Some Open Problems in Fourier Analysis

Stefan Steinerberger

IMPA 2025

UNIVERSITY *of* WASHINGTON

First time at IMPA!

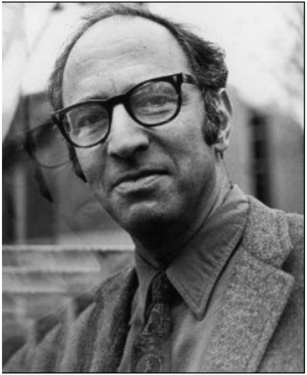
Christmas 2018, **closed doors**!

Second time at IMPA!

Dear Prof. Stefan Steinerberger,

The Brazilian Mathematical Colloquium is the main scientific event of our community, happening every two years at IMPA - Rio de Janeiro, with an attendance of around 1,000 people. This year, from August 2nd to August 6th, this event will reach its 33th edition, it will be by the first time completely online. and we will

2021, **closed doors**!



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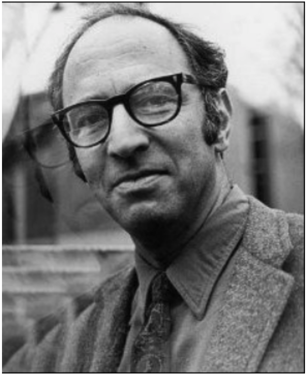

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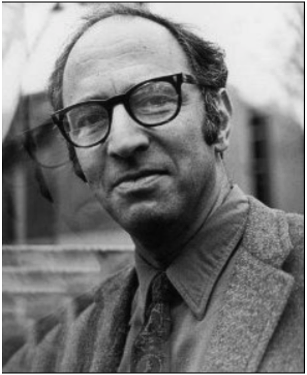

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The point of this talk is to mention some open problems and some partial results. The hope is that these are not (all) insanely hard, that they are not very well known and *that progress is possible*.

Approximate Plan

16 problems (where ≥ 9 are **not** impossible)

Lecture 1

1. The first Erdős problem
2. Littlewood Cosine, Chowla Cosine
3. Hardy-Littlewood Maximal Rigidity
4. Sum of Two Squares

Lecture 2

1. Cosine Sign Correlation, Maximal Geodesic Averages, Signed Orthogonality Problems
2. Directional Poincaré
3. Kakeya on the Sphere, Motzkin-Schmidt
4. Number of Critical Points

Lecture 3

1. Spooky Action at a Distance
2. Opaque Sets
3. The Averaging Problem
4. Kozma-Oravecz Inequalities

The first Erdős problem

This problem is somewhat well known in Additive Combinatorics but not in Fourier Analysis – and Fourier Analysis gives the best results.

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Basic Tensor Construction

If we have $a_1, \ldots, a_n$ with distinct subset sums and $a_n \leq c2^n$, then

$$1, 2a_1, 2a_2, \ldots, 2a_n$$

has the subset sum property and $a_{n+1} \leq c2^{n+1}$.

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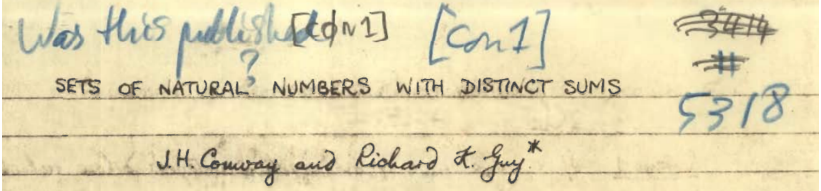

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SETS OF NATURAL NUMBERS WITH DISTINCT SUMS

J.H. Conway and Richard K. Guy*

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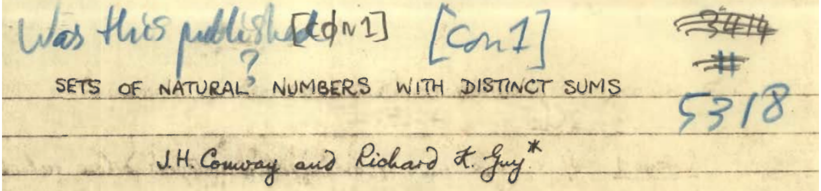

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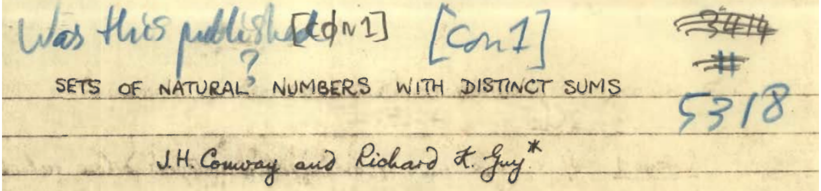
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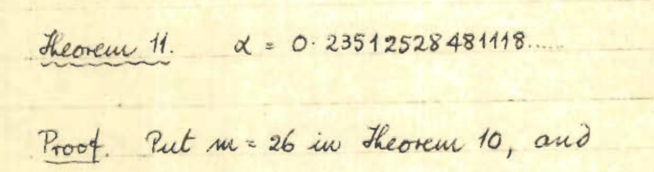
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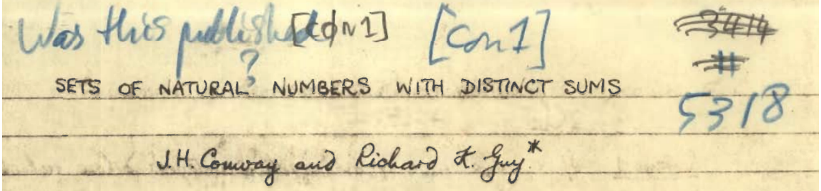

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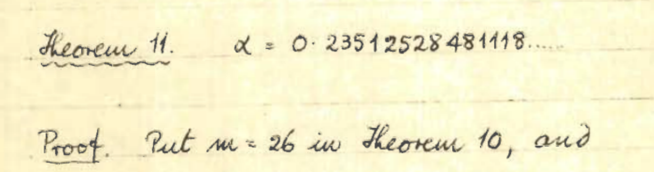

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Further improvements by Lunnon and Bohman: $\leq 0.22002 \cdot 2^n$

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Moser's proof

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$$\mathbb{E}X^2 \geq \sum_{k=-2^{n-1}}^{2^{n-1}} k^2 \cdot \frac{1}{2^n} \sim \frac{2}{3}\left(2^{n-1}\right)^3 \frac{1}{2^n} \sim c \cdot 4^n.$$

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An explicit computation shows

$$\mathbb{E}X^2 = \sum_{i=1}^{n} a_i^2 \leq n \cdot a_n^2 \qquad \text{and we are done.}$$

Lower bound (Moser, 1950s)

$$a_n > c\frac{2^n}{\sqrt{n}}$$

$c \geq 1/4$	Erdős and Moser
$\geq 2/3^{3/2}$	Alon and Spencer
$\geq 1/\sqrt{\pi}$	Elkies
$\geq 1/\sqrt{3}$	Bae, Guy
$\geq \sqrt{3/2\pi}$	Aliev
$\geq \sqrt{2/\pi}$	Elkies, Gleason, unpublished
$\geq \sqrt{2/\pi}$	Dubroff, Fox and Xu
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Where's the Fourier Analysis?

Reformulation (Elkies, 1986)

For any $a_1, \ldots, a_n \in \mathbb{N}$

$$\int_0^1 \prod_{i=1}^n \cos(2\pi a_i x)^2 dx \geq \frac{1}{2^n}.$$

Does equality force $a_n \gtrsim 2^n$?

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with $|x_i - x_j| \geq 2$.

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with $|x_i - x_j| \geq 2$. Then convolve with a bump function and compute L^2-norms. A bit of flexibility in the bump function.

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and use Taylor expansion $\cos x \geq 1 - x^2/2$. Already gives $c2^n/\sqrt{n}$.

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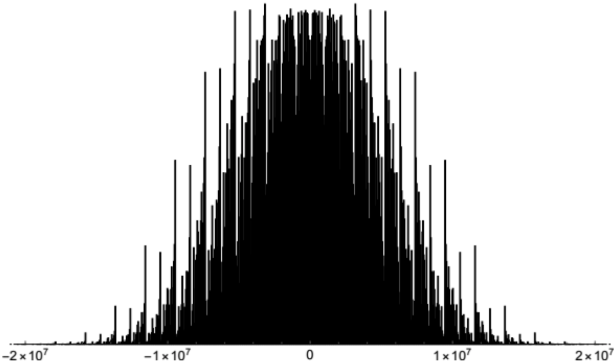

FIGURE 1. A histogram of the discrete measure μ derived from the first 22 terms from the Conway-Guy sequence.

Littlewood's Cosine Problem

Littlewood (1950s)

Given $A \subset \mathbb{N}$, consider $f : [0, 2\pi] \to \mathbb{R}$ given by

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Bedert (2024)

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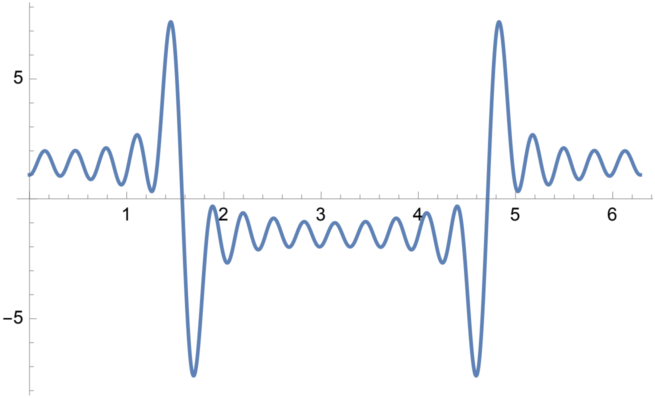
Warning (Sahasrabudhe), the function

$$f(x) = 2\cos(x) + \sum_{k=2}^{2n} \sin\left(\frac{k\pi}{2}\right) \cos(kx)$$

has all coefficients in $\{-1, 0, 1\}$ (and one 2) and only 2 roots.

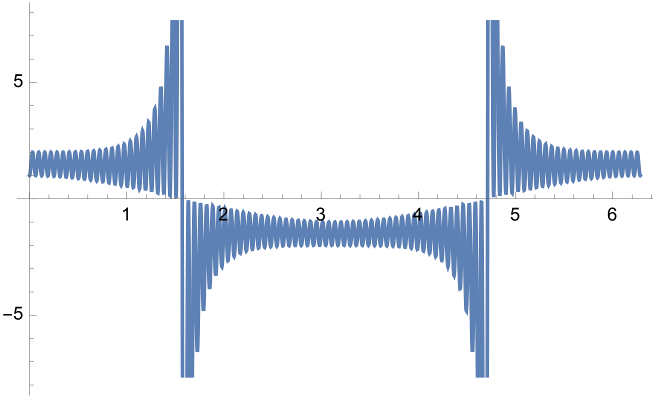
Littlewood's Cosine Problem

$$f(x) = 2\cos(x) + \sum_{k=2}^{20} \sin\left(\frac{k\pi}{2}\right) \cos(kx)$$



Littlewood's Cosine Problem

$$f(x) = 2\cos(x) + \sum_{k=2}^{100} \sin\left(\frac{k\pi}{2}\right)\cos(kx)$$



Chowla Cosine Problem

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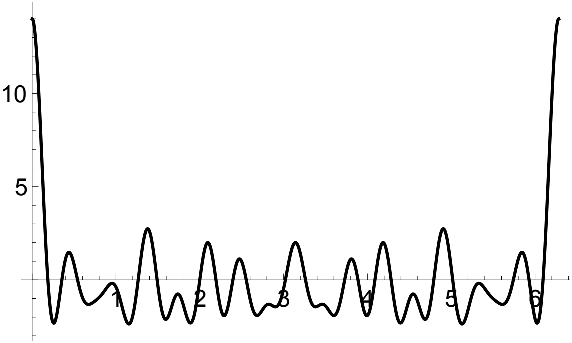
Example. $A = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 18\}$



Bourgain, Ruzsa, Borwein: $\min \leq -c \cdot e^{c\sqrt{\log n}}$

Hardy-Littlewood Maximal Function

Let $f \in L^1(\mathbb{R})$. Everybody knows the **Hardy-Littlewood Maximal Function**

$$(Mf)(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} f(y)dy.$$

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Fairly Natural Question

What are the **maximal radii**?

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Fairly Natural Question

What are the **maximal radii**? What can be said?

Hardy-Littlewood Maximal Function

Here is an amazing and somewhat confusing example

$$\frac{1}{2r}\int_{x-r}^{x+r} \sin(y)dy = \frac{\sin(r)}{r}\sin(x).$$

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This actually tells you a little bit more: when computing integrals one of $\{0, 4.493\ldots\}$ gives you the *maximal* average and the other gives you the *minimal* average

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Suppose $f \in C^{\alpha}(\mathbb{R})$ is periodic and $\alpha > 1/2$. If the function A_x given by

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Rest of the argument: fixed points of tangent $\tan(x) = x$ are linearly independent over $\mathbb{Q}$.

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Suppose $f \in C^\infty$ is periodic and $f(x) \neq a + b\sin(cx + d)$. Does this imply that the maximal radius functions assumes infinitely many values? Or at least 7 values? Or a set of values of Hausdorff dimension at least?

Very few prior results about the basic structure of this function.
Named **frequency function** by Faruk Temur[3]

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Theorem 2. *Let $f \in L^1(\mathbb{R})$. Let $C > 1$ be a real number. Then the set*

$$\left\{x \in \mathbb{R} : \frac{|x|}{2C} \leq \mathcal{T}f(x) \leq \frac{|x|}{C}\right\}$$

is bounded.

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Theorem 4. *For every $\varepsilon > 0$, there exists a function $f \in L^1(\mathbb{R})$ such that*

$$|\{x \in \mathbb{R} : |x| \leq N,\ \mathcal{T} f(x) = 0\}| \geq \frac{1}{8} N / \log^{1+\varepsilon} N$$

for infinitely many values of $N \in \mathbb{N}$.

---

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Sums of two squares

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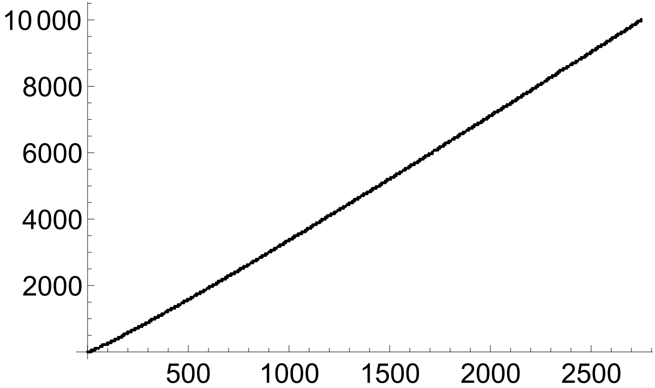
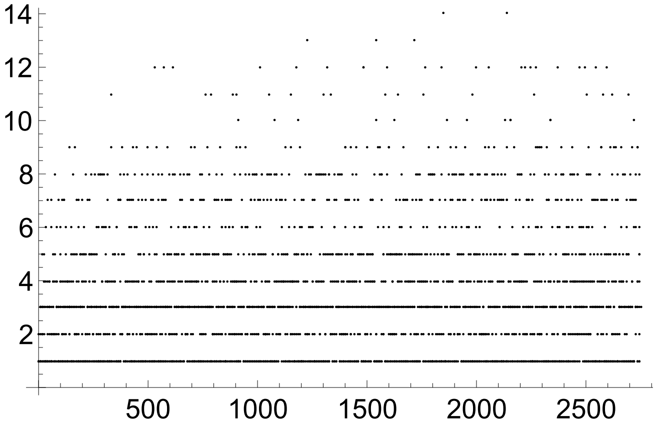
This shows that the **average** gap in the sequence around size $\sim n$ is of order $\sim \sqrt{\log n}$. How large can the gap be?

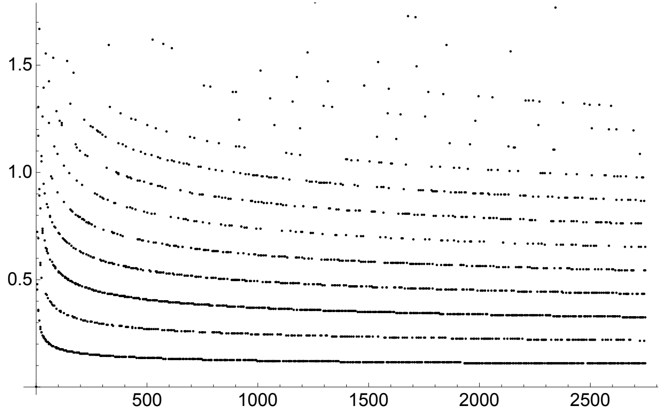
The sequence of sums of two squares (a_n)



The sequence of gaps $(a_{n+1} - a_n)$



The sequence of rescaled gaps $(a_{n+1} - a_n)/\log a_n$



Ram Bambah
(1925 – **26 May 2025**)

Theorem (Bambah-Chowla, '47)

Every interval

$$(n, n + \sqrt{8}n^{1/4} + 2)$$

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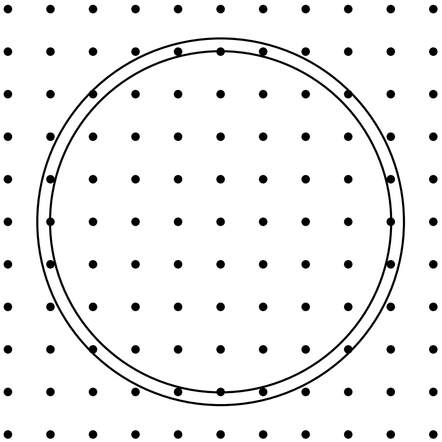
Never been improved, not even the $\sqrt{8}$.



Every ‘wide’ annulus contains lattice points. Lattice points and circles, the Poisson Summation Formula....

Cosine Sign Correlation

The problem started back in 2015 as follows (Fourier Analysis $\to$ Hermite functions $\to$ eigenfunctions of $-\Delta + x^2$).

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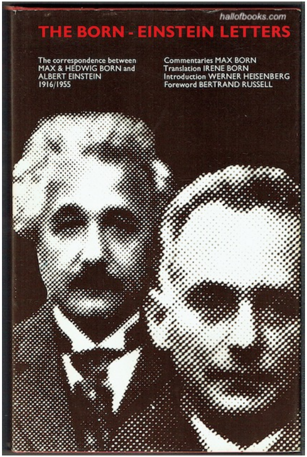
D. Oliveira e Silva (IST Lisboa)

Felipe Gonçalves (UT Austin)

Motivation: Letter Einstein → Born, March 3 1947

[...] that physics should represent reality in time and space, without spooky action at a distance.

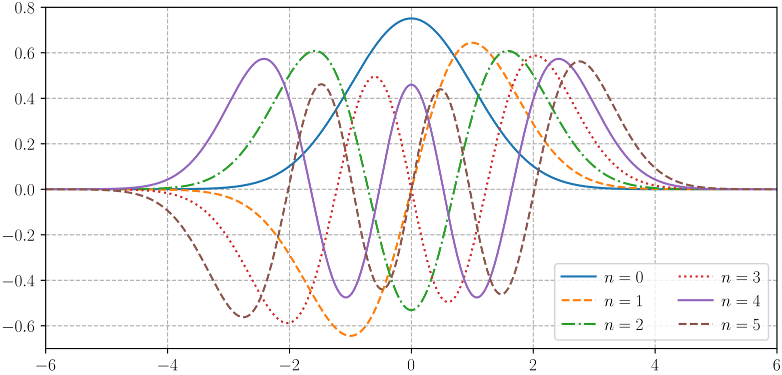


Harmonic Oscillator

The eigenfunctions

$$(-\Delta + x^2)\phi_k = \lambda_k \phi_k$$

are **Hermite functions**.



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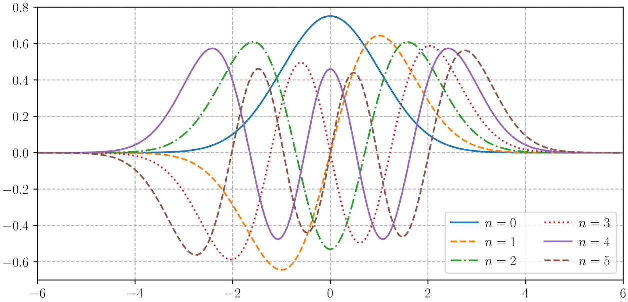

Strange fact. The signs of $\phi_k(1)$ and $\phi_k(5)$ are not independent, they are **positively correlated**.

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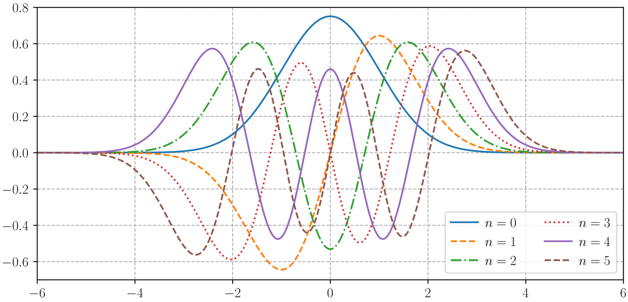

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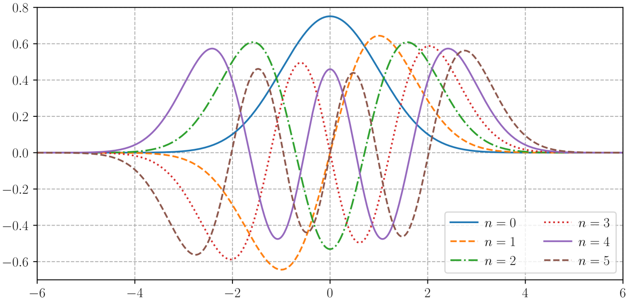

Strange fact. The signs of $\phi_k(1)$ and $\phi_k(5)$ are not independent, they are **positively correlated**. 'Quantum particles located at $x = 1$ and $x = 5$ talk to each other'. This can be made rigorous by using asymptotic expansion of Hermite functions (WKB for physicists) which reduces this to a problem involving only cosines.

Cosine Sign Correlation

Lemma (Gonçalves, Oliveira e Silva, S, J. Spectral Theory '19)

For any integers $a < b$, we have

$$\frac{1}{2\pi}\left|\{x \in [0, 2\pi] : \mathrm{sign}(\cos(ax)) = \mathrm{sign}(\cos(bx))\}\right| \geq \frac{1}{3}$$

with equality if and only if $b = 3a$.

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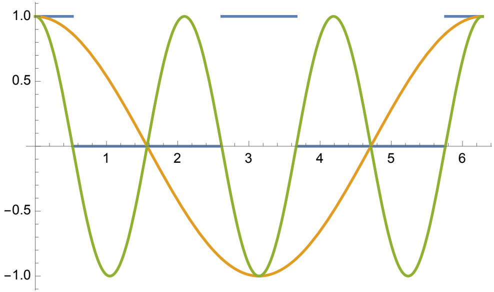

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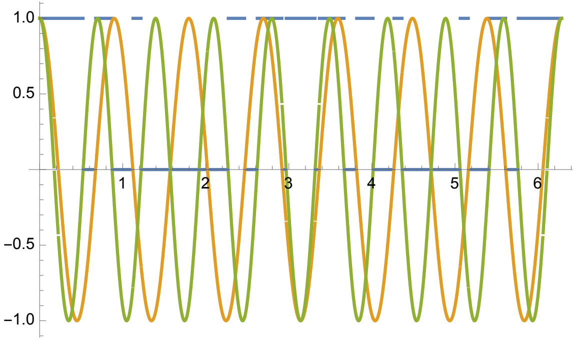

Big Question

Given $a_1, a_2, \ldots, a_n \in \mathbb{N}$ and consider the functions

$$\cos(a_1 x), \cos(a_2 x), \ldots, \cos(a_n x) \quad \text{on } [0, 2\pi].$$

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Consider set $S \subseteq [0, 2\pi]$ where all n functions have the same sign. The set S is non-empty (because $\cos(0) = 1$). How small can it be (as a function of n)?

Let us denote the answer by $F(n)$. We know $F(2) = 1/3$.

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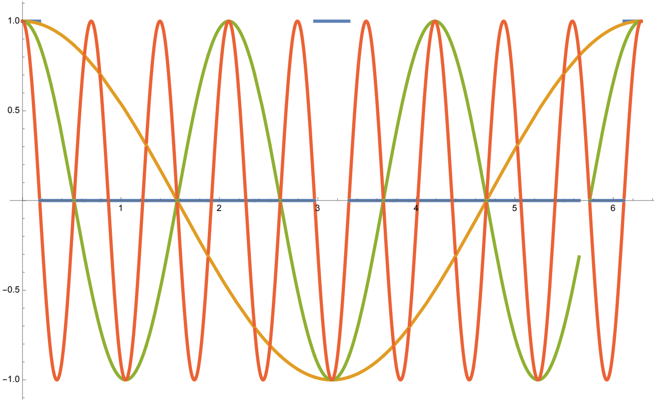
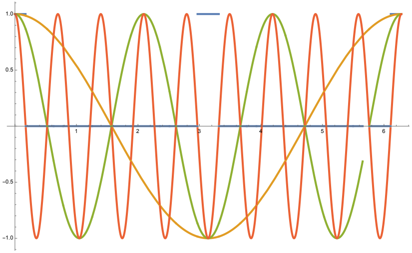


Idea behind proof



$\cos(ax), \cos(bx), \cos(cx)$ and $a < b < c$.

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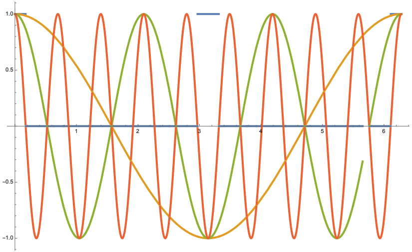

$\cos(ax), \cos(bx), \cos(cx)$ and $a < b < c$. If $c \gg b$, then

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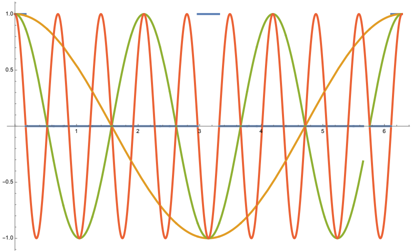

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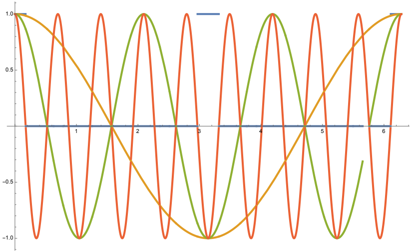

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Likewise if $a \ll b$. Then lots of elementary Fourier Analysis and case distinctions, lots of checking of special cases. No 'clean' proof and no proof that would generalize to $n = 4$.

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by taking $1, 3, 11, 35, 105$. One question I had was whether $F(n) \geq C^{-n}$ and this was recently (last week!) resolved.

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$$\int_0^1 \prod_{k=1}^{n} 1_{[-1/4,1/4]}(a_k x) \geq \int_0^1 \prod_{k=1}^{n} \phi(a_k x) \geq \phi(0)^n.$$

Theorem (Lars Becker)

We have

$$F(n) \geq \frac{1}{4^n}.$$

The measure of the set where all cosines are positive is given by

$$\int_0^1 \prod_{k=1}^n 1_{[-1/4,1/4]}(a_k x).$$

Suppose that $\phi : \mathbb{T} \to \mathbb{R}$ satisfies $\widehat{\phi} \geq 0$ and $\phi \leq 1_{[-1/4,1/4]}$. Then, taking a Fourier transform,

$$\int_0^1 \prod_{k=1}^n 1_{[-1/4,1/4]}(a_k x) \geq \int_0^1 \prod_{k=1}^n \phi(a_k x) \geq \phi(0)^n.$$

Optimal choice of ϕ is given by

$$\phi = 4 \cdot 1_{[-1/8,1/8]} * 1_{[-1/8,1/8]}.$$

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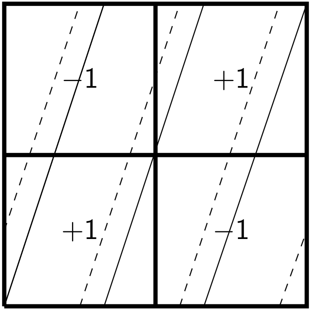

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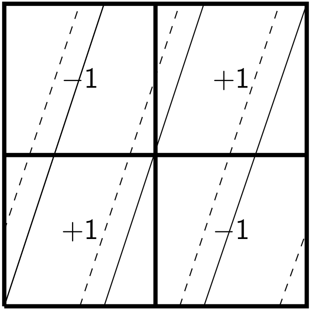

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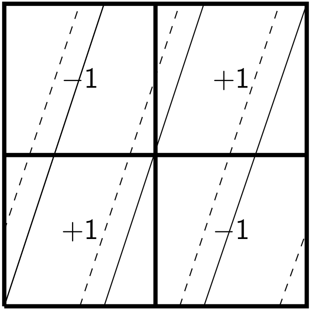

Simple Idea. If the geodesic is very long, then it's probably also fairly uniformly distributed and the average value of the function is probably pretty close to the mean value. Extremal geodesics corresponding to extreme values **have to be short**.

Theorem (S, Bull. Aust. Math. Soc, 2018)

Let $f : \mathbb{T}^2 \to \mathbb{R}$ be at least $s \geq 2$ times differentiable and have mean value 0. Then

$$\sup_{\gamma \text{ closed geodesic}} \frac{1}{|\gamma|} \left| \int_\gamma f \quad d\mathcal{H}^1 \right|,$$

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Directional Poincaré

For any function $f : \mathbb{T}^2 \to \mathbb{R}$ with mean value 0, we have

$$\|\nabla f\|^2_{L^2(\mathbb{T})} \geq \|f\|^2_{L^2(\mathbb{T})}$$

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Only problem when $k = 0$ (excluded), extremizers and near-extremizers have to be low-frequency.

Directional Poincaré

Directional Poincaré (S, Arkiv för Matematik, 2015)

One can replace the gradient by a *directional* derivative

$$\|\nabla f\|_{L^2(\mathbb{T})} \left\| \frac{\partial f}{\partial x} + \frac{1+\sqrt{5}}{2} \frac{\partial f}{\partial y} \right\|_{L^2(\mathbb{T})} \geq c \|f\|^2_{L^2(\mathbb{T})}.$$

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Sharp up to constant at every frequency scale. Take $f_n(x,y) = \sin(F_{n+1}x - F_n y)$ with F_n being the $n-$th Fibonacci number. Then the left-hand side becomes

$$LHS = \sqrt{\frac{F_{n+1}^2}{F_n^2} + 1} \left| \frac{F_{n+1}}{F_n} - \frac{1+\sqrt{5}}{2} \right| F_n^2 \cdot \|f_n\|^2$$

and that expression converges to $1/\sqrt{5}$.

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1. Statement is false if golden ratio is replaced by e.
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3. Version in $\mathbb{T}^d$ exists as well.
4. Proof is actually quite easy. (Too easy).

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Hadamard (1893)

RÉSOLUTION D'UNE QUESTION RELATIVE AUX DÉTERMINANTS;

PAR M. J. HADAMARD.

1. Étant donné un déterminant

$$\Delta = \begin{vmatrix} a_1 & b_1 & \dots & l_1 \\ a_2 & b_2 & \dots & l_2 \\ \dots & \dots & \dots & \dots \\ a_n & b_n & \dots & l_n \end{vmatrix}, \tag{1}$$

dans lequel on sait que les éléments sont inférieurs en valeur absolue à une quantité déterminée A, il y a souvent lieu de chercher une limite que le module de Δ ne puisse dépasser.

Given a determinant [...] with the assumption that the elements are less in absolute value than some known quantity A, we wish to determine a limit that the modulus of Δ does not exceed.

Hadamard's Inequality (1893)

If $A \in \mathbb{R}^{n \times n}$ has all entries bounded, $|A_{ij}| \leq 1$, then

$$|\det(A)| \leq n^{n/2}.$$

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Cases of equality require that each entry is ± 1 and that the rows/columns are orthogonal.

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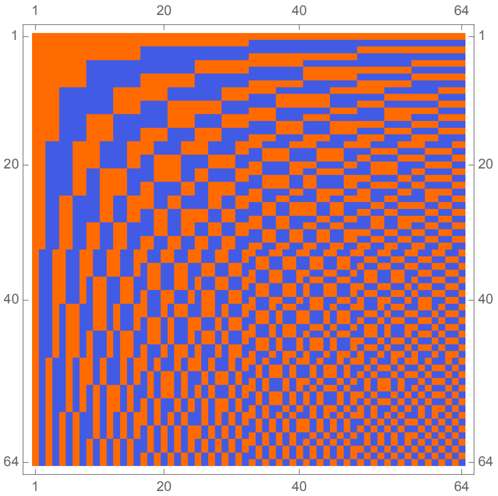
$$H_1 = (1) \quad H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad H_4 = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

A large example (64×64) is



Sylvester Matrices?

When Hadamard talks about Hadamard matrices, he immediately refers to work of James Sylvester (1867)

"Par exemple, pour $n = 4$, M. Sylvester indique les deux types

$$(4) \qquad \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix}$$

et

Sylvester (1867)

THOUGHTS ON INVERSE ORTHOGONAL MATRICES, SIMULTANEOUS SIGN-SUCCESSIONS, AND TESSELLATED PAVEMENTS IN TWO OR MORE COLOURS, WITH APPLICATIONS TO NEWTON'S RULE, ORNAMENTAL TILE-WORK, AND THE THEORY OF NUMBERS.

[*Philosophical Magazine*, XXXIV. (1867), pp. 461—475.]

13. The complete system of relations between the two sets of 2^i quantities given by the theorem in Art. 10 may it is evident be expressed by means of the inverse orthogonal matrix (also orthogonal) whose type corresponds to $2 \,.\, 2 \,.\, 2 \ldots$ (i terms). Thus, for example, for the case of $i=3$, we may write—

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If $A \in \{-1, 1\}^{n \times n}$ satisfies $\det(A) = n^{n/2}$, then

$$B = \begin{pmatrix} A & A \\ A & -A \end{pmatrix}$$

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In particular, $n \times n$ Hadamard exist whenever n is a power of 2.

Back to Hadamard

ments réels. Peut-on trouver de tels déterminants pour d'autres valeurs de n?

Can one find such determinants for other values of n?

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If $A \in \mathbb{R}^{n \times n}$ is a Hadamard matrix and $n \geq 4$, then 4 divides n.

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If $A \in \mathbb{R}^{n \times n}$ is a Hadamard matrix and $n \geq 4$, then 4 divides n.

Proof.

Simple. We can always multiply each column by ± 1. We can thus assume the first row is only 1's. This forces the second row to have the same number of $+1$ as -1 (and thus n is even). We can now reorder the columns so that the first $n/2$ entries in the second row are $+1$. Let us denote the number of $+1$ among the first $n/2$ entries in the third row by a. Then $n/2 - a$ entries are -1 and

$$a - (n/2 - a) - (n/2 - a) + a = 0 \qquad \text{or} \qquad 4a = n.$$

Signed Orthogonality

A Hadamard matrix $H \in \mathbb{R}^{n \times n}$ is a matrix all of whose entries are ± 1 and

$$\forall\, x \in \mathbb{R}^n \qquad \|Hx\| = \sqrt{n}\|x\|.$$

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Amazing Theorem (Dong-Rudelson, 2023)

There exists $C > 0$ such that for all $n \in \mathbb{N}$ there exists a matrix $H \in \{-1, 1\}^{n\times n}$ such that

$$\forall\ x \in \mathbb{R}^n \qquad \frac{\sqrt{n}}{C}\|x\| \leq \|Hx\| \leq C\sqrt{n}\|x\|.$$

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Theorem (S, 2025)

There exists $C > 0$ such that for all $n \in \mathbb{N}$ there exists a **circulant** matrix $H \in \{-1, 1\}^{n \times n}$ such that

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Circulants are matrices of the type

$$A = \begin{pmatrix} a_0 & a_{n-1} & a_{n-2} & \dots & a_1 \\ a_1 & a_0 & a_{n-1} & \dots & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 \end{pmatrix}$$

Are there any nice good explicit constructions of approximate Hadamard matrices? (Circulant or not circulant).

Proof

- Using the Fourier matrix

$$\mathcal{F} = \frac{1}{\sqrt{n}} \left(e^{-2\pi imk/n} \right)_{m,k=0}^{n-1}$$

every circulant matrix can be written as $A = \mathcal{F}^{-1} D \mathcal{F}$.

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- Eigenvalues λ_j given by, where $0 \leq j \leq n-1$,

$$\lambda_j = a_0 + a_1 w^j + a_2 w^{2j} + \cdots + a_{n-1} w^{(n-1)j} \text{ with } w = \exp(2\pi i/n)$$

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- Introducing the polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}$$

  we see that the eigenvalues show up at the roots of unity.

Proof (continued)

- Introducing the polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}$$

$$\frac{\max_{\|x\|=1} \|Ax\|}{\min_{\|x\|=1} \|Ax\|} = \frac{\max_{0 \le j \le n} |\lambda_j|}{\min_{0 \le j \le n} |\lambda_j|}$$
$$= \frac{\max_{z^n=1} |p(z)|}{\min_{z^n=1} |p(z)|} \le \frac{\max_{|z|=1} |p(z)|}{\min_{|z|=1} |p(z)|}.$$

Proof (continued)

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$$= \frac{\max_{z^n=1} |p(z)|}{\min_{z^n=1} |p(z)|} \le \frac{\max_{|z|=1} |p(z)|}{\min_{|z|=1} |p(z)|}.$$

- The question is thus: can we find a polynomial $p(z)$ with coefficients ± 1 such that

$$\frac{\max_{|z|=1} |p(z)|}{\min_{|z|=1} |p(z)|} \le C?$$

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This question is not new (1966)

ON POLYNOMIALS $\sum^{n} \pm z^{m}$, $\sum^{n} e^{\alpha_m i} z^{m}$, $z=e^{\theta i}$

J. E. LITTLEWOOD

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ON POLYNOMIALS $\sum^{n} \pm z^m$, $\sum^{n} e^{\alpha_m i} z^m$, $z = e^{\theta i}$

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and was recently answered (2020)

FLAT LITTLEWOOD POLYNOMIALS EXIST

PAUL BALISTER, BÉLA BOLLOBÁS, ROBERT MORRIS,
JULIAN SAHASRABUDHE, AND MARIUS TIBA

ABSTRACT. We show that there exist absolute constants $\Delta > \delta > 0$ such that, for all $n \geqslant 2$, there exists a polynomial P of degree n, with coefficients in $\{-1, 1\}$, such that

$$\delta\sqrt{n} \leqslant |P(z)| \leqslant \Delta\sqrt{n}$$

for all $z \in \mathbb{C}$ with $|z| = 1$. This confirms a conjecture of Littlewood from 1966.

Signed Orthogonality

Question

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4. Is there a **nice** construction?

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Failed idea: let us take the matrix

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and replace each entry by its sign

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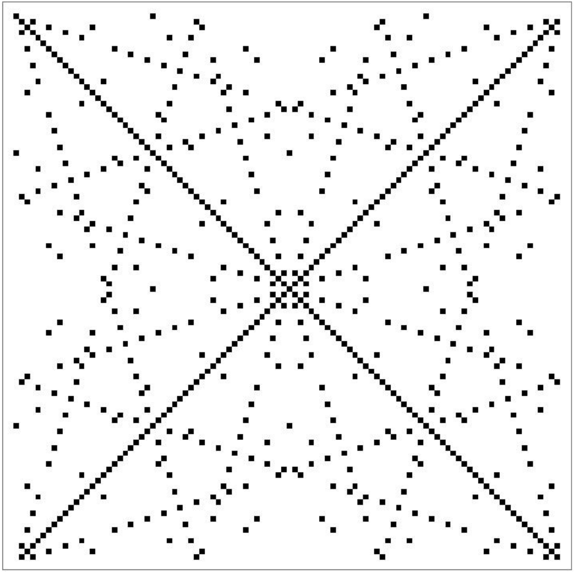
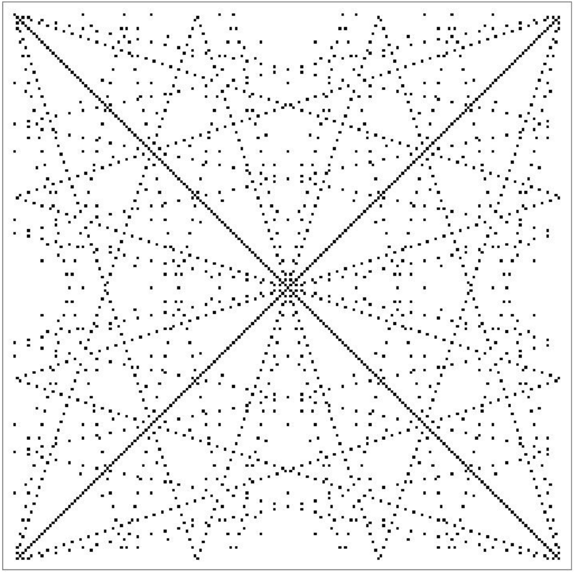
and $A^T A$ should be close to some rescaled identity matrix.

Signed Orthogonality



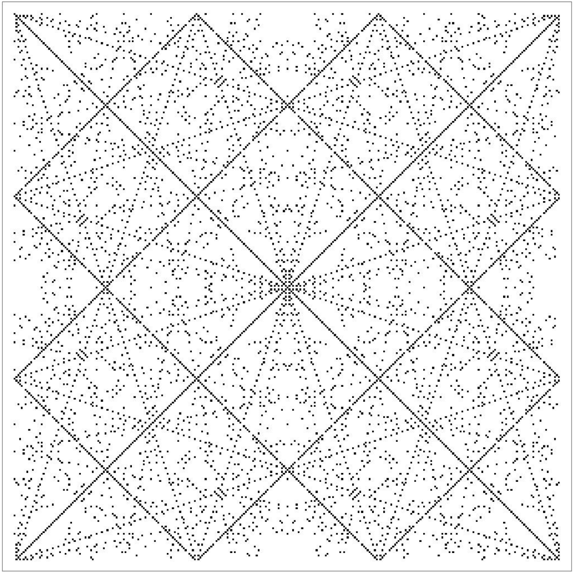
Large entries of $A^T A$ when $n = 100$.

Signed Orthogonality



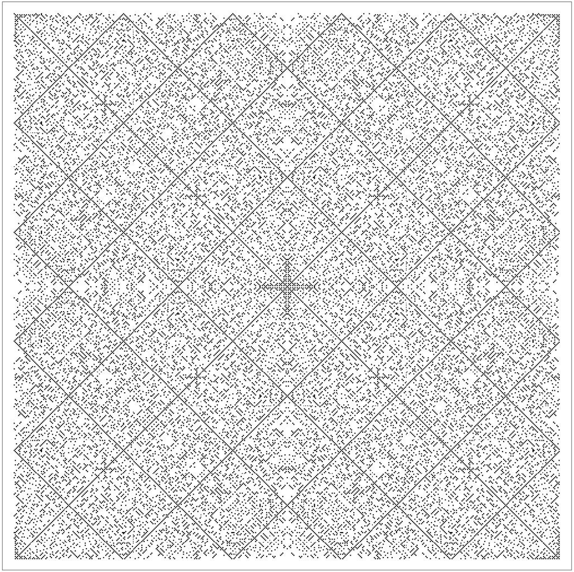
Large entries of $A^T A$ when $n = 200$.

Signed Orthogonality



Large entries of $A^T A$ when $n = 300$.

Signed Orthogonality



Large entries of $A^T A$ when $n = 500$.

Signed Orthogonality

Theorem (François Clément and S, 2025)

Let n be prime and $0 \neq a, b \in \mathbb{F}_n$. Then

$$\Sigma_{a,b} = \left| \sum_{k=1}^{n} \operatorname{sign}\left(\cos\left(\frac{2\pi}{n} ak\right)\right) \operatorname{sign}\left(\cos\left(\frac{2\pi}{n} bk\right)\right) \right|$$

satisfies,

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satisfies, using $\|x\|$ to denote distance from 1,

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Several other more technical results.

Signed Orthogonality

This phenomenon is more general: take the Legendre polyomials

$$p_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n .$$

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Consider

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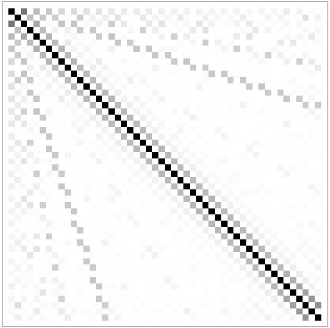
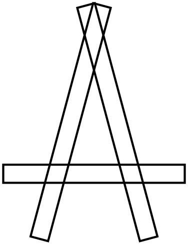


Probably generally true for orthogonal polynomials on the real line...?

Kakeya on the Sphere



Here's a cheap Kakeya-type statement in $\mathbb{R}^2$.

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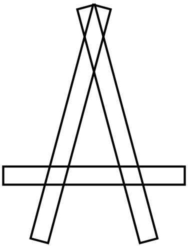

Here's a cheap Kakeya-type statement in $\mathbb{R}^2$. Given $\ell_1, \ldots, \ell_n$ lines (no two parallel), then their $1/n-$neighborhoods $T_1, \ldots, T_n$ satisfy

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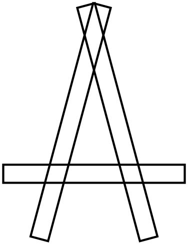

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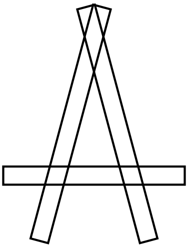

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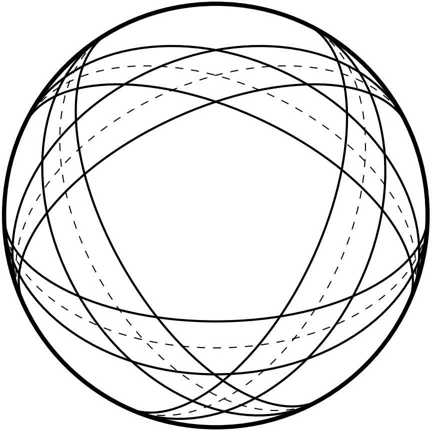
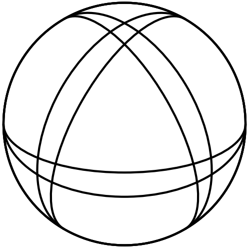
Trivial bound is $\gtrsim 1$. What happens on $\mathbb{S}^2$?

Kakeya on the Sphere





Given $\ell_1, \ldots, \ell_n$ great circles, then their $1/n-$neighborhoods $C_1, \ldots, C_n$ satisfy

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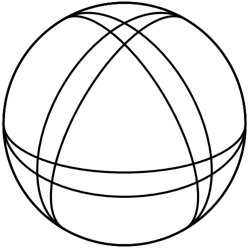
and *this is best possible*[4].

[4] S, Well-Distributed Great Circles on $\mathbb{S}^2$, Disc. Comp. Geometry, 2018.



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and *this is best possible*[4]. Nontrivial sharp bound

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} |C_i \cap C_j|^2 \gtrsim \frac{\log n}{n^2}.$$

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Idea behind proof. Identify a great circle with 'north and south pole'.

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This is reminiscent of the Riesz energy

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$$\left(1 - \langle p_i, p_j \rangle^2\right)^{-s/2} \qquad \text{independently (Chen-Hardin-Saff, 2020).}$$

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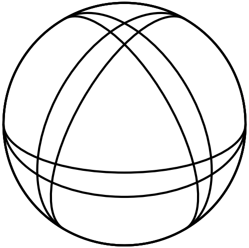

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We have (from Hölder and $p = 1$) that for all $p \geq 1$,

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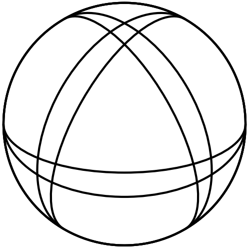

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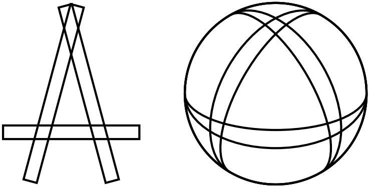
Sharp for $1 \leq p \leq 2$. *Probably not when $p > 2$* **?**

Speaking of curvature?



The curvature of $\mathbb{S}^2$ increases transversality, which decreases the area of intersection.

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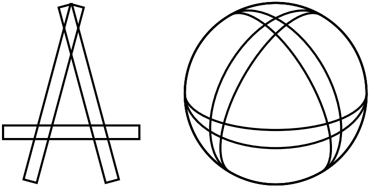

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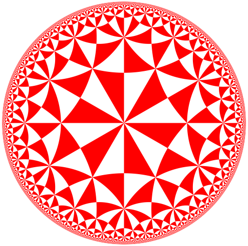

?

The Motzkin-Schmidt Problem

Jean was very generous with problems. In December 2013, there was a very nice conference in Jerusalem that I especially remember for two things. We had the snowstorm of the century and that was the last time I met Jean in person. He told me there the following problem, which to the best of my knowledge, is still open. Is it possible to find n points in the unit square such that the $1/n$-neighborhood of any line contains no more than C of them for some absolute constant C? The motivation for this problem comes from a possible construction of spherical harmonics as a combination of Gaussian beams, which would have L^∞ norm bounded by a constant independently of the degree.

(Peter Varju, Remembering Jean Bourgain, AMS Notices 2021)

The Motzkin-Schmidt Problem

A Trivial Statement

Given n points in $[0,1]^2$, there exists exists a line $\ell \in \mathbb{R}^2$ whose $3/n-$neighborhood contains at least 3 points.

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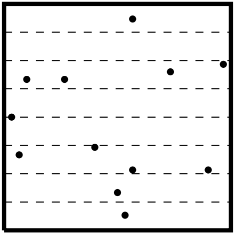

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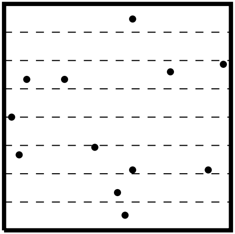

The Motzkin-Schmidt Problem

Is this true with a $o(1/n)$ neighborhood?

A quick one: number of critical points

Trivial

Let $f : \mathbb{T} \to \mathbb{R}$ be smooth. Then

$$(\#\text{number of critical points}) \cdot \|f\|_{L^\infty(\mathbb{T})} \gtrsim \|f'\|_{L^1(\mathbb{T})}.$$

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Open problem. Is there a one-parameter family? Interpolation? Higher dimensions? Or it's just an accident?

Spooky Action at a Distance

When doing Fourier Analysis, there is no real difference between $\sin(x)$ and $-\sin(x)$, the sign cancels

$$f = \sum_{k \in \mathbb{N}} \langle f, \phi_k \rangle \phi_k.$$

[5] S, Quantum entanglement and the growth of Laplacian eigenfunctions. Communications in Partial Differential Equations, 48 (2023), 511-541.

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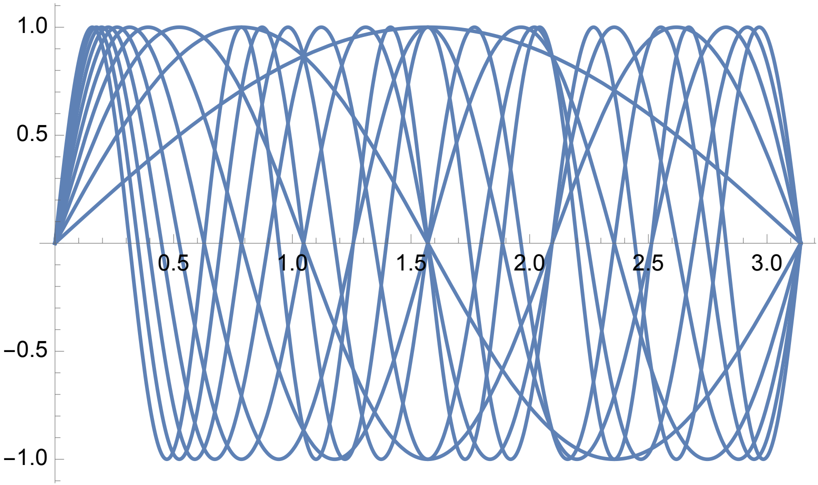
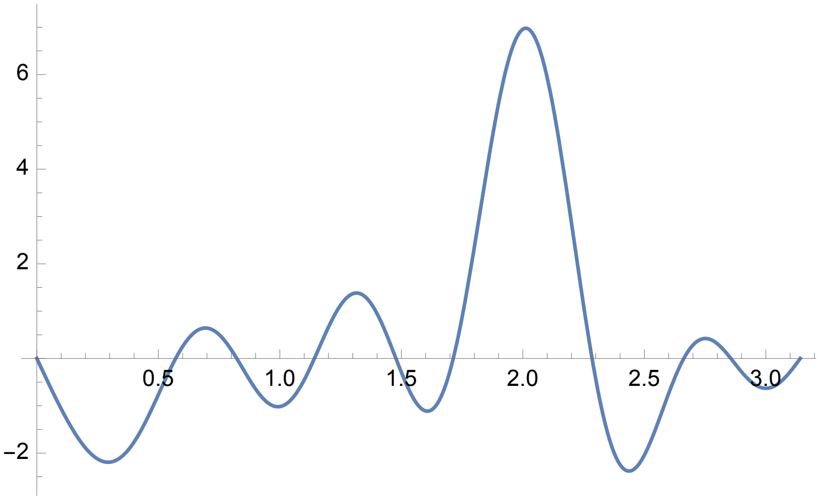
Motivated by some question on Laplacian eigenfunctions[5], we can flip all the signs so that the functions are all positive in x

$$f(x,y) = \sum_{k=1}^{n} \operatorname{sign}(\phi_k(x)) \phi_k(y).$$

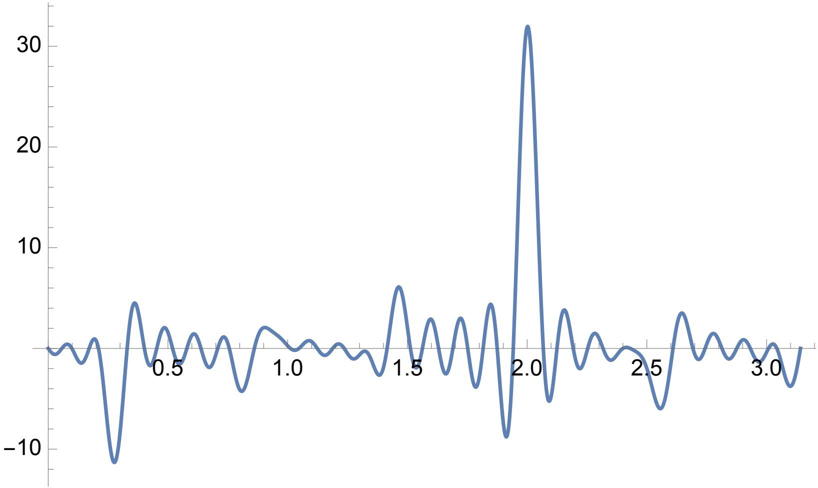
[5] S, Quantum entanglement and the growth of Laplacian eigenfunctions. Communications in Partial Differential Equations, 48 (2023), 511-541.



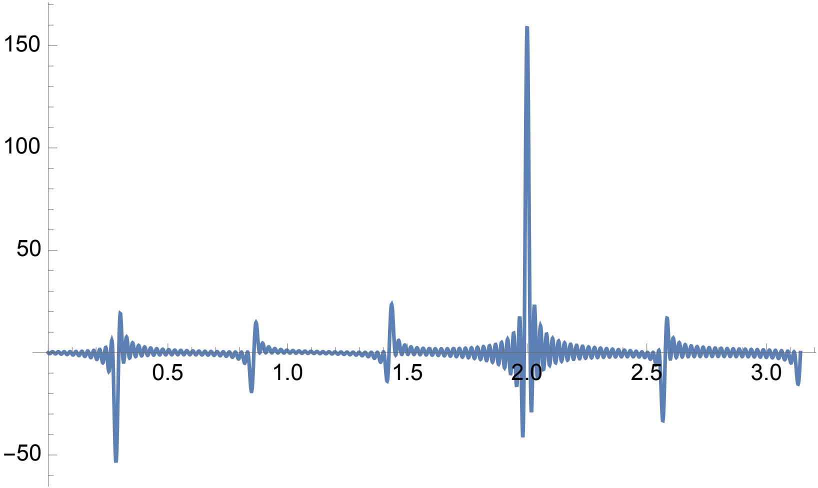
The functions $\sin(kx)$ on $[0, \pi]$.



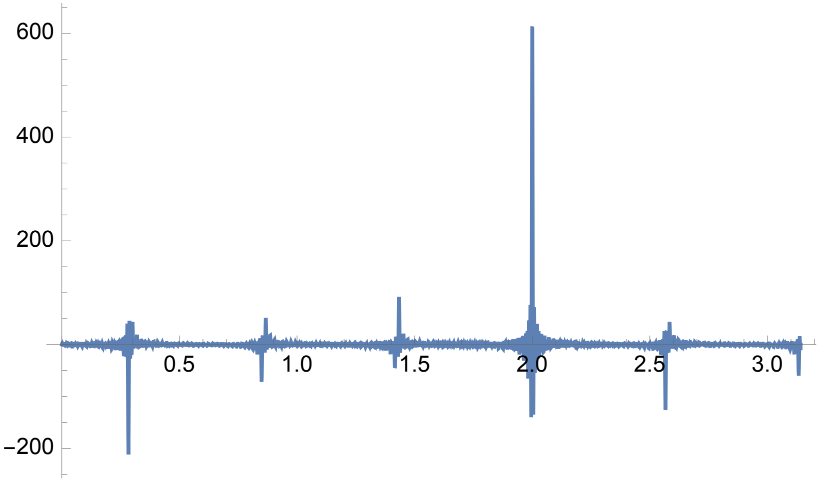
Summed sines (sign so that positive in $x = 2$), 10 terms.



Summed sines (sign so that positive in $x = 2$), 50 terms.



Summed sines (sign so that positive in $x = 2$), 250 terms.



Summed sines (sign so that positive in $x = 2$), 1000 terms.

Spooky: General Orthogonal Functions

Eigenfunctions of Laplacian on $1/4-$disk with Dirichlet boundary conditions are an orthonormal family of basis functions in L^2.

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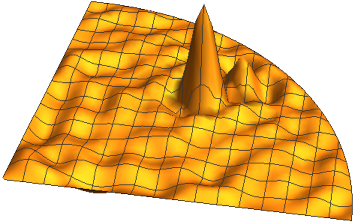

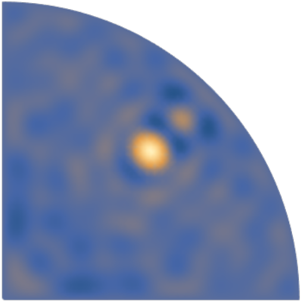

Bessel Function Identities?

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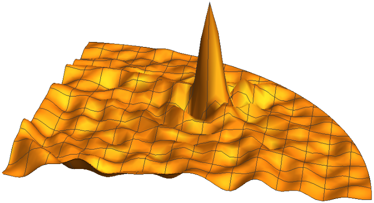

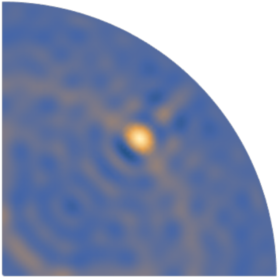

Bessel Function Identities? Other examples?

Opaque Sets and Barriers

Opaque Square (Mazurkiewicz 1916, Bagemihl 1959)

Consider a one-dimensional set in $\mathbb{R}^2$ such that every line that intersects $[0,1]^2$ also intersects the set.

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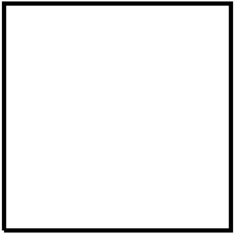

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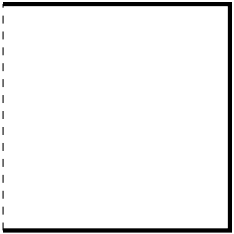

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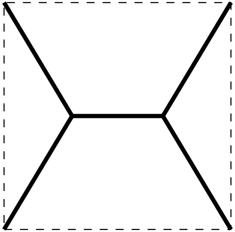

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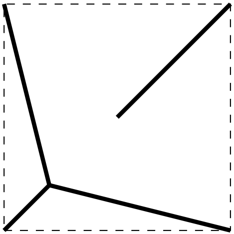

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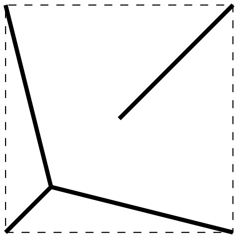

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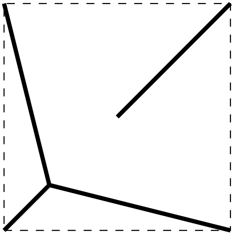

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This is probably the minimizer; people are somewhat amazed by this because it is not connected. Very little progress in the last 30 years.

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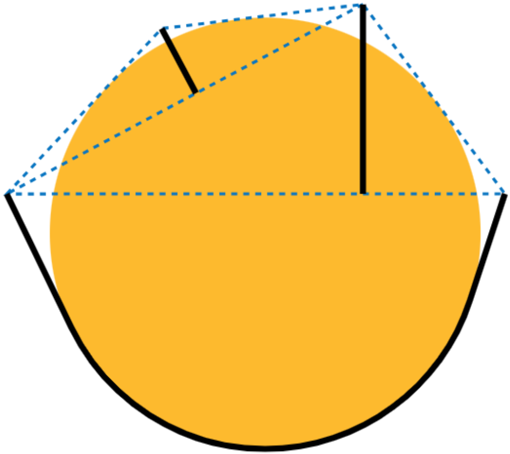
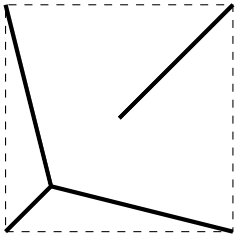
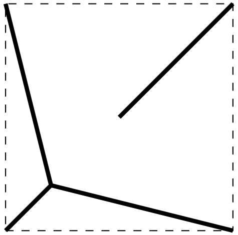




Lower Bound (Jones 1962)

For any convex $\Omega \subset \mathbb{R}^2$, then length is at least half the boundary.



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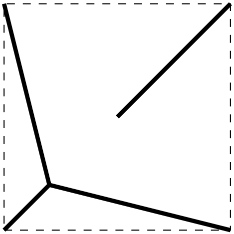

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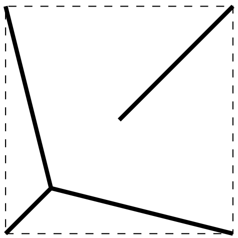

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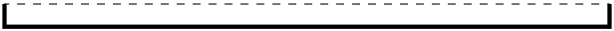

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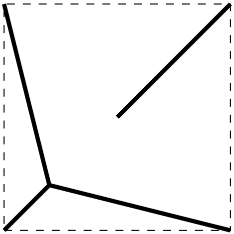

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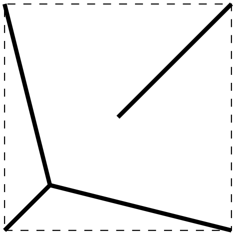

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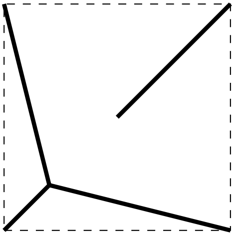

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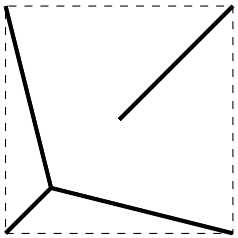

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The mean-width of a line segment of length ℓ is

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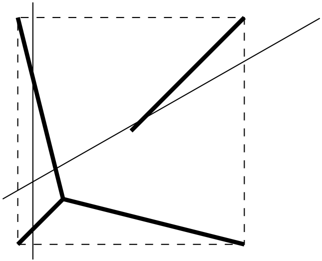

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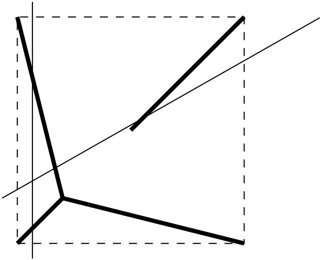


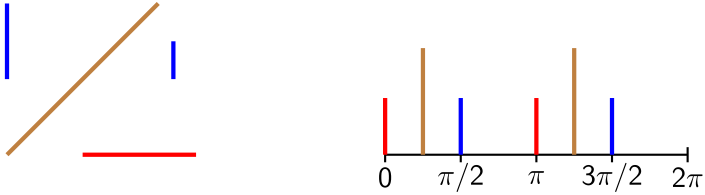
Hard to get improvements from this.

Theorem (Kawamura-Moriyama-Otachi-Pach, 2019)

An opaque set in the square has length at least 2.00002.

Near-Extremizers of Jones bound

Given line segments in the plane, can associate measure on $[0, 2\pi]$.



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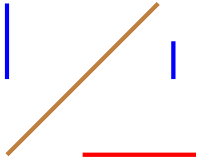

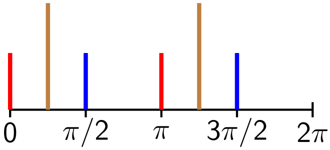

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$$\|\mu_{\mathcal{O}} - \mu_{\partial\Omega}\|_{\dot{H}^{-2}(\mathbb{T})} \leq \frac{L^{1/4}}{\sqrt{2}} \cdot \left(L - \frac{|\partial\Omega|)}{2}\right)^{3/4}.$$

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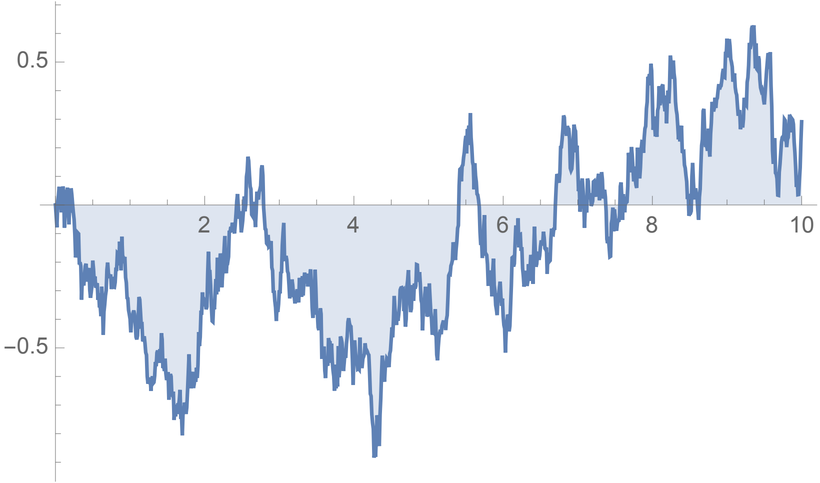
This allows us to make the question more specific: suppose

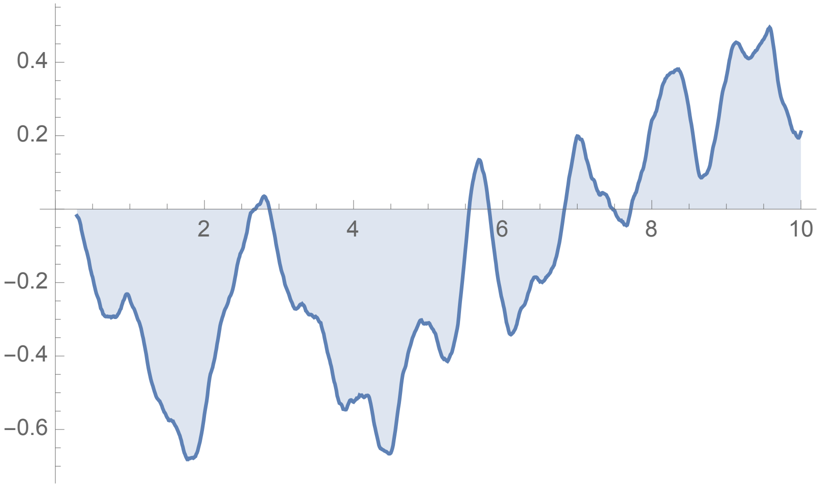
$$[Av](f)(x) = (f * g)(x) = \int_{\mathbb{R}} f(x+y)g(y)dy.$$

What is the best g?

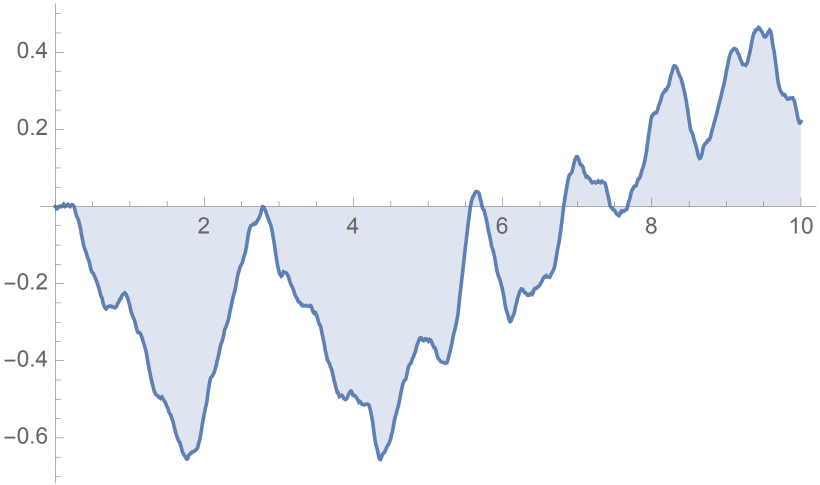
An example: we want understand local averages of



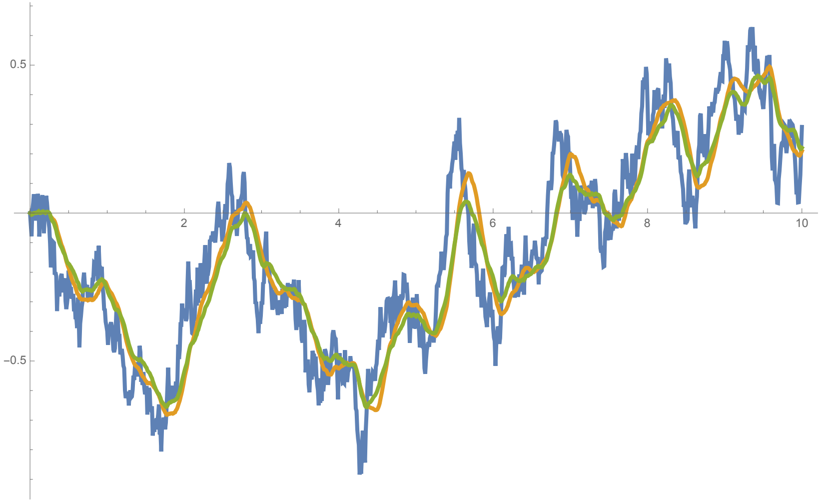
Averaging with characteristic functions



Averaging with exponential distribution



So which one is the right one?



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But generally, all these questions are open.

Some New Uncertainty Principles

Theorem (S, Math Res Lett, 2021)

For $\alpha > 0$ and $\beta > n/2$, there exists $c_{\alpha,\beta,n} > 0$ such that for all $u \in L^1(\mathbb{R}^n)$

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- If we additionally fix the variance, i.e. $\alpha = 2$, then

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Extremizers

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This would mean that the extremizers are **not** smooth.

Open Problem

Is the classical simple averaging

$$\mathrm{Av}(f)(x) = \frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(x+y)dy$$

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Note. This is the discrete analogue of the characteristic function (like in the continuous case!)

Theorem (Noah Kravitz and S, Bull. London Math Soc, 2021)

Let $u : \{-n, \ldots, n\} \to \mathbb{R}$ be a symmetric function with normalization $\sum_{k=-n}^{n} u(k) = 1$ and nonnegative Fourier transform.

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Open problem. For $\ell = 3, 4, \dots$

$$\sup_{0 \neq f \in \ell^2(\mathbb{Z})} \frac{\|\Delta^{\ell}(f * u)\|_{\ell^2(\mathbb{Z})}}{\|f\|_{\ell^2}} \geq ?$$

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$$p(x) = \frac{1}{(n+1)^2}\cdot\frac{1-T_{n+1}(x)}{1-x},$$

where T_n is the $n-$th Chebychev polynomial.

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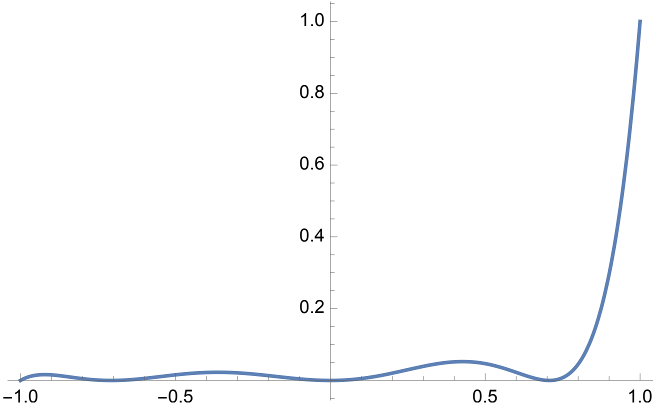

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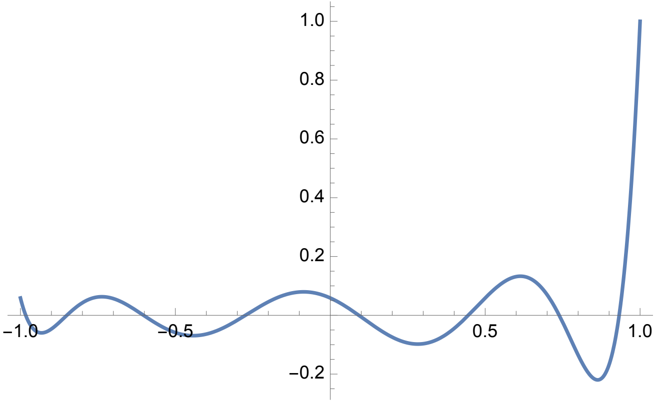

Kozma-Oravecz Inequalities

The function $f : \mathbb{T}^d \to \mathbb{R}$ given by

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Theorem (Taikov, 1965)

The function $f : \mathbb{T} \to \mathbb{R}$ given by

$$f(x) = \sum_{k=A}^{A+B} a_k \cos(kx)$$

has a root in every interval of length $(B+1)/(2A+B)$.

Theorem (Kozma-Oravecz, 2003)

The function $f : \mathbb{T} \to \mathbb{R}$ given by $f(x) = \sum_{k \in S} a_k \cos(kx)$ has a root in each ball of radius

$$r(f) = \frac{1}{4} \sum_{k \in S} \frac{1}{\|k\|}.$$

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The function $f : \mathbb{T} \to \mathbb{R}$ given by $f(x) = \sum_{k \in S} a_k \cos(kx)$ has a root in each ball of radius

$$r(f) = d^{3/2} \sum_{\lambda \in \{\|k\| : k \in S\}} \frac{1}{\lambda}.$$

The proof is simple and based on the convolution identity and if

$$f(x) = \sum_{k \in S} a_k \exp\left(2\pi i \langle x, k \rangle\right),$$

then

$$(f * h_{\delta^*})(x) = \sum_{k \in S} a_k \cdot \widehat{h_{\delta^*}}(k) \cdot \exp\left(2\pi i \langle x, k \rangle\right).$$

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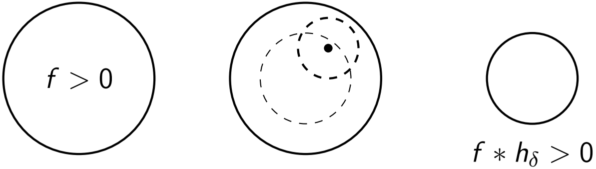

If $f > 0$ on a ball of radius r and we convolve f with a positive function supported on a ball of radius δ^*, then the convolution is positive on a ball of radius $r - \delta^*$.

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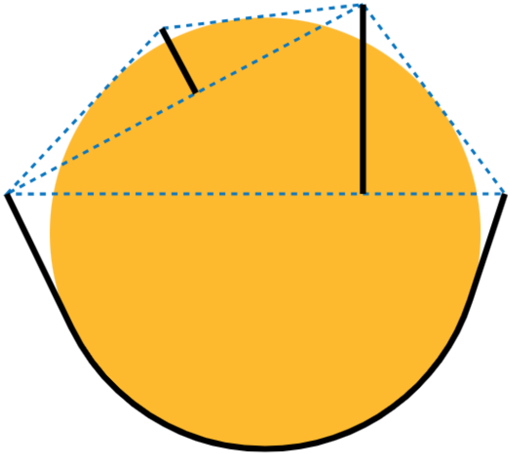
Problem. I don't see any indication that the correct scaling should be the **sum** of the inverse frequencies (inverse frequencies themselves are okay). Maybe something like

$$\left(\sum_{\lambda \in \{\|k\| : k \in S\}} \frac{1}{\lambda^2} \right)^{1/2} \qquad ?$$



Thank you!