

A PDE describing Roots of Polynomials under Differentiation

Stefan Steinerberger

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If you are outside the convex hull, the charges 'push you away'.

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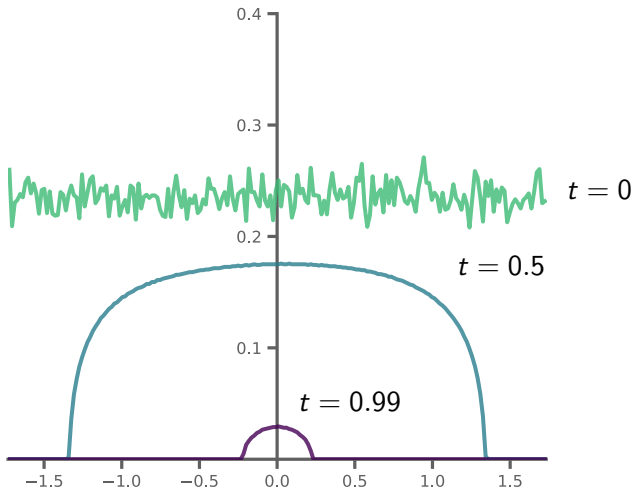
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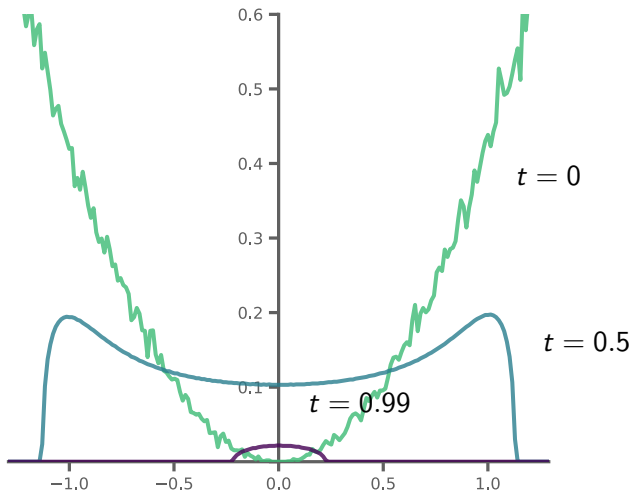
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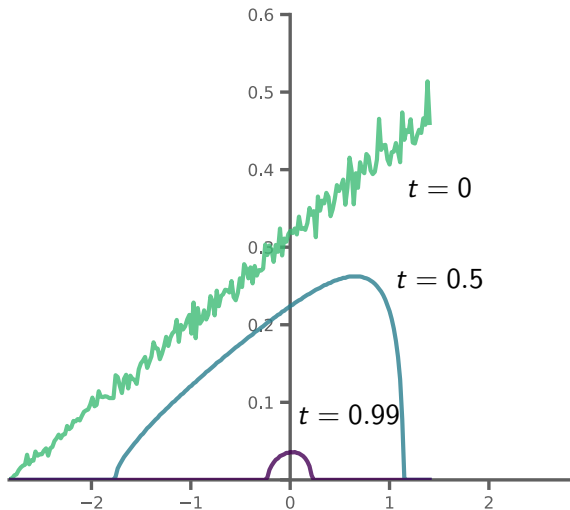


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we have (Riesz, Sz-Nagy, Walker, 1920s)

$$G(p'_n) \geq G(p_n).$$

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What can one say about $u(t, x)$?

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Theorem (S. 2018)

If there exists a nice and continuous evolution, then it is described by

$$\frac{\partial u}{\partial t} + \frac{1}{\pi} \frac{\partial}{\partial x} \arctan \left(\frac{Hu}{u} \right) = 0 \quad \text{on } \text{supp}(u)$$

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Closed Form Solutions

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Hermite polynomials $H_n : \mathbb{R} \rightarrow \mathbb{R}$ satisfy a nice recurrence relation

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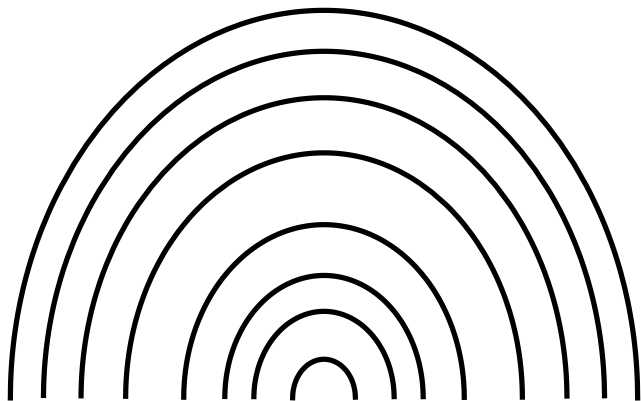
This suggests that

$$u(t, x) = \frac{2}{\pi} \sqrt{1 - t - x^2} \cdot \chi_{|x| \leq \sqrt{1-t}} \quad \text{for } t \leq 1$$

should be a solution of the PDE (and it is).

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The roots converge in distribution to the Marchenko-Pastur distribution

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Indeed,

$$u_c(t, x) = \nu\left(\frac{c+t}{1-t}, \frac{x}{1-t}\right)$$

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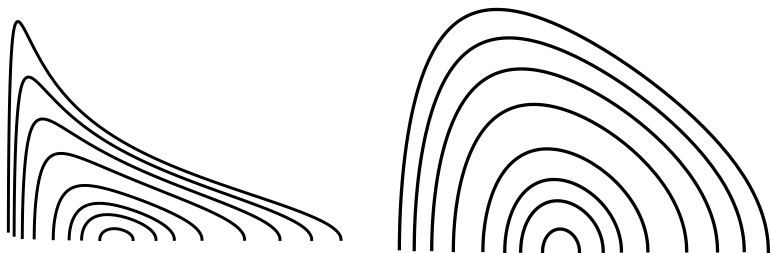


Figure: Marchenko-Pastur solutions $u_c(t, x)$: $c = 1$ (left) and $c = 15$ (right) shown for $t \in \{0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.99\}$.

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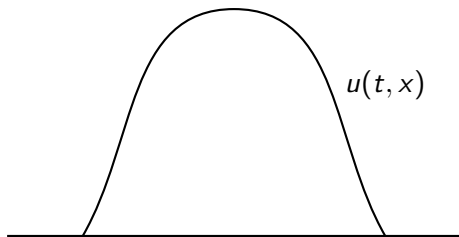
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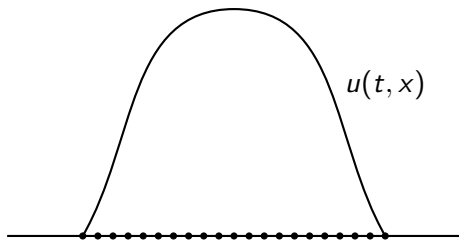
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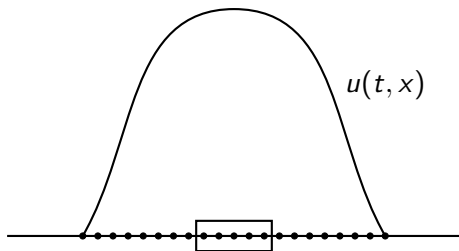
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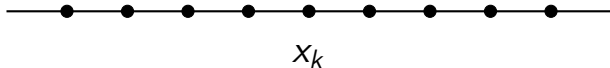
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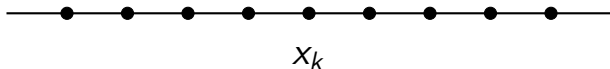
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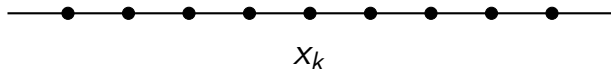
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$$\sum_{k=1}^n \frac{1}{x-x_k} = \sum_{|x_k-x| \text{ large}} \frac{1}{x-x_k} + \sum_{|x_k-x| \text{ small}} \frac{1}{x-x_k}$$

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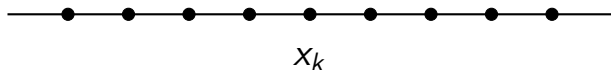


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$$\sum_{|x_k-x| \text{ large}} \frac{1}{x-x_k} \sim n \int_{\mathbb{R}} \frac{1}{x-y} \cdot u(t,y) dy = n \cdot [Hu](t,x).$$

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It thus remains to understand the behavior of the local term.

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$$\sum_{|x_k - x| \text{ small}} \frac{1}{x - x_k} \sim \sum_{\ell \in \mathbb{Z}} \frac{1}{x - \left(x_k + \frac{\ell}{u(t,x)n}\right)}.$$

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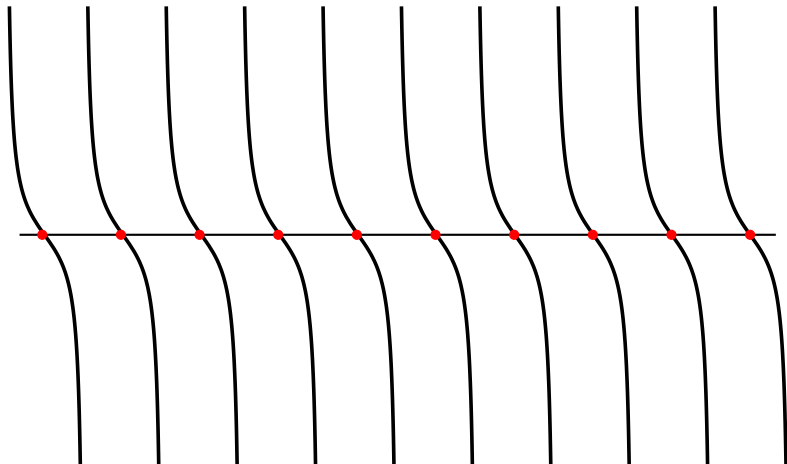
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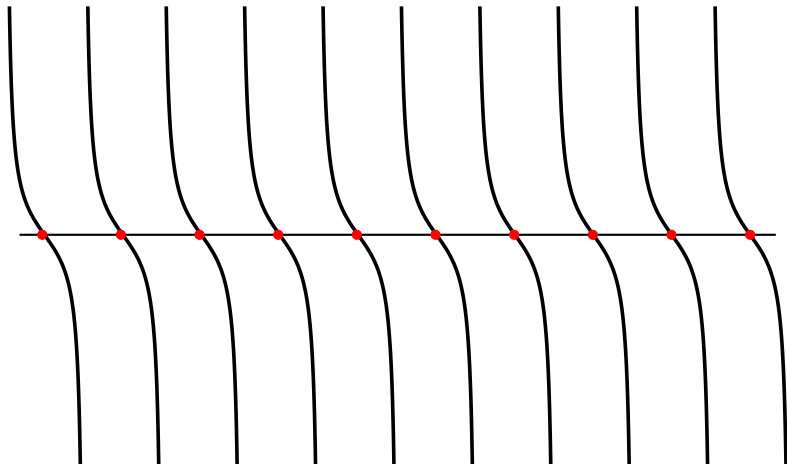
We are in luck: this sum has a closed-form expression due to Euler

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right) \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Z}.$$

The Local Field



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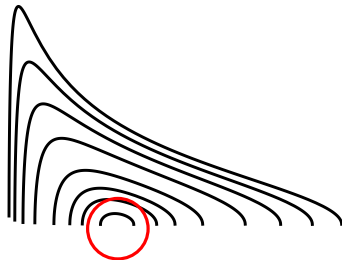
We can then predict the behavior of the roots of the derivative: they are in places where the local (near) field and the global (far) field cancel out. This leads to the desired equation.

A Fast Numerical Algorithm



Jeremy Hoskins (U Chicago) used the electrostatic interpretation to produce an algorithm that can compute all derivatives of polynomials up to degree ~ 100.000 .

A Fast Numerical Algorithm



Jeremy Hoskins (U Chicago) used the electrostatic interpretation to produce an algorithm that can compute all derivatives of polynomials up to degree $\sim 100,000$. **Semicircles at the end.**

Informal Theorem (Jeremy Hoskins and S, 2020)

Let X be a random variable on \mathbb{R} such that all moments are finite and $\mathbb{E}X = 0$ as well as $\mathbb{V}X = 1$.

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2. If $x_1, x_2, \dots, x_n \sim X$, then

$$\frac{x_1 + \dots + x_n}{\sqrt{n}} \sim \mathcal{N}(0, 1)$$

and the mean of the roots is preserved under differentiation (that's why there is a random shift).

Sep 3, 2020

Sep 3, 2020

Fractional free convolution powers

Dimitri Shlyakhtenko, Terence Tao

The extension $k \mapsto \mu^{\boxplus k}$ of the concept of a free convolution power to the case of non-integer

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The same PDE in a supposedly different context is presumably not a coincidence.

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Infinitely many conserved quantities.

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$u(t, x)$ should be close to a semicircle for t close to 1. That's exactly the Theorem that Jeremy and I proved for polynomials.

Sep 3, 2020

Sep 4, 2020

Sep 4, 2020

[Submitted on 4 Sep 2020]

Universal objects of the infinite beta random matrix theory

Vadim Gorin, Victor Kleptsyn

Sep 4, 2020

[Submitted on 4 Sep 2020]

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which proves that, in a certain setting, the crystallization assumption for roots is justified in the bulk

Oct 27, 2020

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DYNAMICS OF ZEROES UNDER REPEATED DIFFERENTIATION

JEREMY HOSKINS AND ZAKHAR KABLUCHKO

ABSTRACT. Consider a random polynomial P_n of degree n whose roots are independent random variables sampled according to some probability distribution μ_0 on the complex plane \mathbb{C} . It is natural to conjecture that, for a fixed $t \in [0, 1)$ and as $n \rightarrow \infty$, the zeroes of the $[tn]$ -th derivative of P_n are distributed according to some measure μ_t on \mathbb{C} . Assuming either that μ_0 is concentrated on the real line or that it is rotationally invariant, Steinerberger [Proc. AMS, 2019] and O'Rourke and Steinerberger [arXiv:1910.12161] derived nonlocal transport equations for the density of roots. We introduce a different method to treat such problems. In the rotationally invariant case, we obtain a closed formula for μ_t and the asymptotic density of the radial parts of the roots of the $[tn]$ -th derivative of P_n .

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- ▶ (rigorously) proved the connection to free convolution
- ▶ study of the complex equation (see below)
- ▶ and many examples

Dec 16, 2020

THE FLOW OF POLYNOMIAL ROOTS UNDER DIFFERENTIATION

ALEXANDER KISELEV AND CHANGHUI TAN

ABSTRACT. The question about the behavior of gaps between zeros of polynomials under differentiation is classical and goes back to Marcel Riesz. In this paper, we analyze a nonlocal nonlinear partial differential equation formally derived by Stefan Steinerberger [55] to model dynamics of roots of polynomials under differentiation. Interestingly, the same equation has also been recently obtained formally by Dimitri Shlyakhtenko and Terence Tao as the evolution equation for free fractional convolution of a measure [51] - an object in free probability that is also related to minor processes for random matrices. The partial differential equation bears striking resemblance to hydrodynamic models used to describe the collective behavior of agents (such as birds, fish or robots) in mathematical biology. We consider periodic setting and show global regularity and exponential in time convergence to uniform density for solutions corresponding to strictly positive smooth initial data. In the second part of the paper we connect

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- ▶ $(-\Delta)^{1/2}$ appears quite naturally!

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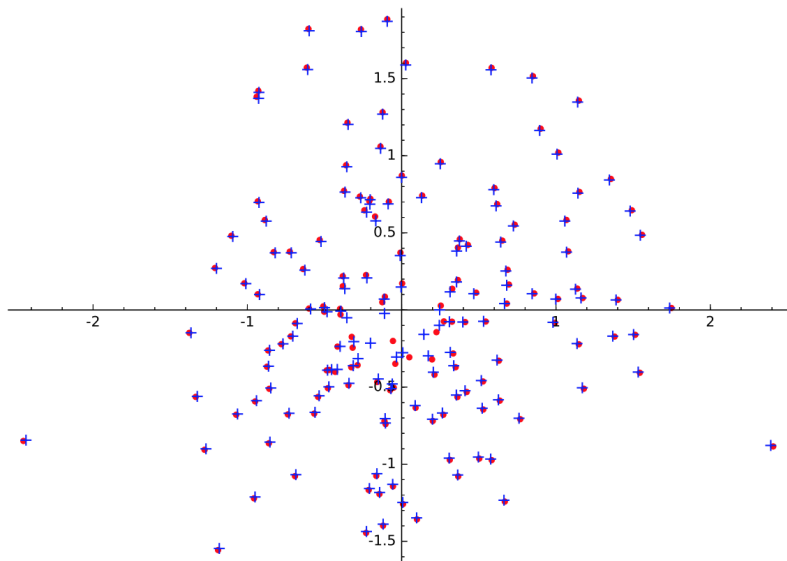
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- ▶ **What about the complex case?**
- ▶ There is a PDE but it's more complicated (essentially a complex Burger's equation), the radial case becomes a really nice one-dimensional transport equation (joint work with Sean O'Rourke)

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picture from O'Rourke and Williams (2018)

A Nonlocal Transport Equation

Sean O'Rourke and I tried to see whether the equation simplifies if we assume that the initial distribution is radial around the origin.

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If the density is $\psi(t, x)$, then

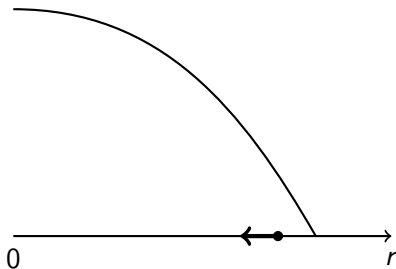
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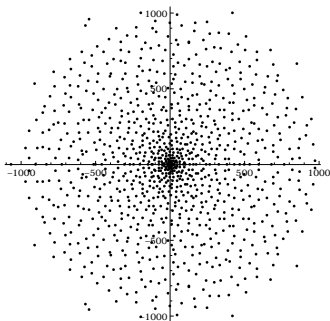
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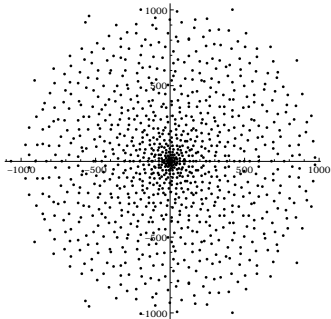
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This corresponds to **Random Taylor Polynomials**.

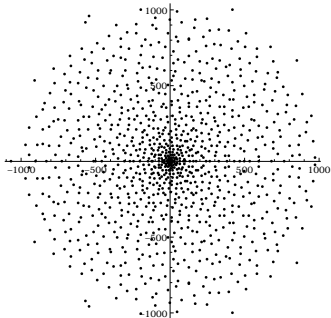




Random Taylor polynomials are defined by

$$p_n = \sum_{k=0}^n \gamma_k \frac{z^k}{k!},$$

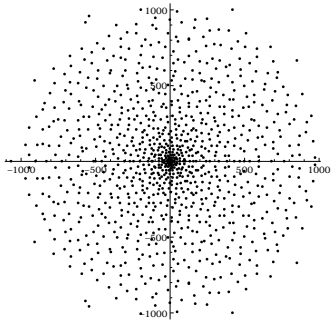
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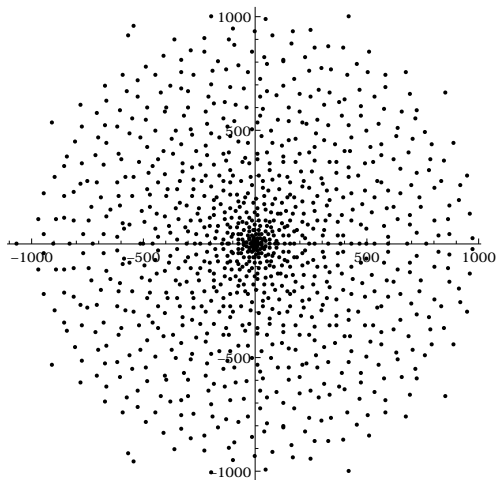
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$$p_n = \sum_{k=0}^n \gamma_k \frac{z^k}{k!},$$

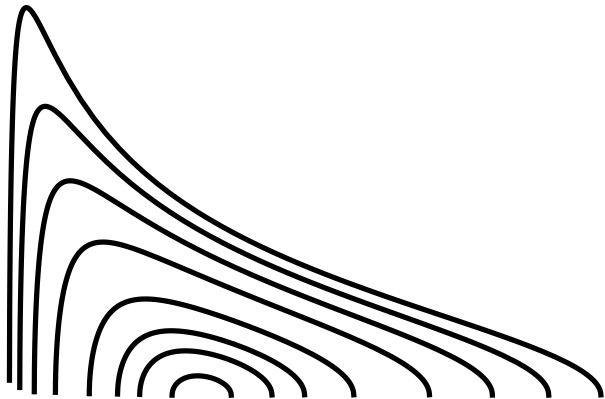
where $\gamma_k \sim \mathcal{N}(0, 1)$. They are preserved under differentiation.

Theorem (Kabluchko & Zaporozhets)

$$\frac{1}{n} \sum_{k=1}^n \delta_{z_k n^{-1}} \rightarrow \frac{\chi_{|z| \leq 1}}{2\pi|z|} \quad \text{as } n \rightarrow \infty.$$



Kabluchko & Hoskins give many more examples of solutions.



THANK YOU!