For the simple 2-box model discussed in class, these are the equations we want to solve. \( v_1 \) and \( v_2 \) are the amounts of material in each reservoir. \( v_{1,0} \) and \( v_{2,0} \) are the initial amounts at \( t = 0 \). \( k_{12} \) is the exchange coefficient for transport from reservoir 1 to reservoir 2; \( k_{21} \) is the coefficient for transport from 2 to 1. We want solutions for \( v_1(t) \) and \( v_2(t) \), i.e. the amounts of material in each reservoir as a function of time.*

\[
\begin{align*}
\text{DSolve}[ & \{ v_1'[t] = -k_{12} v_1[t] + k_{21} v_2[t], \\
v_2'[t] = k_{12} v_1[t] - k_{21} v_2[t], \\
v_1[0] = v_{1,0}, \\
v_2[0] = v_{2,0}, \} \}, \{v_1[t], v_2[t]\}, t] \\
\end{align*}
\]

**Solutions for \( v_1(t) \) and \( v_2(t) \)**

The solutions are:

\[
\begin{align*}
v_1(t) &= \frac{e^{-k_{12} t} k_{12} v_{1,0} + k_{21} v_{1,0} + k_{21} v_{2,0} - e^{-k_{12} t} k_{21} v_{2,0}}{k_{12} + k_{21}}, \\
and \quad v_2(t) &= \frac{-k_{12} v_{1,0} + e^{-k_{12} t} k_{12} v_{1,0} - k_{12} v_{2,0} - e^{-k_{12} t} k_{21} v_{2,0}}{k_{12} + k_{21}}
\end{align*}
\]

which is a bit complicated. No need to see how we got to this, or memorize it for the exam, etc.

Instead, let’s just look at the solutions graphically:
Starting from steady state ...

(*Let's set the total amount of exchangeable material \( v_1 + v_2 = 8 \), with \( v_1 \) at 5, and \( v_2 \) at 3. To keep the system at steady state, we need exchange coefficients that will make the fluxes the same in both directions. To make this simple, choose coefficients \( k_{12} = 1/5 \ \text{yr}^{-1} \) and \( k_{21} = 1/3 \ \text{yr}^{-1} \). With these values, the system remains in steady state because the fluxes are the same in both directions: \( k_{12} \times v_{1,0} = 1 \) and \( k_{21} \times v_{2,0} = 1 \), so the net fluxes back and forth balance and the amounts remain at their initial (steady-state) values.*)

\[
\text{Plot}[\{v_1[t]/.\{k_{12}\to0.2, k_{21}\to0.333333, v_{1,0}\to5, v_{2,0}\to3\}, v_2[t]/.\{k_{12}\to0.2, k_{21}\to0.333333, v_{1,0}\to5, v_{2,0}\to3\}\}, \{t, 0, 20\}, \text{PlotRange}\to\{0, 8\}, \text{Frame}\to\text{True}, \text{PlotStyle}\to\{\text{Blue, Red}\}]
\]

Starting from different initial conditions ...

(*Now start with different amounts in each reservoir: \( v_{1,0} = 2 \), \( v_{2,0} = 6 \). The total amount of exchangeable material remains the same (\( v_1 + v_2 = 8 \)). Nonetheless, both reservoirs end up at the same steady state values as in the previous run.*)

\[
\text{Plot}[\{v_1[t]/.\{k_{12}\to0.2, k_{21}\to0.333333, v_{1,0}\to2, v_{2,0}\to6\}, v_2[t]/.\{k_{12}\to0.2, k_{21}\to0.333333, v_{1,0}\to2, v_{2,0}\to6\}\}, \{t, 0, 20\}, \text{PlotRange}\to\{0, 8\}, \text{Frame}\to\text{True}, \text{PlotStyle}\to\{\text{Blue, Red}\}]
\]

Why? In this case the initial fluxes are out of balance: The initial flux from reservoir 1 to 2,
\( \phi_{12} = k_{12} \times v_{1,0} = 0.2 \times 2 = 0.4 \), whereas the flux \( \phi_{21} \) from reservoir 2 to 1 = \( k_{21} \times v_{2,0} = 0.333 \times 6 = 2 \). This imbalance increases the amount of material in reservoir 1 and decreases the amount in reservoir 2. Eventually the amount in reservoir 1 reaches 5 and the amount in reservoir 2 drops to 3, at which point the fluxes come back into balance and the system remains in steady state.

\[ \text{Plot}\left[\{v_1[t] / . \{k_{12} \rightarrow 0.2, k_{21} \rightarrow 0.333333, v_{1,0} \rightarrow 6, v_{2,0} \rightarrow 2\}, \right. \]
\[ \left. v_2[t] / . \{k_{12} \rightarrow 0.2, k_{21} \rightarrow 0.333333, v_{1,0} \rightarrow 6, v_{2,0} \rightarrow 2\}\right], \]
\[ \{t, 0, 20\}, \text{PlotRange} \rightarrow \{0, 8\}, \text{Frame} \rightarrow \text{True}, \text{PlotStyle} \rightarrow \{\text{Blue, Red}\}\]
How fast does the system return to steady state, and why?

(*Look back at the solutions for \(v_1(t)\) and \(v_2(t)\).
and notice that the time-dependent part of each equation is \(e^{(k_{12} + k_{21}) t}\). This indicates
that after any perturbation to \(v_1\) and \(v_2\), the system will return to steady-
state exponentially with a "characteristic timescale" \(\tau = 1/(k_{12} + k_{21})\). This is
the time required to reduce the perturbation by a factor of \(1/e (\approx 0.368)\). In
the 3 examples shown above the timescale is \(1/(0.2 + 0.333) = 1.88\) years.

By doubling both exchange coefficients we can preserve the same steady state
distribution of material between the reservoirs, but the system will respond
twice as fast to any imbalance in the fluxes. With \(k_{12} = 0.4\) yr\(^{-1}\) and \(k_{21} =
0.667\) yr\(^{-1}\), we have \(\tau = 1/(0.4 + 0.667) = 0.94\) yr. In the example below,
the return to steady state is twice as fast as in the last example above.*

\[
\text{Plot}[[v_1[t] / . \{k_{12} \to 0.4, k_{21} \to 0.666667, v_{1,0} \to 6, v_{2,0} \to 2\},
            v_2[t] / . \{k_{12} \to 0.4, k_{21} \to 0.666667, v_{1,0} \to 6, v_{2,0} \to 2\}],
  \{t, 0, 20\}, \text{PlotRange} \to \{0, 8\}, \text{Frame} \to \text{True}, \text{PlotStyle} \to \{\text{Blue, Red}\}]
\]

Exchange material faster ...

... or slower

(*Halving the original exchange coefficients again preserves the same steady state
distribution of material, but slows down the response time of the system.*)

\[
\text{Plot}[[v_1[t] / . \{k_{12} \to 0.1, k_{21} \to 0.1666667, v_{1,0} \to 6, v_{2,0} \to 2\},
            v_2[t] / . \{k_{12} \to 0.1, k_{21} \to 0.1666667, v_{1,0} \to 6, v_{2,0} \to 2\}],
  \{t, 0, 20\}, \text{PlotRange} \to \{0, 8\}, \text{Frame} \to \text{True}, \text{PlotStyle} \to \{\text{Blue, Red}\}]
\]