Radioactive decay:

Statistical law of radioactivity: *The probability that a radioactive atom will decay in a given time interval is constant and not influenced by the past history or present circumstances of the atom.*

• The decay rate is an immutable nuclear process, insensitive to the physical and chemical environment of the atom (temperature, pressure, chemical state, physical state, magnetic fields, etc have no perceptible effect on the decay rate). Radioactive clocks keep exceptionally good time.

From the statistical law, we can derive a fairly simple expression describing the change in the number of radioactive parent atoms with time. For a derivation, see the the back of this handout. If the initial number of parent atoms is N_0 at time 0, then the number remaining (N) at time t is:

$$N = N_0 \, e^{-\lambda t} \qquad\qquad\qquad \text{... (1)}$$

Or equivalently, the fraction remaining, N/N_0, at time t is

$$N/N_0 = e^{-\lambda t} \qquad\qquad\qquad \text{... (2)}$$

This is the exponential law of radioactive decay – no matter what the initial abundance of a radioactive species, both the number of atoms and the radioactivity decline exponentially with time. λ in these equations is referred to as the "decay constant". It is the probability that any individual atom will decay in a given time interval. Commonly, λ is specified in units of yr^{-1} or "per year", though for short-lived radioactive atoms, you might see units of day^{-1}, min^{-1} or even sec^{-1}. Think of λ as "the fraction of atoms that decay *per year*". In a population of atoms large enough to smooth out the random variations in individual atomic lifetimes, λ is the fraction of atoms that will decay in a given time interval.

Example: The decay constant of ^{14}C is $1.210 \times 10^{-4} \, yr^{-1}$. If a sample contains 10^6 atoms of ^{14}C initially, how many will it contain after 1 year? The decay constant represents the fraction of a sample of ^{14}C atoms that will decay per year - so the number that we expect to decay is $1.210 \times 10^{-4} \times 10^6 = 121$. At the end of 1 year, we would expect to find 999, 879 atoms of ^{14}C remaining.

Notes: (a) We can only use this simple calculation in place of equation (1) if the period of time we're considering is $\ll 1/\lambda$ (8260 years, in this case). (b) The decay of individual atoms is a statistically random process. For a fairly small sample of 10^6 atoms and a rather short period of time such as a year, we might see a significant departure from the "predicted" number of decays. In fact, if we counted the number of decays year after year, we'd expect to see fewer than 110, or more than 132, about once every three years. More about the statistics of radioactive decay later.

We commonly use two timescales when thinking about radioactive decay. The <u>half-life</u> is the time required for one half of an initial population of radioactive atoms to decay. In terms of the exponential law, $t = t_{1/2}$ when $N/N_0 = 1/2$:

$$N/N_0 \ = \ 1/2 \ = \ e^{-\lambda t_{1/2}} \qquad\qquad \text{... (3)}$$

<u>Homework Q.1</u>: Solve the equation above for $t_{1/2}$, to show that the half-life, $t_{1/2}$, is equal to $ln(2) / \lambda$.

The <u>mean life</u> is the average lifetime of a radioactive atom, given by $\tau = 1/\lambda$. By looking at the exponential law, you can also see that τ is the time required for a population of radioactive atoms to decrease by a factor of $1/e$ (= 0.368).

<u>Homework Q.2</u>: Explain why the mean life is longer than the half life.

Exploring the probabilistic nature of radioactivity

Let's use a simple analogy to look at the probabilistic nature of radioactive decay, and how the exponential law arises from the (random) decay probability of individual atoms.

We'll use a collection of six-sided dice as our "radioactive" atoms. Here's how the analogy works:

Think of the dice as atoms of a radioactive parent element (Element 121: Walmartium). Each roll of the the dice represents an increment of time, let's say, one year. Dice that land on "6" are considered to have "decayed" (to daughter element 119: Microsoftium; evidently we've entered the era of sponsored chemistry). We'll record the number that decay and remove them from the population. Then we roll the remaining dice again (another "year" goes by). Once again, we record the number that decay, remove them from the population, then roll again. And so on ....

The probability that one of our individual "atoms" "decays" each "year" is constant and not influenced by its past history or present circumstances. In this case, the probability is $1/6$ per year:

$$\lambda \ = \ 1/6 \ \text{yr}^{-1} \ = \ 0.1667 \ \text{yr}^{-1}$$

Now let's see how the abundance of the parent and daughter atoms evolve over a period of 12 "years". Because we'll be working with a small number of "atoms", expect to see quite a bit of random noise on top of the signal we're looking for.

Break into three groups, and each group take a set of dice. There are 80 each white, green and red dice. Roll them, and record the results on the following pages. We'll group and tabulate all three sets of results on the board as we go.

After the third run through, calculate the overall averages for the number of atoms (out of 240) that decayed each year.

Q.3 Plot these data on the linear 12 years x 240 atoms graph.

Q.4 On the same graph, but using a different color, plot the number of atoms predicted to remain by the exponential law, at the end of each year ($N_0 = 240$, $\lambda = 0.1667$).

Q.5 How closely does the experiment match the prediction? [Note that for a population of atoms this small, the random distribution of the outcomes each time we roll the dice will result in deviations from the precise predictions of the exponential law. As a rough guide, we expect the actual number of decays that takes place in successive experiments to be distributed around the predicted values, with a *variance* of $\sim \sqrt{N} / N$, where N is the number of predicted decay events.]

Q.6 How long (in dice roll "years") after the beginning of the experiment does it take for 120 of the initial 240 "atoms" to decay? How long until 180 have decayed? Until 210 have decayed? How well do these values compare with the predicted half life for the parent element? Remember that the half-life is given by $t_{1/2} = \ln(2) / \lambda$.

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What we've been working to show is that the abundance of a radioactive parent element decays exponentially through time. A neat way to examine whether a process is exponential is to plot its graph on a "semi-log" graph. On semi-log graph paper, the x-axis is linear, but values on the y-axis are plotted at their natural logarithm. Because the exponential and logarithm functions are inverse functions, exponential curves plot as straight lines on semi-log paper. In this case, if we start with the exponential law and take the logarithm of both sides:

$$\ln \left(N / N_0 \right) = -\lambda t$$
$$\ln(N) = \ln(N_0) - \lambda t \qquad \text{... (5)}$$

Notice that equation (5) has the form of a straight-line equation y = mx + b in our co-ordinates (where y = ln(N) and x = t).

Q7. Plot the final (average) data on the semi-log 12 years x 240 atoms graph. Estimate or calculate a best-fit straight line through the data points. Measure the slope of this line in units of $(\ln(N_2) - \ln(N_1)) / (t_2 - t_1)$ Note – you can't just measure the slope with a ruler, because the X- and Y-scales aren't equal. Instead take the natural log of the abundance N at the beginning and end of the experiment, subtract $\ln(N_{final})$ from $\ln(N_{initial})$, and divide by the time interval. This will give you the slope in units of yr^{-1}. What is this slope? What does it represent in equation (5) above?

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The decay rate of a sample of radioactive material is referred to as its <u>activity</u>. Activity is a measure of the number of decays that occur per unit time. The formal unit for expressing activity is the Bequerel (Bq), equal to 1 decay per second (s^{-1}). For practical, everyday use in geology labs, the "dpm" (decay per minute) is more common.

Intuitively, the decay rate or activity of a radioactive sample is given by:

$$A = N\lambda \qquad\qquad (6)$$

i.e Activity = number of atoms × fraction that decay per unit time.

Go back to the cumulative results of experiments 1, 2 and 3. At the far-right in the data tables we recorded the number of dice that "decayed" in each "year", out of an initial total recorded one column to the left. The number of dice that "decayed" (i.e. the number that turned up 6 in a given roll) is analogous to the *activity* of our parent element. Obviously, the activity decreases with time, as the total number of surviving parent atoms decreases. From equation (6), we can see that a linear plot of activity (on the y-axis) *vs* the number of parent atoms (on the x-axis) should also give us the decay constant for our parent nuclide.

Q.8 Go back to your linear graph paper and plot the activity (A) *vs* parent abundance (N) data from the three experimental runs. Best not to work with the average data here, to avoid plotting a lot of fractional data – don't worry if the plot looks very noisy and scattered, especially at low values of N. This is just a consequence of the random nature of the process. Plot or calculate a best-fit line through your data – it's fair to assume that when N = 0, A must also equal zero, so your line should pass through the origin. What is the slope of this line? What is the significance of the slope?

<u>Extra – a derivation of the radioactive decay equation:</u>

Assume that for sufficiently short time intervals, the probability of decay is proportional to the length of the time interval. We also assume that the decay rate is constant – an immutable nuclear process, insensitive to the physical and chemical environment of the atom (temperature, pressure, chemical state, physical state, magnetic fields, etc have no effect on the decay rate).

Based on the statement above, we can write the probability of a radioactive atom decaying in a time interval Δt:

$$p_{decay} = \lambda \, \Delta t$$

consequently, the probability of survival is:

$$p_{survival} = (1 - \lambda \, \Delta t)$$

For an atom to survive two successive time intervals, the probability is:

$$p_{survival} = (1 - \lambda \, \Delta t) \times (1 - \lambda \, \Delta t)$$

and for n intervals:

$$p_{survival} = (1 - \lambda \, \Delta t)^n$$

Let the total time elapsed after n such intervals, $t = n \, \Delta t$, so that the probability of finding the atom alive and well at time t is:

$$p_{survival}(t) = \left(1 - \frac{\lambda t}{n}\right)^n$$

Now it so happens that ...

$$e^x = \lim_{n \to \infty} \left(1 + x/n\right)^n$$

Hence as we consider shorter and shorter time intervals Δt, and let $n \to \infty$, we get :

$$p_{survival}(t) = e^{-\lambda t}$$

For a large, statistically representative number of atoms, we can equate the probability of individual survival up to time t with the fraction of the original population that remains at time t. Let the original number of atoms be N_0, then the fraction remaining, N/N_0, at time t is:

$$N/N_0 = e^{-\lambda t}$$

This is the basic equation describing the decay of a radioactive species.