Parabolic Techniques for Elliptic PDEs in Mathematical Physics

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Nonlinear Problems of Mathematical Physics
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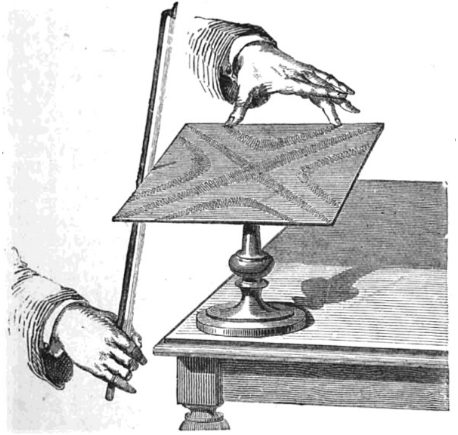
Ernst Florens Friedrich Chladni (1756 - 1827)



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The elliptic equation

$$-\Delta u = \lambda u$$

is a fixed point (in time) of the parabolic equation

$$\frac{\partial u}{\partial t} = (\Delta + \lambda)u.$$

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Use parabolic techniques to study elliptic problems!

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- An upper bound on the Hot Spots constant

Laplacian eigenfunctions: a short proof of (Cheng, 1976)

Theorem (Cheng, 1976)

If $-\Delta u = \lambda u$ on some two-dimensional domain, then any nodal domain – nodal domain being a connected component of

$$\{x : u(x) \neq 0\}$$

– satisfies an open cone condition with $\alpha \gtrsim \lambda^{-1/2}$.

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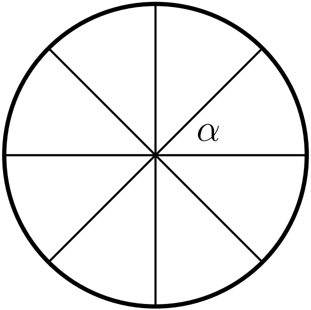

(see also Lipman Bers, 1955)

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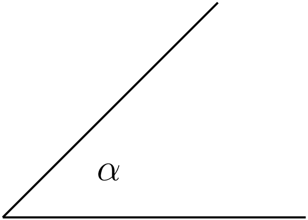
Idea. Start a heat equation with Dirichlet conditions:

$$u(t,x) = e^{-\lambda t} u(x).$$



At the same time: solve via Brownian motion!

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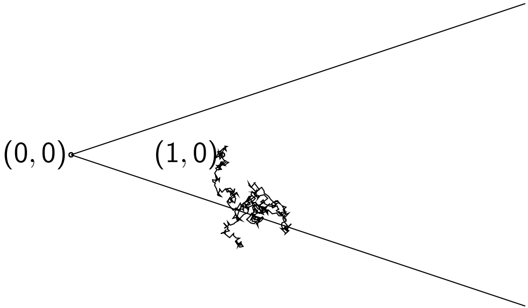
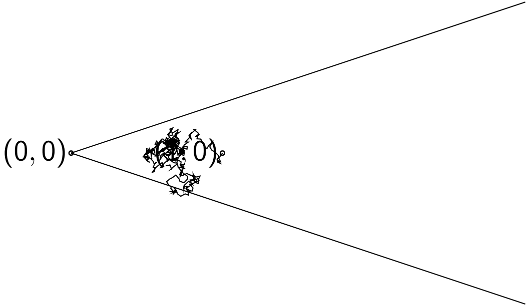
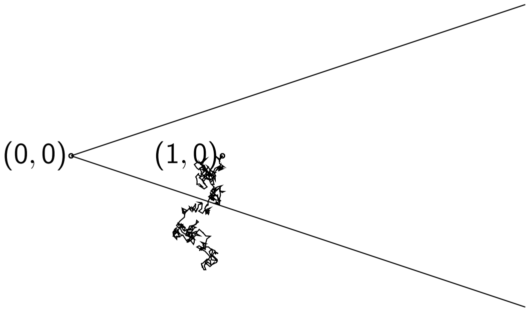


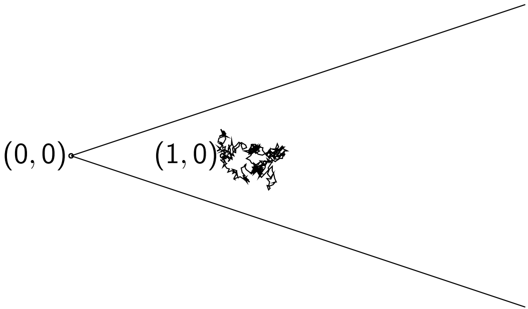
Figure: If the angle is too narrow...


(0, 0)



Suffices: Let $B(t)$ be a Brownian motion started in $(1, 0)$. Define a stopping time

$$T(r) = \inf \{ t \geq 0 : |B(t)| = r \} .$$



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Then, for $r > 1$,

$$\mathbb{P}\left(B[0, T(r)] \subset W(\alpha)\right) = \frac{2}{\pi} \arctan\left(\frac{2r^{\frac{\pi}{\alpha}}}{r^{\frac{2\pi}{\alpha}} - 1}\right).$$

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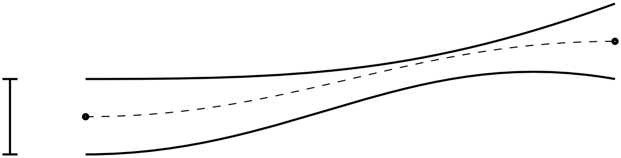
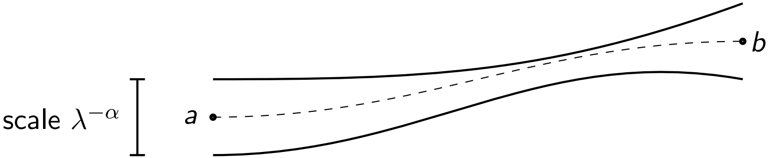


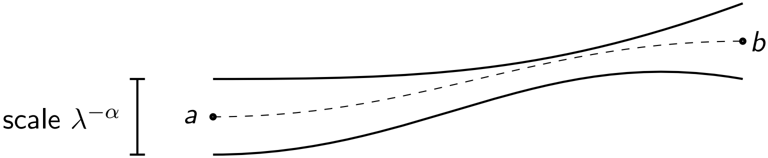
The idea is that nodal lines cannot run in parallel for arbitrarily long time.

Avoided crossings



Theorem (S, Comm. PDE, 2014)

Suppose $-\Delta u = \lambda u$ on a two-dimensional manifold and $\{x : u(x) = 0\}$ has the local structure as seen in the picture.



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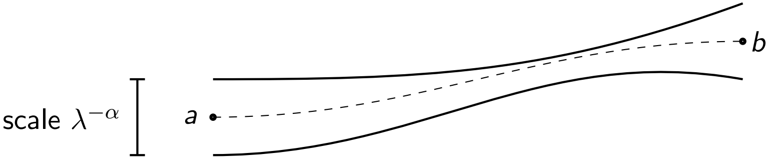
Suppose $-\Delta u = \lambda u$ on a two-dimensional manifold and $\{x : u(x) = 0\}$ has the local structure as seen in the picture. Then

$$d(a, b) \leq C\lambda^{1/2-\alpha} \log \lambda$$

for some constant $C < \infty$ depending only on (M, g).



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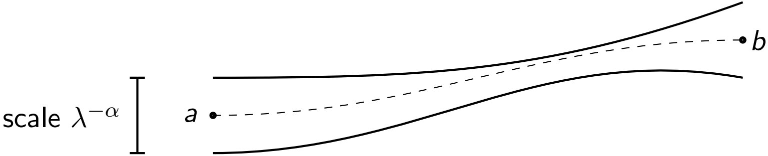
(see also Donnelly & Fefferman (1990) and Mangoubi (2010))



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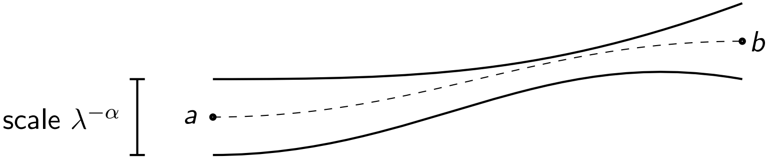

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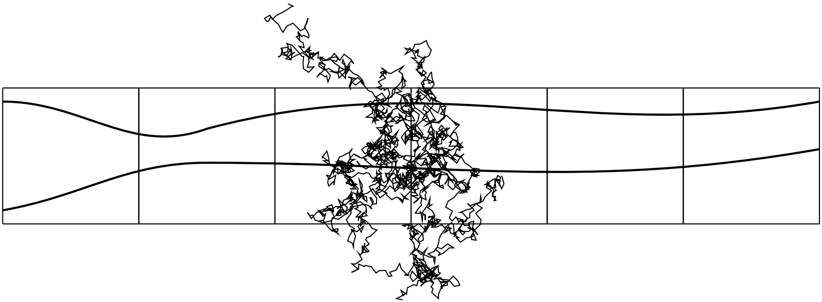
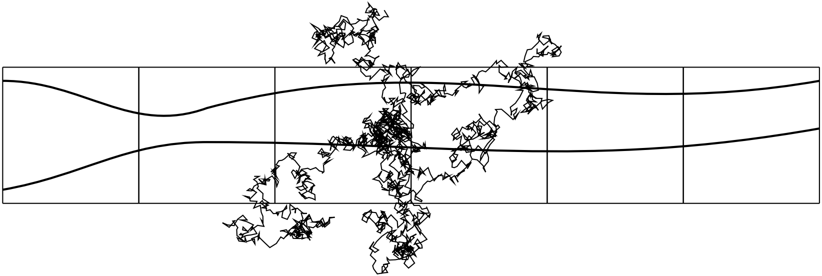




Neighboring square has to have slightly larger function values to balance the massive decay induced by absorbtion on the boundary ('rapid growth of elliptic equations in narrow channels', cf. the work of Landis).

Energy Landscape of Schrödinger operators

Schrödinger operators

Consider a nice domain $\Omega \subset \mathbb{R}^2$ and a potential $V : \mathbb{R}^2 \to [0, \infty)$. *Where* are the eigenfunctions of

$$-\Delta + V?$$

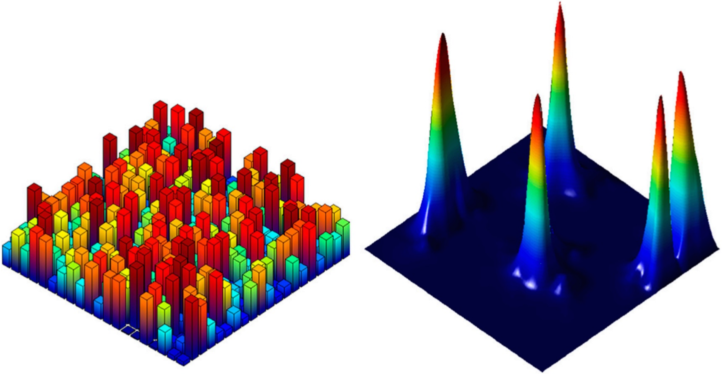
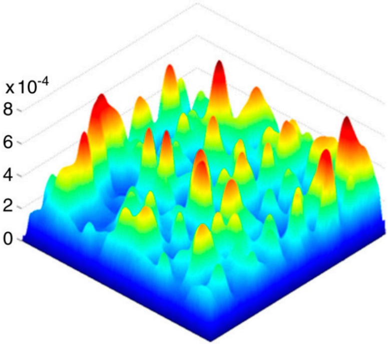


Figure: Filoche & Mayboroda, (PNAS, 2012)

The Landscape function

Mayboroda & Filoche: associate the solution of

$$(-\Delta + V)v = 1.$$



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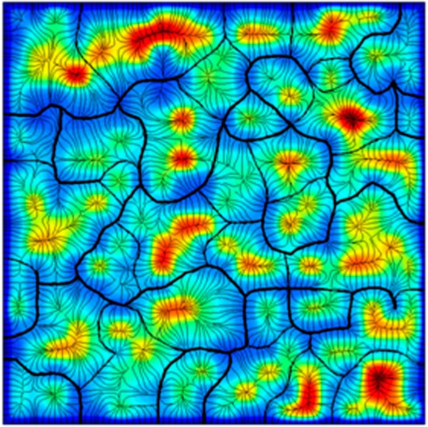

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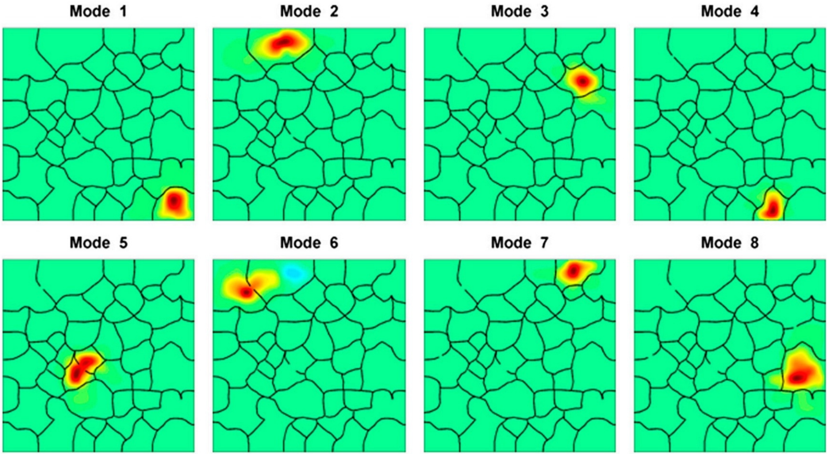

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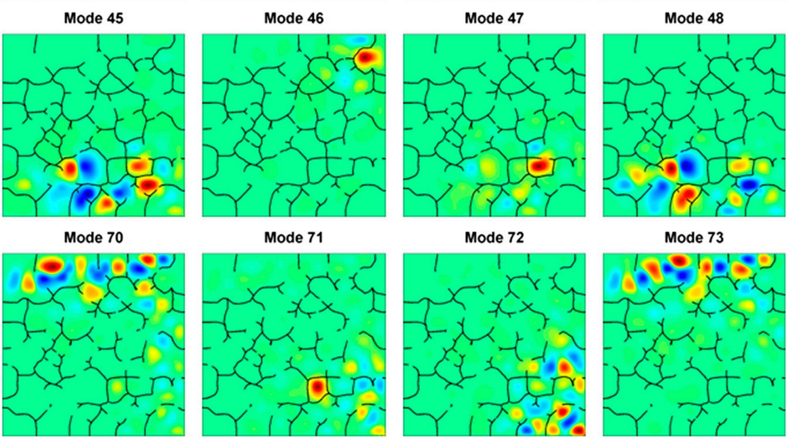

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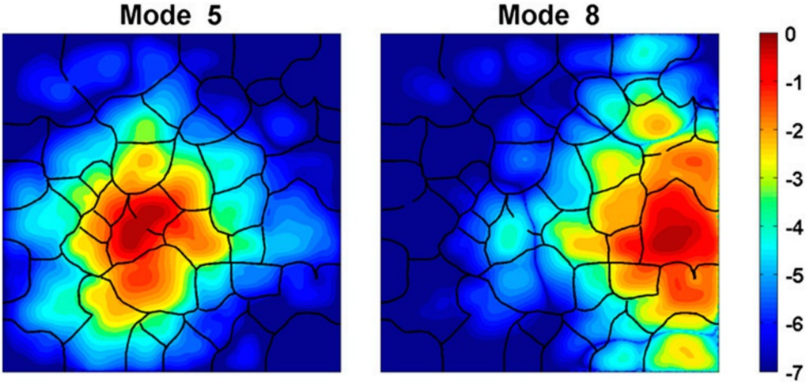
Whenever an eigenfunction crosses a barrier: exponential decay.



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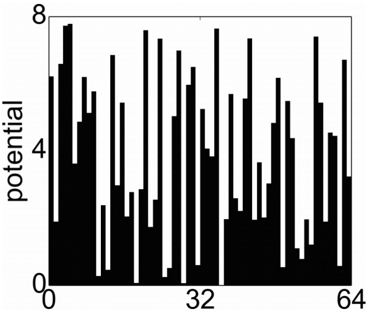
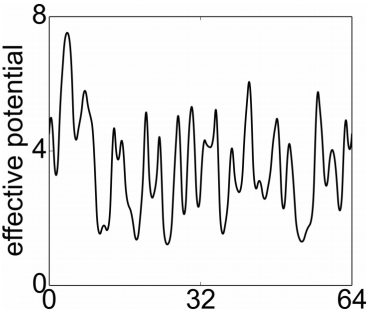
Known results

Theorem (Arnold, G. David, Filoche, Jerison & Mayboroda, Phys Rev 2016)

Exponential decay related to Agmon's inequality.





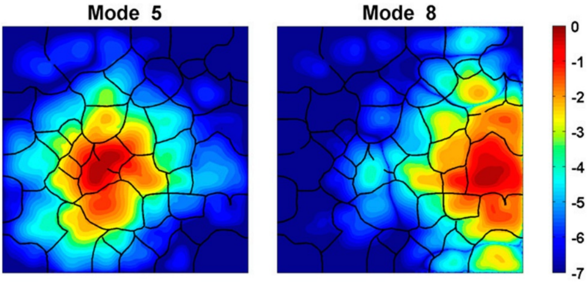
$1/v(x)$ is an effective effective potential.

Apply the heat equation and counteract the heat equation at the same time.

$$u(x) = e^{\lambda t} u(t,x)$$

Using Feynman-Kac gives, for every $t > 0$,

$$u(x) = \mathbb{E}_x \left(u(\omega(t)) e^{\lambda t - \int_0^t V(\omega(z))dz} \right).$$



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This relates the landscape function and the path integral via

$$t \sim v(x)\mathbb{E}_x \int_0^t V(\omega(z))dz.$$

(S, Proc. Amer. Math. Soc, 2017)

Making this precise

Recall that

$$\langle(-\Delta + V)u, u\rangle = \int_\Omega |\nabla u|^2 dx + \int_\Omega V u dx.$$

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This means we are looking for a type of averaging kernel k_t such that

$$(k_t * V)(x) = \int V(x+y) k_t(y) dy$$

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Making this precise

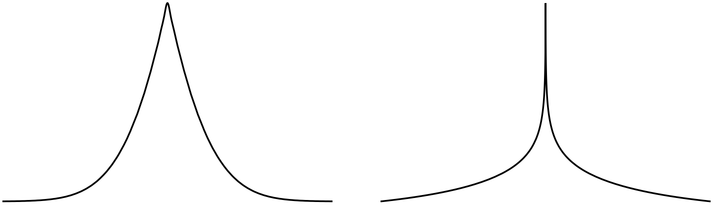

Figure: The radial profiles of the convolution kernel $k_t(r)$ in $d = 1$ dimensions (left) and $d = 2$ (right).

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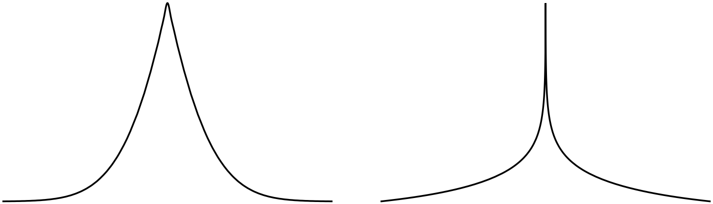

Figure: The radial profiles of the convolution kernel $k_t(r)$ in $d = 1$ dimensions (left) and $d = 2$ (right).

These kernels have different closed forms in different dimensions, for example

$$k_t(r) = \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{r^2}{4t}\right) - \frac{r}{2t} \operatorname{erfc}\left(\frac{r}{2\sqrt{t}}\right) \qquad (d = 1)$$

$$k_t(r) = \frac{1}{4\pi t} \Gamma\left(0, \frac{r^2}{4t}\right) \qquad (d = 2).$$

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In the next picture: we show (1) the behavior of the Filoche-Mayboroda landscape function, (2) the behavior of $k_t * V$ and (3) the localization of the first few eigenfunctions of $-\Delta + V$.

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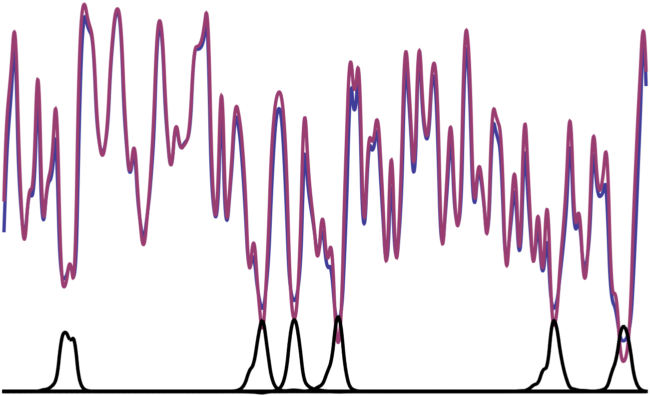

Making this precise

Theorem (S., Comm. PDE, 2021)

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain with smooth boundary, let $0 \leq V \in C(\overline{\Omega})$ be a continuous potential and let ϕ be a solution of

$$\begin{aligned} (-\Delta + V)\phi &= \lambda\phi \qquad \text{in } \Omega \\ \phi &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

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Then, for any fixed $x \in \Omega$, as $t \to 0$, we have, for k_t as above,

$$-\Delta\phi(x) + (V * k_t)(x)\phi(x) = \lambda\phi(x) + \mathcal{O}_{\phi, \|V\|_{L^\infty}}(t),$$

where the implicit constant depends *only* on ϕ and $\|V\|_{L^\infty}$.

Lieb's inradius result and the Polya-Szegő conjecture

Polya

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Mathematically, for $\Omega \subset \mathbb{R}^2$, we have the lowest frequency

$$\lambda_1(\Omega) = \inf_{f \neq 0} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx} \sim \frac{1}{\text{inradius}^2}$$

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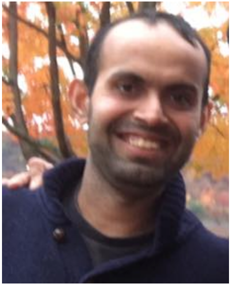
The Orchester Principle (Makai 1965, Hayman 1978)

There exist constants c_1, c_2 such that

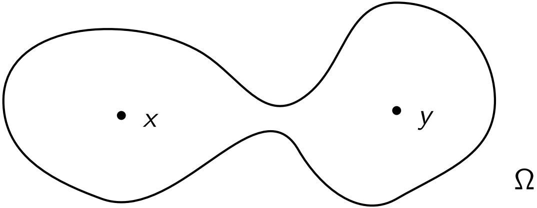
$$\frac{c_1}{\text{inradius}^2} \leq \lambda_1(\Omega) \leq \frac{c_2}{\text{inradius}^2}.$$

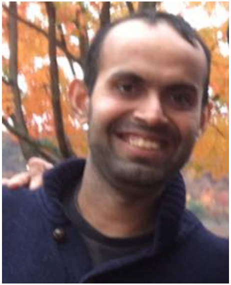


Theorem (M. Rachh and S, Comm. Pure Applied Math. 2017)

Let $\Omega \subset \mathbb{R}^2$ be simply connected and $u : \Omega \to \mathbb{R}^2$ vanish on $\partial\Omega$. If u assumes a global extremum in $x_0 \in \Omega$, then

$$\inf_{y \in \partial\Omega} \|x_0 - y\| \geq c \left\| \frac{\Delta u}{u} \right\|_{L^{\infty}(\Omega)}^{-1/2}.$$





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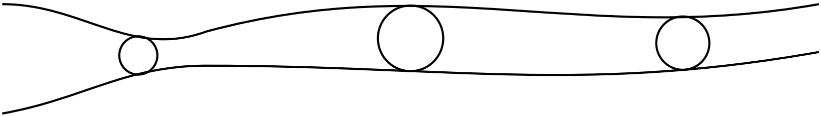
Idea behind the proof. If an eigenfunction assumes a maximum in $x_0 \in \Omega$, then any Brownian motion started there has likelihood $< 70\%$ of hitting the boundary within time $t = \lambda^{-1}$.

Lieb's theorem

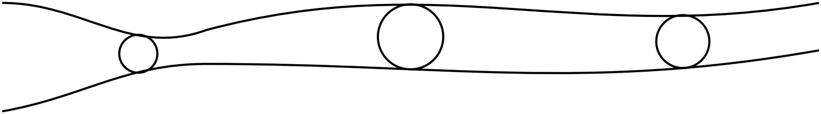
Such results are impossible in dimensions ≥ 3: one can take a ball and remove one-dimensional lines without affecting the PDE.



Theorem (Elliott Lieb, 1984, Inventiones)

Ω contains a $(1-\varepsilon)$-fraction of a ball with radius

$$r \sim \frac{c_\varepsilon}{\sqrt{\lambda_1(\Omega)}}$$



Theorem (Rachh and S, Comm. Pure Applied Math. 2017)

Let $-\Delta u = Vu$ with Dirichlet conditions. Then Ω contains a $(1-\varepsilon)$-fraction of a ball with radius

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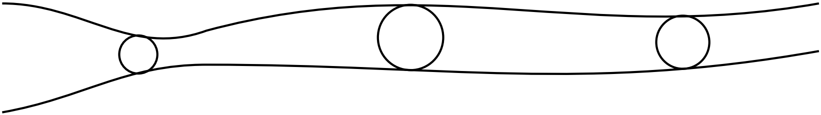
centered around the maximum of u.



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(Lierl & S, Comm. PDE 2018: the L^∞ can, in some sense, be replaced by the Lorentz space $L^{n/2,1}$)

Quantilized Donsker-Varadhan estimates

?

M. Donsker and S. Varadhan, On a variational formula for the principal eigenvalue for operators with maximum principle, PNAS 1975

Donsker-Varadhan

Setup: $\Omega \subset \mathbb{R}^n$ and

$$Lu = -\mathrm{div}(a(x)\nabla u) + \nabla V \cdot \nabla u.$$

Donsker-Varadhan

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$$Lu = -\mathrm{div}(a(x)\nabla u) + \nabla V \cdot \nabla u.$$

Question. What is the smallest $\lambda > 0$ for which

$$Lu = \lambda u \qquad \text{has a solution with } u\big|_{\partial\Omega} = 0?$$

Donsker-Varadhan: associate a drift diffusion process.

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Donsker-Varadhan inequality

$$\lambda_1 \geq \frac{1}{\sup_{x \in \Omega} \mathbb{E}_x \ \tau_{\Omega^c}}.$$

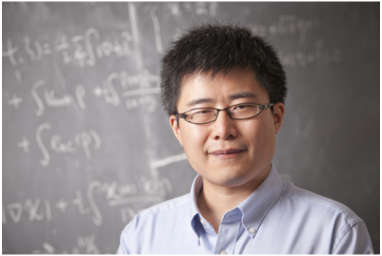
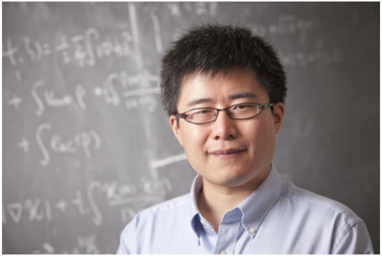


Figure: Jianfeng Lu

Instead of looking at the mean of the first exist time, we study quantiles:



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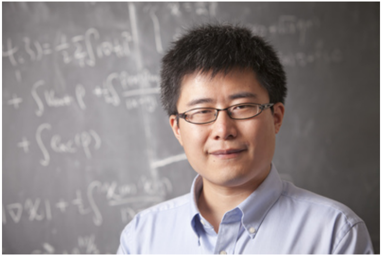
Instead of looking at the mean of the first exist time, we study quantiles: let $d_{p,\partial\Omega} : \Omega \to \mathbb{R}_{\geq 0}$ be the smallest time t such that the likelihood of exiting within that time is p.



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J. Lu and S., Proc. Royal Soc. 2016

$$\lambda_1 \geq \frac{\log(1/p)}{\sup_{x\in\Omega} d_{p,\partial\Omega}(x)}.$$

Moreover, as $p \to 0$, the lower bound converges to λ_1.

Example 1

Let us consider

$$L = -\Delta \qquad \text{on } [0,1].$$

Then $\lambda_1 = \pi^2$.

p	$1/2$	$1/4$	10^{-1}	10^{-2}	10^{-8}	Donsker-Varadhan
lower bound	7.28	8.40	8.92	9.39	9.74	8

Example 2

Let us consider

$$L = -\Delta + \nabla \left(\frac{1}{2} x^2 \right) \qquad \text{on } [0, 1].$$

Then $\lambda_1 = 2$.

p	0.5	0.3	0.2	0.1	0.05	Donsker-Varadhan
lower bound	1.52	1.67	1.74	1.79	1.83	1.678

An upper bound on the Hot Spots constant

Let $\Omega \subset \mathbb{R}^d$ and consider the first nontrivial eigenfunction

$$\begin{aligned} -\Delta u &= \lambda_2 u \qquad \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

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For 'nice' domains, the maxima and minima of a solution of this equation are 'usually' on the boundary.

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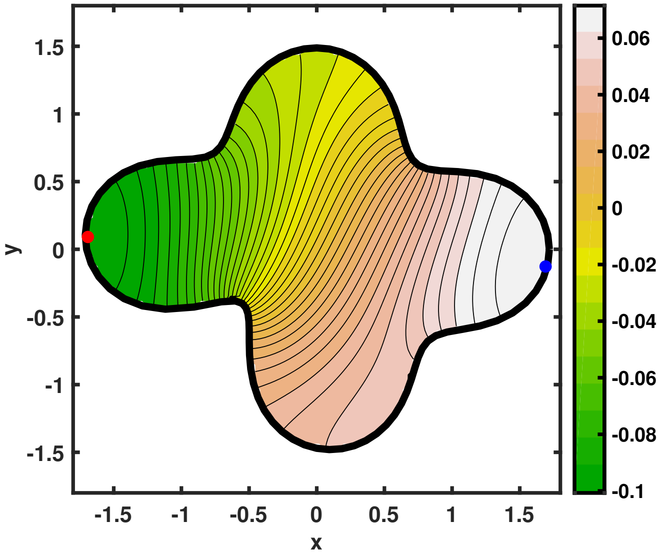
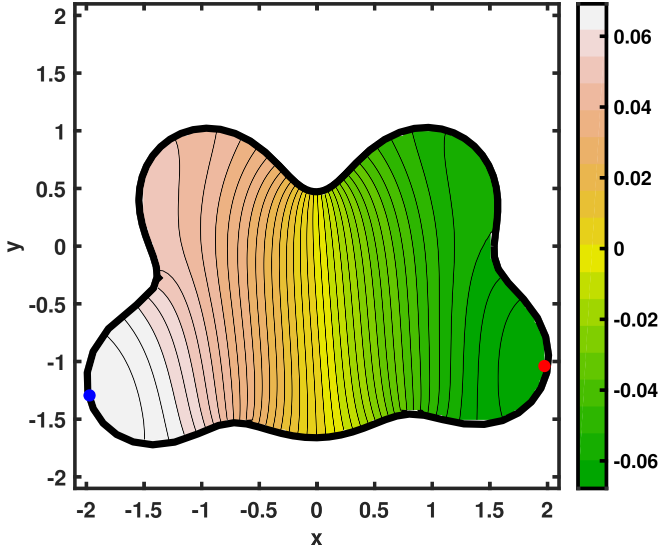
For 'nice' domains, the maxima and minima of a solution of this equation are 'usually' on the boundary.

Question: what is 'nice'? Probably convex is enough, maybe even simply connected.



(Picture produced by Andreas Kleefeld)



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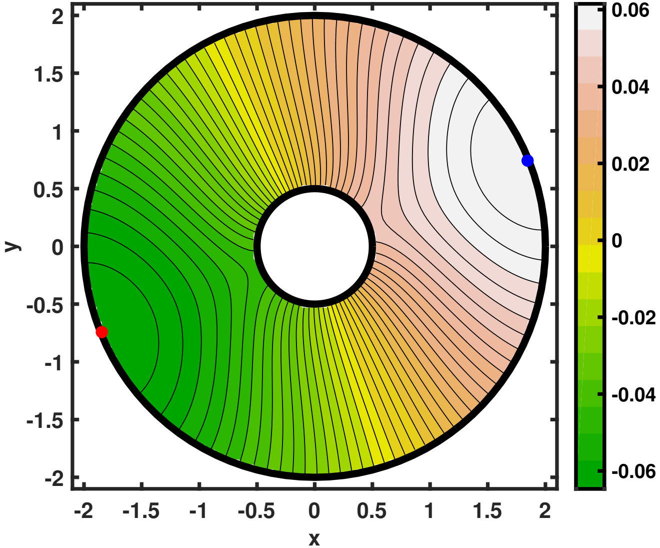

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The Hot Spots Conjecture

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Let ϕ_1 denote the first nontrivial eigenfunction of the Laplacian with Neumann boundary conditions. Then ϕ_1 assumes its maximum and minimum on the boundary.

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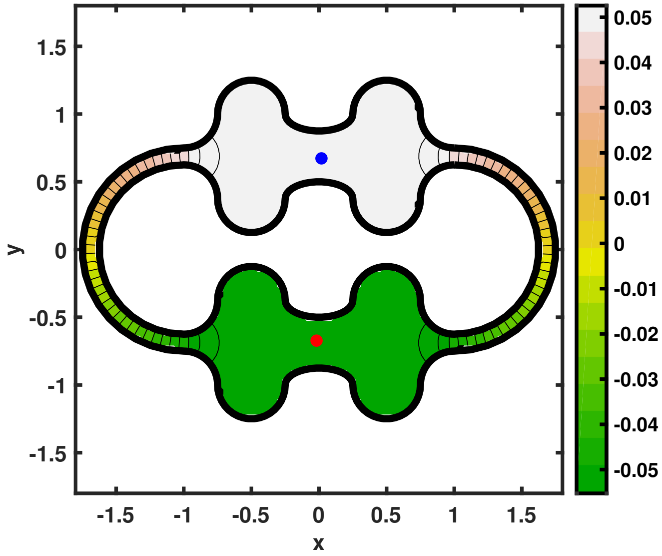
- Judge & Mondal (2018) showed it for triangles.



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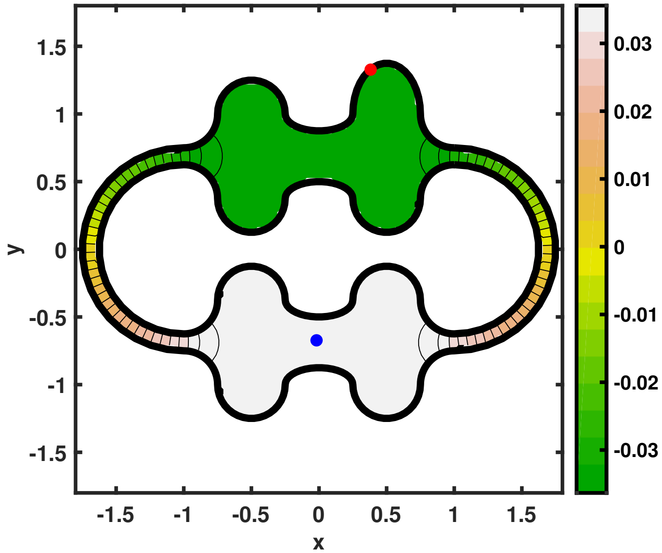

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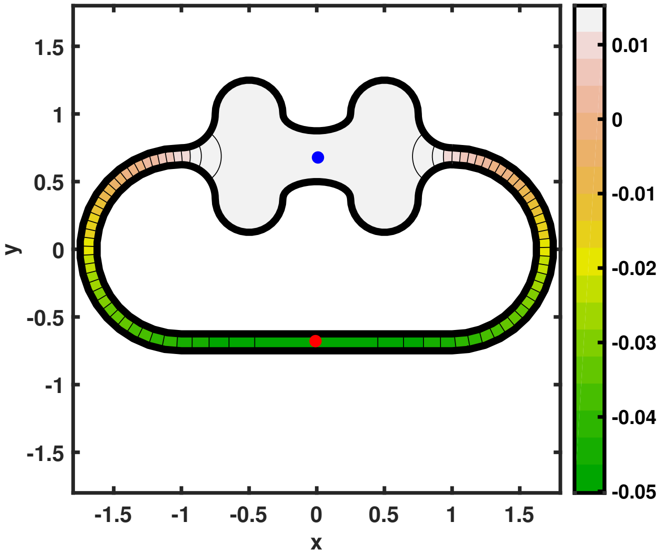

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So a natural question is: can the maximum be a lot bigger?

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Theorem (S, arXiv 2021)

Let $\Omega \subset \mathbb{R}^d$ be simply connected with smooth boundary. Then

$$\max_{x \in \Omega} u(x) \le 60 \cdot \max_{x \in \partial\Omega} u(x).$$

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What is really nice is that the result is uniform in the domain and the dimension.

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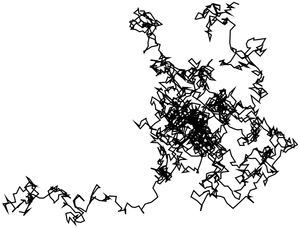
What is really nice is that the result is uniform in the domain and the dimension. As $d \to \infty$, the constant converges to $\sqrt{e^e} \sim 3.89\ldots$.



Thank you!