U(x)$ to be 0 at large distance.)

The binding energy is just $U_0 - E_n$, which is the energy that must be supplied to the particle in state n to bring it to the top of the well.