Growth Models in the Plane

Stefan Steinerberger

Academia Sinica, Taiwan, July 2025

UNIVERSITY of WASHINGTON

Phlyctis argena on a tree

How does it grow?

Eden Model (1961)

A TWO-DIMENSIONAL GROWTH PROCESS

MURRAY EDEN

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

1. Introduction

It is the purpose of this paper to examine certain of the properties of populations of cells, in particular, properties relating to the architecture of cell colonies. We imagine that the underlying process in the growth of an organism begins with a single cell (perhaps derived from the fusion of two germ cells), and then continues by a process in which the initial cell divides into daughter cells. These in

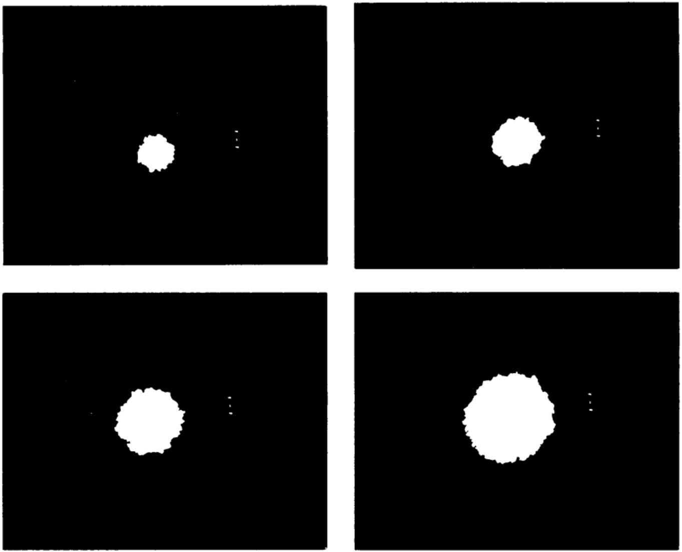
The Eden Model

4. A "symmetrical" growth process

The first model of "growth" which we will consider is perhaps the simplest. Distinguish a single node of the square lattice and assume that it is occupied by a cell γ_1. Assign equal probability to the configurations obtained by adjoining one other cell to the nodes adjacent to γ_1 (there are four of these that are obviously equivalent under rotation). This two-celled configuration has six adjacent nodes. Again assign equal probability to the six 3-configurations obtained by adjoining a single cell. This procedure can be carried out indefinitely, each time adjoining a single cell.

M. Eden, A two-dimensional growth process, Proceedings of Fourth Berkeley Symposium on Mathematics, Statistics, and Probability, 1961

The Eden Model



The Eden Model

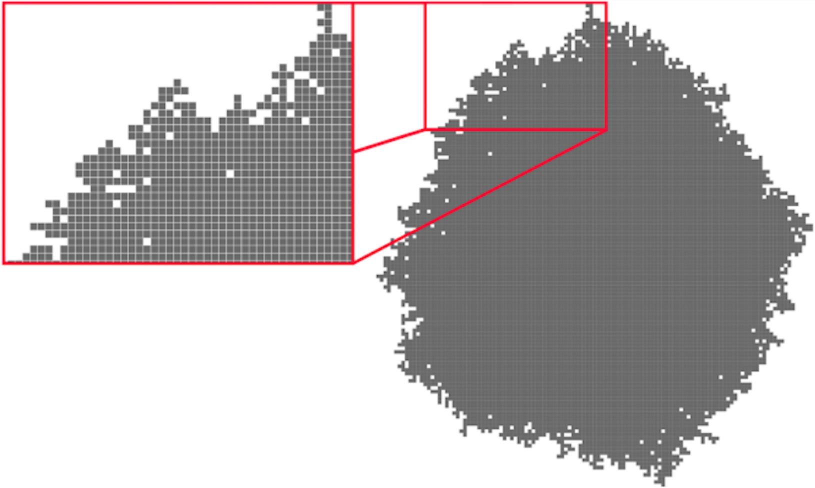


Image: Manin, Roldán & Schweinhart, Topology and Local Geometry of the Eden Model, Discrete & Computational Geometry 69, p. 771-799 (2023)

Anistropy in the Eden Model (1985)

J. Phys. A: Math. Gen. **18** (1985) L1163–L1168. Printed in Great Britain

LETTER TO THE EDITOR

Surface structure and anisotropy of Eden clusters

P Freche†, D Stauffer†‡ and H E Stanley‡

† Institute of Theoretical Physics, Cologne University, 5000 Köln 41, West Germany
‡ Center for Polymer Studies, Boston University, Boston, MA 02215, USA

Received 10 September 1985

Abstract. The simple Eden model is simulated with clusters which are orders of magnitude larger than those of some previous work. The 'surface' (perimeter) is slightly anisotropic and feels the underlying structure of the square lattice even for 17 million cluster sites.

Anistropy in the Eden Model (1985)

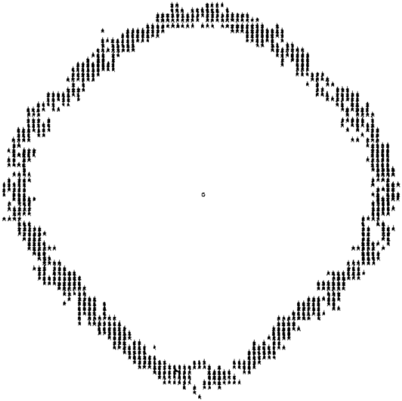


Figure 2. The centre of perimeter region, averaged over nine clusters with 4 million sites each. To make the very slight anisotropy more visible, most of the inner space is omitted; actually the radius is two orders of magnitude larger than the width of the ring.

Vold-Sutherland

Particles travel along random linear trajectories and stop the moment they first hit the existing cluster.

Vold-Sutherland

Particles travel along random linear trajectories and stop the moment they first hit the existing cluster.

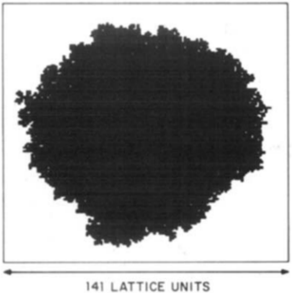


FIG. 2. A cluster of 10,000 particles grown on a two-dimensional lattice using the Eden model.

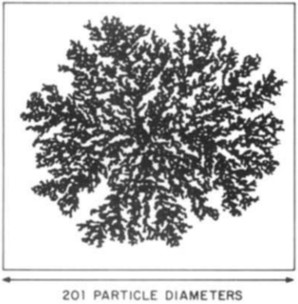


FIG. 1. A typical cluster of 10,000 particles grown in a two-dimensional simulation using the Vold–Sutherland (VS) model.

Image: Paul Meakin, The Vold-Sutherland and Eden Models of Cluster Formation
Journal of Colloid and Interface Science, Vol. 96, No. 2, December 1983

Diffusion-Limited Aggregation (DLA)

VOLUME 47, NUMBER 19 PHYSICAL REVIEW LETTERS 9 NOVEMBER 1981

Diffusion-Limited Aggregation, a Kinetic Critical Phenomenon

T. A. Witten, Jr.[a]
Groupe de Physique de la Matière Condensée, Collège de France, F-75231 Paris, France

and

L. M. Sander
Physics Department, University of Michigan, Ann Arbor, Michigan 48109
(Received 31 August 1981)

Idea: a particle travels by diffusion (a random walk) and stops the moment it first touches the existing set of particles.

Witten-Sanders, 1981

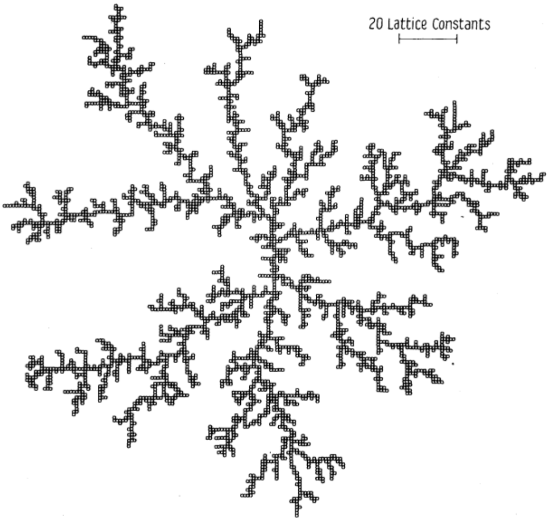


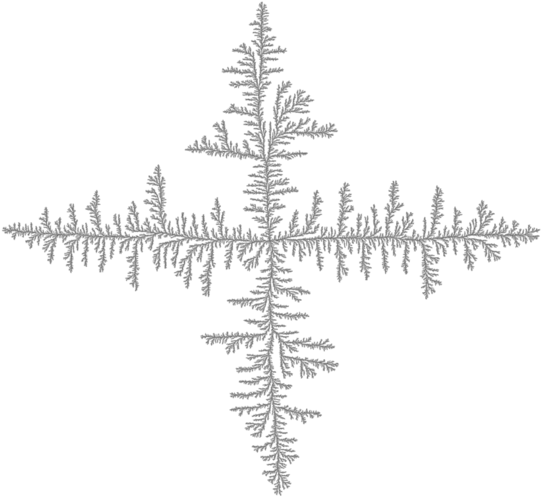
FIG. 1. Random aggregate of 3600 particles on a square lattice.

Grebenkov-Beliaev, 2017

DLA is often considered on lattice $\mathbb{Z}^2$ vs. $\mathbb{R}^2$. There seems to be a difference.

Grebenkov-Beliaev, 2017

DLA is often considered on lattice $\mathbb{Z}^2$ vs. $\mathbb{R}^2$. There seems to be a difference. 145.199.976 particles



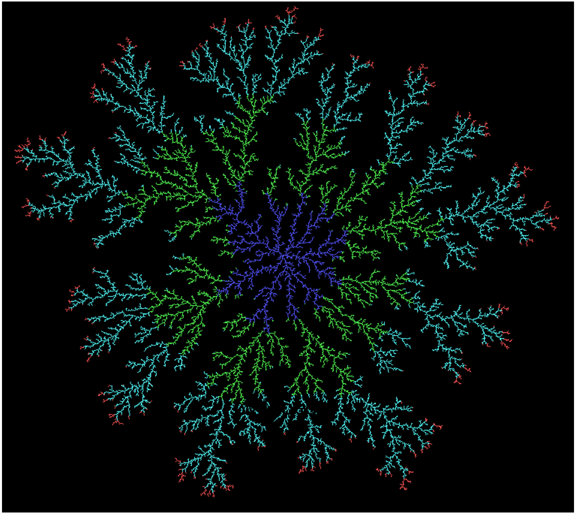


Image: wikipedia

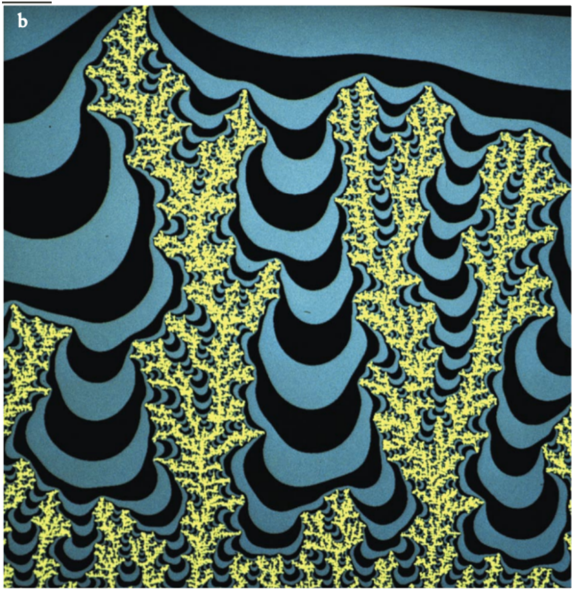


Image: Halsey, 2000

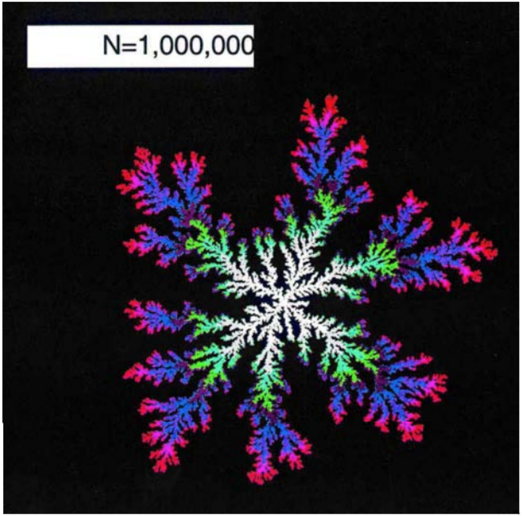


Figure 3. A large DLA cluster, N= 100 000. The radius of this object is roughly 15 times that of the cluster in figure 1. The colours represent the time of arrival: thus white is the first 1/10 of N, grey the second 1/10, etc. There are ten colours in all.

Image: Sanders, 2000

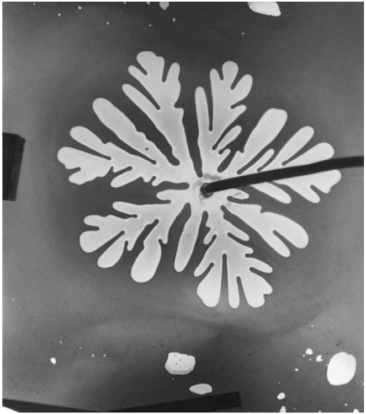


Figure 5. A radial viscous fingering pattern. Air is injected through the tube in the centre and displaces fluid (glycerin) which is confined between two plates held 1 mm apart. The pattern is about 20 cm across.

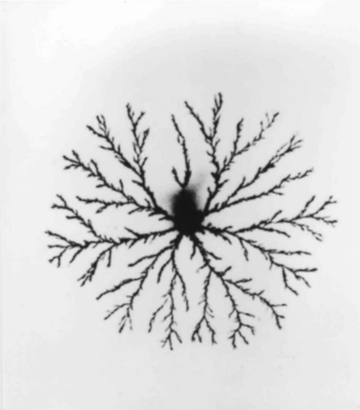


Figure 6. A zinc electrodeposit produced in a thin cell. The electrolyte is confined between two plexiglass plates held 0.1 mm apart. The cathode is inserted through a hole in the plate, and there is a ring anode (not shown). The pattern is about 3 cm across.

Image: Sanders, 2000

Figure 7. A colony of ***Paenibacillus dendritiform is*** bacteria, T morphotype, grown on hard agar and under severe starvation. The pattern is about 10 cm across. ***Courtesy of E. Ben-Jacob***.

Image: Sanders, 2000

wikipedia: Manganese oxide dendrites on a limestone bedding plane

Steffen Rohde with DLA

DLA: Existing Theory

Mostly due to

Harry Kesten (1931 – 2019)

with contributions by Lawler, Makarov, Smirnov, ...

DLA: Existing Theory

We consider the particles as disks of size $1/2$ in the plane identified with their center $x_1, \ldots, x_n, \ldots$ and normalize $x_1 = (0, 0)$.

DLA: Existing Theory

We consider the particles as disks of size $1/2$ in the plane identified with their center $x_1, \ldots, x_n, \ldots$ and normalize $x_1 = (0, 0)$. One excepts with high likelihood

$$\frac{\sqrt{n}}{1000} \leq \|x_n\| \leq n,$$

DLA: Existing Theory

We consider the particles as disks of size $1/2$ in the plane identified with their center $x_1, \dots, x_n, \dots$ and normalize $x_1 = (0,0)$. One excepts with high likelihood

$$\frac{\sqrt{n}}{1000} \leq \|x_n\| \leq n,$$

The first is true for Eden or Vold-Sutherland, the second one would be accurate if it grows along a line. DLA is probably in between.

DLA: Existing Theory

We consider the particles as disks of size $1/2$ in the plane identified with their center $x_1, \ldots, x_n, \ldots$ and normalize $x_1 = (0,0)$. One excepts with high likelihood

$$\frac{\sqrt{n}}{1000} \leq \|x_n\| \leq n,$$

The first is true for Eden or Vold-Sutherland, the second one would be accurate if it grows along a line. DLA is probably in between.

Theorem (Kesten, 1987)

We have, with high probability,

$$\|x_n\| \leq c \cdot n^{2/3}.$$

DLA: Kesten's argument

Theorem (Kesten, 1987)

$$\|x_n\| \leq c \cdot n^{2/3}.$$

We use

$$R(n) = \max\left\{\|x_i\| : 1 \leq i \leq n\right\}.$$

DLA: Kesten's argument

Theorem (Kesten, 1987)

$$\|x_n\| \leq c \cdot n^{2/3}.$$

We use

$$R(n) = \max\left\{\|x_i\| : 1 \leq i \leq n\right\}.$$

There are two steps.

DLA: Kesten's argument

Theorem (Kesten, 1987)

$$\|x_n\| \leq c \cdot n^{2/3}.$$

We use

$$R(n) = \max\{\|x_i\| : 1 \leq i \leq n\}.$$

There are two steps.

1. **Beurling's estimate.** The likelihood of hitting any fixed already existing particle is $\leq 1/\sqrt{R(n)}$.

DLA: Kesten's argument

Theorem (Kesten, 1987)

$$\|x_n\| \le c \cdot n^{2/3}.$$

We use

$$R(n) = \max\left\{\|x_i\| : 1 \le i \le n\right\}.$$

There are two steps.

1. **Beurling's estimate.** The likelihood of hitting any fixed already existing particle is $\le 1/\sqrt{R(n)}$.
2. Not entirely true but

$$R(n+1) \lesssim R(n) + \frac{1}{\sqrt{R(n)}}$$

DLA: Kesten's argument

Theorem (Kesten, 1987)

$$\|x_n\| \leq c \cdot n^{2/3}.$$

We use

$$R(n) = \max \{\|x_i\| : 1 \leq i \leq n\}.$$

There are two steps.

1. **Beurling's estimate.** The likelihood of hitting any fixed already existing particle is $\leq 1/\sqrt{R(n)}$.
2. Not entirely true but

$$R(n+1) \lesssim R(n) + \frac{1}{\sqrt{R(n)}}$$

would imply $R'(n) \leq R(n)^{-1/2}$

DLA: Kesten's argument

Theorem (Kesten, 1987)

$$\|x_n\| \leq c \cdot n^{2/3}.$$

We use

$$R(n) = \max\{\|x_i\| : 1 \leq i \leq n\}.$$

There are two steps.

1. **Beurling's estimate.** The likelihood of hitting any fixed already existing particle is $\leq 1/\sqrt{R(n)}$.
2. Not entirely true but

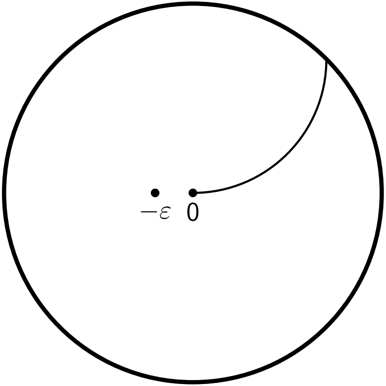
$$R(n+1) \lesssim R(n) + \frac{1}{\sqrt{R(n)}}$$

would imply $R'(n) \leq R(n)^{-1/2}$ and thus $R(n) \sim n^{2/3}$.

Intermezzo: Arne Beurling (1905 – 1986)

It is not unusual that the same mathematical idea will surface, independently, in several places, when the time is ripe. [...] Neither could I have known that Arne Beurling had found a different proof in 1929 while hunting alligators in Panama. (Lars Ahlfors)

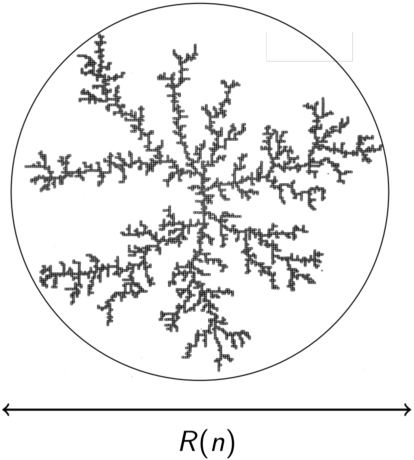
Beurling's estimate



Theorem (Beurling)

$\mathbb{P}$ of Brownian motion escaping disk before hitting the curve $\lesssim \sqrt{\varepsilon}$.

Beurling's estimate II



Maximize chances of hitting any specific point by pruning the tree.

Dielectric Breakdown Model (DBM)

VOLUME 52, NUMBER 12 PHYSICAL REVIEW LETTERS 19 MARCH 1984

Fractal Dimension of Dielectric Breakdown

L. Niemeyer, L. Pietronero,[a] and H. J. Wiesmann
Brown Boveri Research Center, CH-5405 Baden, Switzerland
(Received 23 November 1983)

Don't look at the following equations, look at my handwaving!

Dielectric Breakdown Model (DBM)

VOLUME 52, NUMBER 12 PHYSICAL REVIEW LETTERS 19 MARCH 1984

Fractal Dimension of Dielectric Breakdown

L. Niemeyer, L. Pietronero,[a] and H. J. Wiesmann
Brown Boveri Research Center, CH-5405 Baden, Switzerland
(Received 23 November 1983)

Don't look at the following equations, look at my handwaving! DLA: our existing set of particles Ω has a probability measure μ sitting on $\partial\Omega$.

Dielectric Breakdown Model (DBM)

VOLUME 52, NUMBER 12 PHYSICAL REVIEW LETTERS 19 MARCH 1984

Fractal Dimension of Dielectric Breakdown

L. Niemeyer, L. Pietronero,[a] and H. J. Wiesmann
Brown Boveri Research Center, CH-5405 Baden, Switzerland
(Received 23 November 1983)

Don't look at the following equations, look at my handwaving! DLA: our existing set of particles Ω has a probability measure μ sitting on $\partial\Omega$. If $u : \mathbb{R}^2 \setminus \Omega$ satisfies $\Delta u = 0$, then

$$u(\infty) = \int_{\partial\Omega} u(x) d\mu(x).$$

Dielectric Breakdown Model (DBM)

VOLUME 52, NUMBER 12 PHYSICAL REVIEW LETTERS 19 MARCH 1984

Fractal Dimension of Dielectric Breakdown

L. Niemeyer, L. Pietronero,[a] and H. J. Wiesmann
Brown Boveri Research Center, CH-5405 Baden, Switzerland
(Received 23 November 1983)

Don't look at the following equations, look at my handwaving! DLA: our existing set of particles Ω has a probability measure μ sitting on $\partial\Omega$. If $u : \mathbb{R}^2 \setminus \Omega$ satisfies $\Delta u = 0$, then

$$u(\infty) = \int_{\partial\Omega} u(x) d\mu(x).$$

DBM. Take that density $\partial\mu/\partial\sigma$ and raise it to a power $\eta > 0$

$$\frac{\partial\mu}{\partial\sigma} \to \left|\frac{\partial\mu}{\partial\sigma}\right|^{\eta}$$

Dielectric Breakdown Model (DBM)

VOLUME 52, NUMBER 12 PHYSICAL REVIEW LETTERS 19 MARCH 1984

Fractal Dimension of Dielectric Breakdown

L. Niemeyer, L. Pietronero,[a] and H. J. Wiesmann
Brown Boveri Research Center, CH-5405 Baden, Switzerland
(Received 23 November 1983)

Don't look at the following equations, look at my handwaving! DLA: our existing set of particles Ω has a probability measure μ sitting on $\partial\Omega$. If $u : \mathbb{R}^2 \setminus \Omega$ satisfies $\Delta u = 0$, then

$$u(\infty) = \int_{\partial\Omega} u(x) d\mu(x).$$

DBM. Take that density $\partial\mu/\partial\sigma$ and raise it to a power $\eta > 0$

$$\frac{\partial\mu}{\partial\sigma} \to \left|\frac{\partial\mu}{\partial\sigma}\right|^{\eta}$$

$\eta = 0$ is Eden model, $\eta = 1$ is DLA.

Dielectric Breakdown Model (DBM)

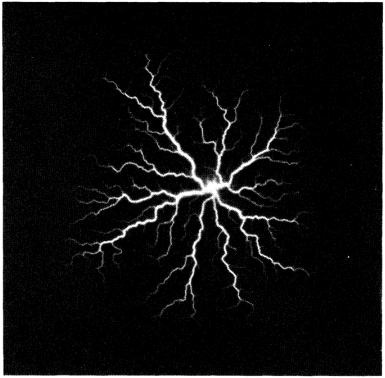
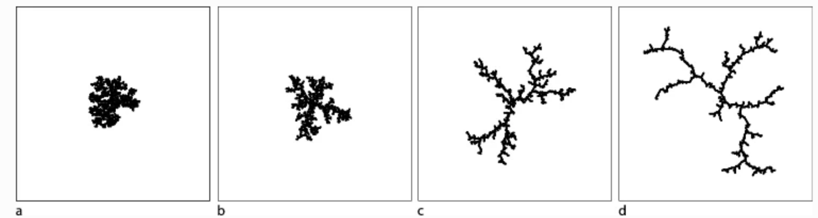


FIG. 1. Time-integrated photograph of a surface leader discharge (Lichtenberg figure) on a 2-mm glass plate in 0.3-MPa SF_6. Applied voltage pulse: 30 kV × 1 μs (Ref. 5). This experiment corresponds to an equipotential channel system growing in a plane with radial electrode.

Niemeyer, Pietronero, Wiesmann

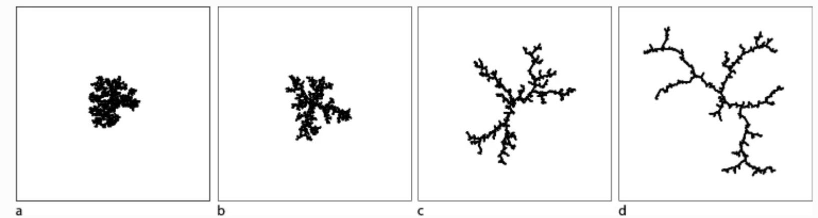
Dielectric Breakdown Model (DBM)



$\eta = 0.5$ $\eta = 1$ $\eta = 2$ $\eta = 3$

(Image: L. Sander, Fractal Growth Processes, Springer 2011)

Dielectric Breakdown Model (DBM)

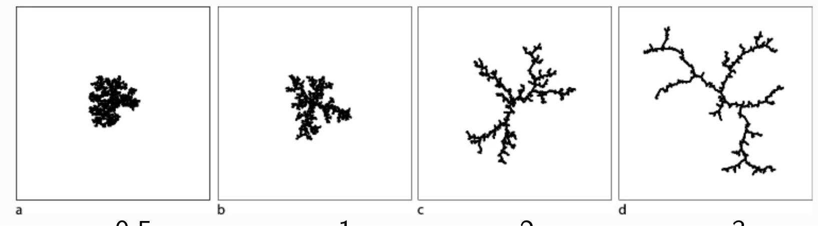


$\eta = 0.5$ $\eta = 1$ $\eta = 2$ $\eta = 3$

(Image: L. Sander, Fractal Growth Processes, Springer 2011)

Larger η leads to faster growth of the tips.

Dielectric Breakdown Model (DBM)



$\eta = 0.5$ $\eta = 1$ $\eta = 2$ $\eta = 3$

(Image: L. Sander, Fractal Growth Processes, Springer 2011)

Larger η leads to faster growth of the tips.

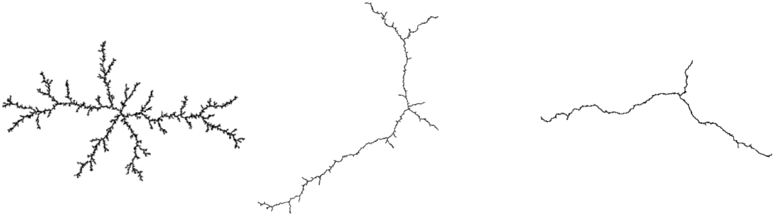
Theorem (Losev, Smirnov, 2023)

For $0 < \eta < 2$ in $\eta-$DBM one has, up to log factors,

$$\|x_n\| \lesssim_{\log} n^{\frac{2}{4-\eta}}.$$

Dielectric Breakdown Model (DBM)

$\eta = 4$ *might* be the critical exponent for ballistic growth.

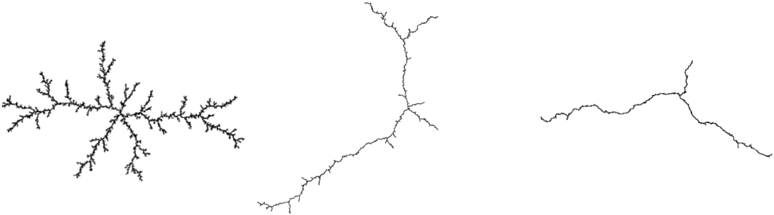


$\eta = 2$ $\eta = 3.5$ $\eta = 4.5$

(Images: Hastings, Physical Review Letters, 2001)

Dielectric Breakdown Model (DBM)

$\eta = 4$ *might* be the critical exponent for ballistic growth.



$\eta = 2$ $\eta = 3.5$ $\eta = 4.5$

(Images: Hastings, Physical Review Letters, 2001)

I am skipping Hastings-Levitov (Riemann mapping, complex analysis).

Gradient Flow Aggregation (GFA)

Goal: a 'low-tech' version of these models.

Gradient Flow Aggregation (GFA)

Goal: a 'low-tech' version of these models.

$$u(\infty) = \int_{\partial\Omega} u(x) d\mu(x).$$

Everybody loves harmonic measure!

Gradient Flow Aggregation (GFA)

Goal: a 'low-tech' version of these models.

$$u(\infty) = \int_{\partial\Omega} u(x) d\mu(x).$$

*E*verybody loves harmonic measure!

but....

Gradient Flow Aggregation (GFA)

Goal: a 'low-tech' version of these models.

$$u(\infty) = \int_{\partial\Omega} u(x) d\mu(x).$$

*E*verybody loves harmonic measure!

but....

1. not so easy to numerically simulate (at least DBM)

Gradient Flow Aggregation (GFA)

Goal: a 'low-tech' version of these models.

$$u(\infty) = \int_{\partial\Omega} u(x) d\mu(x).$$

*E*verybody loves harmonic measure!

but....

1. not so easy to numerically simulate (at least DBM)
2. not always so easy to explain

Gradient Flow Aggregation (GFA)

Goal: a 'low-tech' version of these models.

$$u(\infty) = \int_{\partial\Omega} u(x) d\mu(x).$$

*E*verybody loves harmonic measure!

but....

1. not so easy to numerically simulate (at least DBM)
2. not always so easy to explain
3. and it's using a lot of dice!! (pure philosophy)

Gradient Flow Aggregation (GFA)

Given n particles $x_1, \ldots, x_n$, a new particle spawns at '∞' (far away)

Gradient Flow Aggregation (GFA)

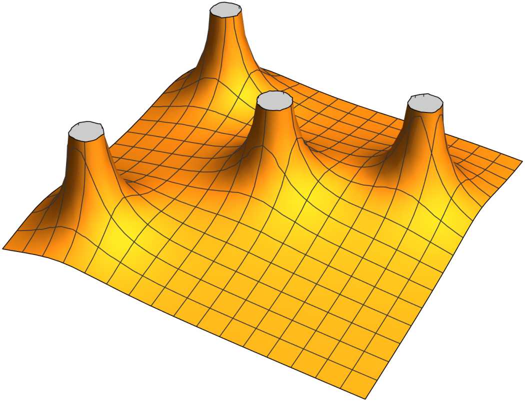
Given n particles $x_1, \ldots, x_n$, a new particle spawns at '∞' (far away) and then does a gradient *ascent* for the function

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^{\alpha}}.$$

Gradient Flow Aggregation (GFA)

Given n particles $x_1, \ldots, x_n$, a new particle spawns at '∞' (far away) and then does a gradient *ascent* for the function

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^{\alpha}}.$$



Gradient Flow Aggregation (GFA)

Gradient *ascent* for the function

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^{\alpha}}.$$

Gradient Flow Aggregation (GFA)

Gradient *ascent* for the function

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^{\alpha}}.$$

$\alpha = 0$ $\alpha = 1$ $\alpha = 2$

GFA for $n = 1000$ particles with various parameters of α.

Gradient Flow Aggregation (GFA)

Given n particles $x_1, \ldots, x_n$, a new particle spawns at ∞ and then does a gradient *ascent* for the function

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^{\alpha}}.$$

Conventions. When $\alpha = 0$, we define

$$f(x) = \sum_{i=1}^{n} \log \left(\frac{1}{\|x - x_i\|} \right).$$

Gradient Flow Aggregation (GFA)

Given n particles $x_1, \ldots, x_n$, a new particle spawns at ∞ and then does a gradient *ascent* for the function

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^{\alpha}}.$$

Conventions. When $\alpha = 0$, we define

$$f(x) = \sum_{i=1}^{n} \log\left(\frac{1}{\|x - x_i\|}\right).$$

When $\alpha = \infty$, we define

$$f(x) = \max_{1 \le i \le n} \frac{1}{\|x - x_i\|}.$$

Gradient Flow Aggregation (GFA)

Given n particles $x_1, \ldots, x_n$, **a new particle spawns at** ∞ and then does a gradient *ascent* for the function

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^{\alpha}}.$$

Gradient Flow Aggregation (GFA)

Given n particles $x_1, \ldots, x_n$, **a new particle spawns at** ∞ and then does a gradient *ascent* for the function

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^{\alpha}}.$$

You pick the new particle uniformly at random from a big disk with radius $R \gg 1$ and let $R \to \infty$.

Gradient Flow Aggregation (GFA)

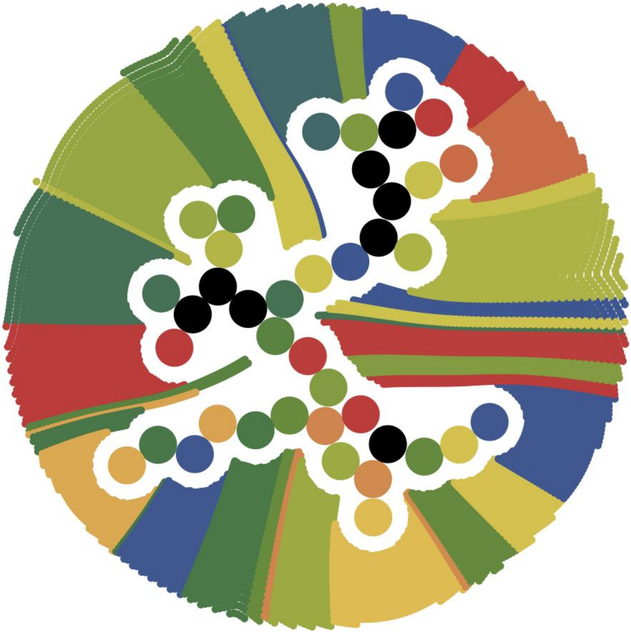
Given n particles $x_1, \ldots, x_n$, **a new particle spawns at** ∞ and then does a gradient *ascent* for the function

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^\alpha}.$$

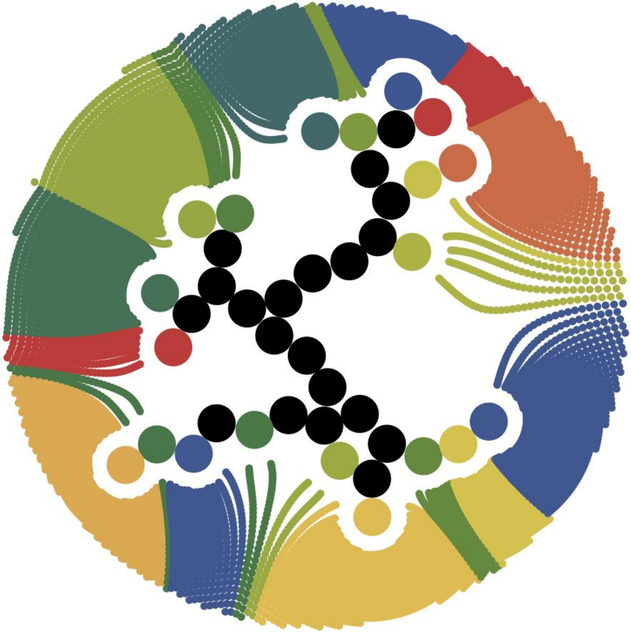
You pick the new particle uniformly at random from a big disk with radius $R \gg 1$ and let $R \to \infty$.

Proposition

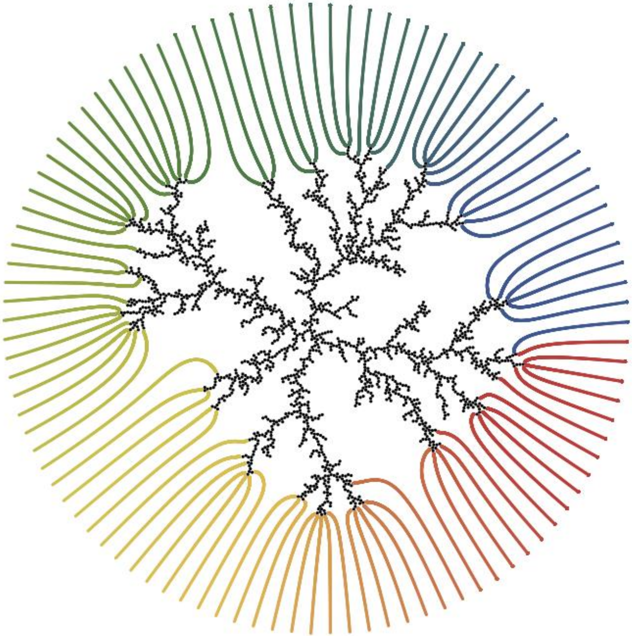
The probabilities converge as $R \to \infty$.



$\alpha = 0$



$\alpha = 4$

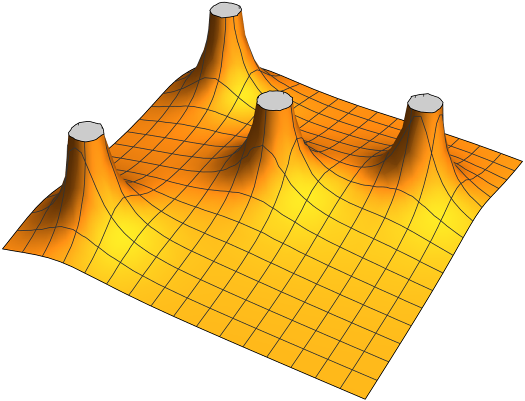


$\alpha = 2$

Disclaimer

A potential issue

We define things via gradient ascent. What if the gradient flow gets stuck in a critical point?



Disclaimer

A potential issue

We define things via gradient ascent. What if the gradient flow gets stuck in a critical point?

Local analysis shows that there is at most one trajectory per critical point, so not a big issue. Except...

Disclaimer

A potential issue

We define things via gradient ascent. What if the gradient flow gets stuck in a critical point?

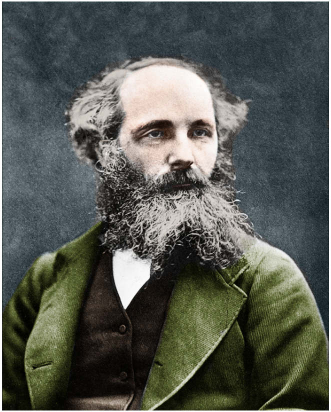
Local analysis shows that there is at most one trajectory per critical point, so not a big issue. Except...

Question

How many critical points are there? Formally, given $0 < \alpha < \infty$ and $x_1, \ldots, x_n \in \mathbb{R}^2$, how many critical points can

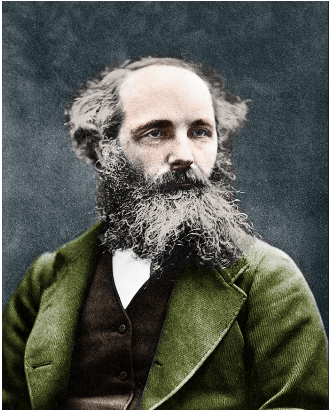
$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^{\alpha}} \qquad \text{have?}$$

A treatise on electricity and magnetism



James Clerk Maxwell

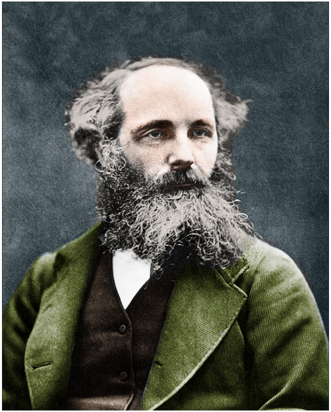
A treatise on electricity and magnetism



James Clerk Maxwell

113.] To determine the number of the points and lines of equilibrium, let us consider the surface or surfaces for which the potential is equal to C, a given quantity. Let us call the regions in which the potential is less than C the negative regions, and those in which it is greater than C the positive regions. Let V_0 be the lowest, and V_1 the highest potential existing in the

A treatise on electricity and magnetism



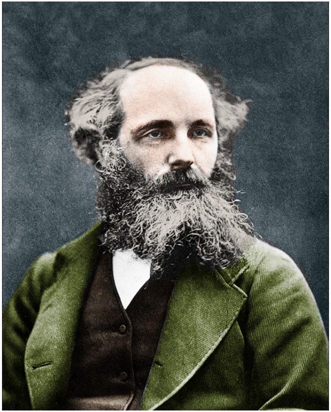
James Clerk Maxwell

113.] To determine the number of the points and lines of equilibrium, let us consider the surface or surfaces for which the potential is equal to C, a given quantity. Let us call the regions in which the potential is less than C the negative regions, and those in which it is greater than C the positive regions. Let V_0 be the lowest, and V_1 the highest potential existing in the

[...]

If the form of the electrified bodies or conductors is arbitrary, we can only assert that the number of these additional points or lines is even, but if they are electrified points or spherical conductors, the number arising in this way cannot exceed $(n-1)(n-2)$, where n is the number of bodies.

A treatise on electricity and magnetism



James Clerk Maxwell

113.] To determine the number of the points and lines of equilibrium, let us consider the surface or surfaces for which the potential is equal to C, a given quantity. Let us call the regions in which the potential is less than C the negative regions, and those in which it is greater than C the positive regions. Let V_0 be the lowest, and V_1 the highest potential existing in the

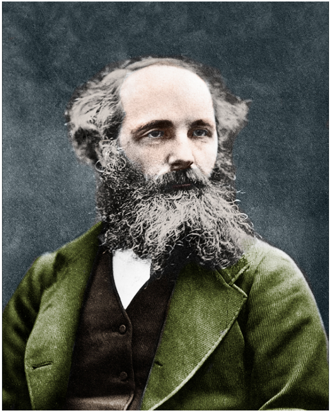
[...]

If the form of the electrified bodies or conductors is arbitrary, we can only assert that the number of these additional points or lines is even, but if they are electrified points or spherical conductors, the number arising in this way cannot exceed $(n-1)(n-2)$, where n is the number of bodies.

(First edition)

ductors, the number arising in this way cannot exceed $(n-1)(n-2)$, where n is the number of bodies*.

A treatise on electricity and magnetism



James Clerk Maxwell

113.] To determine the number of the points and lines of equilibrium, let us consider the surface or surfaces for which the potential is equal to C, a given quantity. Let us call the regions in which the potential is less than C the negative regions, and those in which it is greater than C the positive regions. Let V_0 be the lowest, and V_1 the highest potential existing in the

[...]

If the form of the electrified bodies or conductors is arbitrary, we can only assert that the number of these additional points or lines is even, but if they are electrified points or spherical conductors, the number arising in this way cannot exceed $(n-1)(n-2)$, where n is the number of bodies.

(First edition)

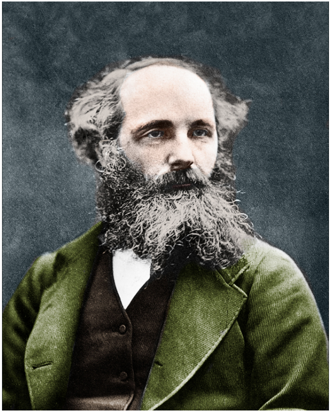
172 POINTS AND LINES OF EQUILIBRIUM. [115.

ductors, the number arising in this way cannot exceed $(n-1)(n-2)$, where n is the number of bodies*.

* {I have not been able to find any place where this result is proved.}

(Second edition, commented by J. J. Thomson)

A treatise on electricity and magnetism


James Clerk Maxwell

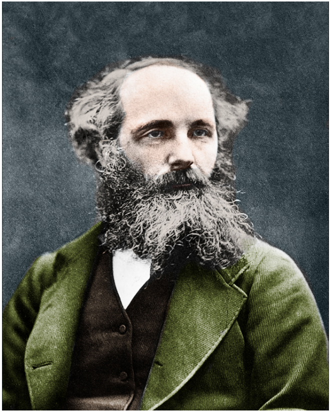
Maxwell's Conjecture

$f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^2}$$

has at most $(n-1)^2$ critical points.

A treatise on electricity and magnetism



James Clerk Maxwell

Maxwell's Conjecture

$f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^2}$$

has at most $(n-1)^2$ critical points.

Tsai (2015)

True for $n = 3$.

(Lee-Tsai, 2022)

$f : \mathbb{R}^2 \to \mathbb{R}$ given by, where $a_i > 0$,

$$f(x) = \sum_{i=1}^{4} \frac{a_i}{\|x - x_i\|^2}$$

can have 9 critical points.

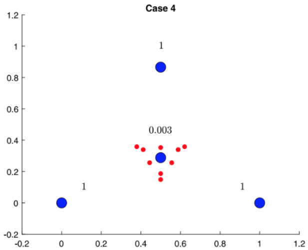


Image: Lee & Tsai, Nine equilibrium points of four point charges on the plane, Applied Math Letters (2022)

Maxwell's Conjecture

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^2}$$

has at most $(n-1)^2$ critical points.

Maxwell's Conjecture

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^2}$$

has at most $(n-1)^2$ critical points.

Competing Conjecture (Gabrielov, Novikov, Shapiro, 2004)

It has at most $5n - 11$ critical points.

Maxwell's Conjecture

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^2}$$

has at most $(n-1)^2$ critical points.

Competing Conjecture (Gabrielov, Novikov, Shapiro, 2004)

It has at most $5n - 11$ critical points.

Conjecture 2 (folklore, very irritating). *For any set of charges of the same sign in $\mathbb{R}^n$, the set of its points of equilibrium is finite.*

(Boris Shapiro, 2015)

Maxwell's Conjecture

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^2}$$

has at most $(n-1)^2$ critical points.

Competing Conjecture (Gabrielov, Novikov, Shapiro, 2004)

It has at most $5n - 11$ critical points.

Conjecture 2 (folklore, very irritating). *For any set of charges of the same sign in $\mathbb{R}^n$, the set of its points of equilibrium is finite.*

(Boris Shapiro, 2015)

I will always assume that the set of critical points is finite.

Maxwell's Conjecture

$$f(x) = \sum_{i=1}^{n} \frac{1}{\|x - x_i\|^2}$$

has at most $(n-1)^2$ critical points.

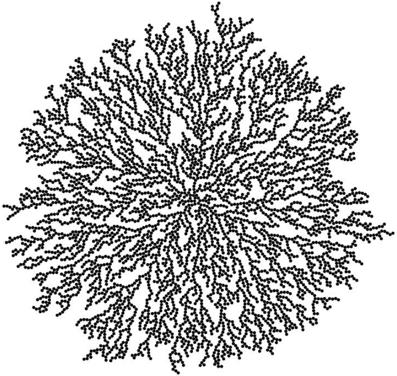
Competing Conjecture (Gabrielov, Novikov, Shapiro, 2004)

It has at most $5n - 11$ critical points.

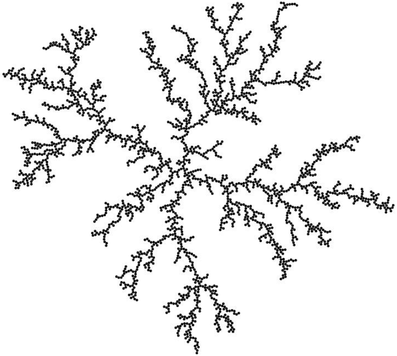
Conjecture 2 (folklore, very irritating). *For any set of charges of the same sign in $\mathbb{R}^n$, the set of its points of equilibrium is finite.*

(Boris Shapiro, 2015)

I will always assume that the set of critical points is finite.
(Countable would actually already be enough, any ideas?)



$\alpha = 0$



$\alpha = 2$

GFA – Main Results

Theorem (Beurling-type estimate)

Let $0 \leq \alpha \leq 1$ and suppose $\{x_1, \dots, x_n\} \subset \mathbb{R}^2$. Then, for some $c_\alpha > 0$ depending only on α,

$$\max_{1 \leq i \leq n} \mathbb{P}\,(\text{new particles hits } x_i) \leq c_\alpha \cdot n^{\frac{\alpha-1}{2\alpha+2}}.$$

GFA – Main Results

Theorem (Beurling-type estimate)

Let $0 \leq \alpha \leq 1$ and suppose $\{x_1, \ldots, x_n\} \subset \mathbb{R}^2$. Then, for some $c_\alpha > 0$ depending only on α,

$$\max_{1 \leq i \leq n} \mathbb{P}\,(\text{new particles hits } x_i) \leq c_\alpha \cdot n^{\frac{\alpha-1}{2\alpha+2}}.$$

Especially nice: we get the optimal rate $n^{-1/2}$ for $\alpha = 0$.

GFA – Main Results

Theorem (Beurling-type estimate)

Let $0 \le \alpha \le 1$ and suppose $\{x_1, \ldots, x_n\} \subset \mathbb{R}^2$. Then, for some $c_\alpha > 0$ depending only on α,

$$\max_{1 \le i \le n} \mathbb{P}(\text{new particles hits } x_i) \le c_\alpha \cdot n^{\frac{\alpha-1}{2\alpha+2}}.$$

Especially nice: we get the optimal rate $n^{-1/2}$ for $\alpha = 0$.

Theorem (Beurling-type estimate)

Let $0 \le \alpha \le 1$. We have (whp) that

$$\|x_n\| \le c_\alpha n^{\frac{3\alpha+1}{2\alpha+2}}.$$

GFA – Main Results

Theorem (Beurling-type estimate)

Let $0 \le \alpha \le 1$ and suppose $\{x_1, \dots, x_n\} \subset \mathbb{R}^2$. Then, for some $c_\alpha > 0$ depending only on α,

$$\max_{1 \le i \le n} \mathbb{P}\left(\text{new particles hits } x_i\right) \le c_\alpha \cdot n^{\frac{\alpha-1}{2\alpha+2}}.$$

Especially nice: we get the optimal rate $n^{-1/2}$ for $\alpha = 0$.

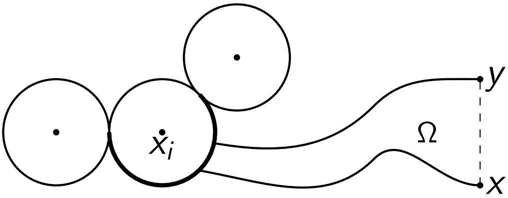
Theorem (Beurling-type estimate)

Let $0 \le \alpha \le 1$. We have (whp) that

$$\|x_n\| \le c_\alpha n^{\frac{3\alpha+1}{2\alpha+2}}.$$

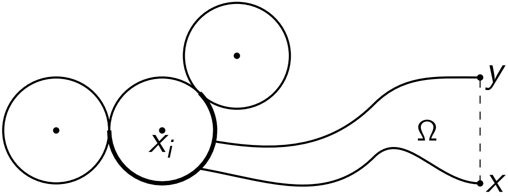
Optimal rate $n^{1/2}$ for $\alpha = 0$.

Proof of the Beurling-type estimate



$f(x) = \sum_{i=1}^{n} \|x - x_i\|^{-\alpha}$

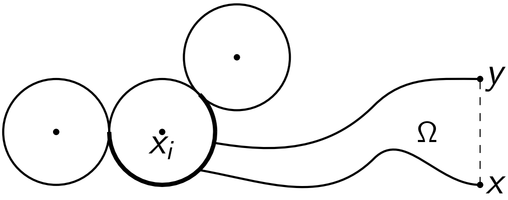
Proof of the Beurling-type estimate



$f(x) = \sum_{i=1}^{n} \|x - x_i\|^{-\alpha}$

Subharmonic + Green $\quad 0 \leq \int_{\Omega} \Delta f(x) dx = \int_{\partial\Omega} \frac{\partial f}{\partial \nu} d\sigma.$

Proof of the Beurling-type estimate

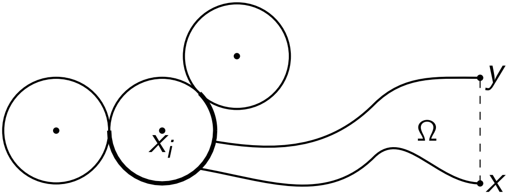


$f(x) = \sum_{i=1}^{n} \|x - x_i\|^{-\alpha}$

Subharmonic + Green $\quad 0 \leq \int_\Omega \Delta f(x) dx = \int_{\partial\Omega} \frac{\partial f}{\partial \nu} d\sigma.$

Gradient flows implies $\partial f / \partial \nu = 0$ on the long gradient flow lines.

Proof of the Beurling-type estimate

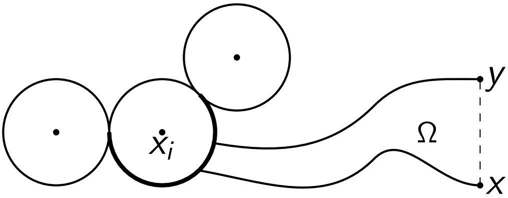


$f(x) = \sum_{i=1}^{n} \|x - x_i\|^{-\alpha}$

Subharmonic + Green $\qquad 0 \leq \int_\Omega \Delta f(x) dx = \int_{\partial\Omega} \frac{\partial f}{\partial \nu} d\sigma.$

Gradient flows implies $\partial f / \partial \nu = 0$ on the long gradient flow lines.
On one endpoint there's an asymptotic expansion

Proof of the Beurling-type estimate

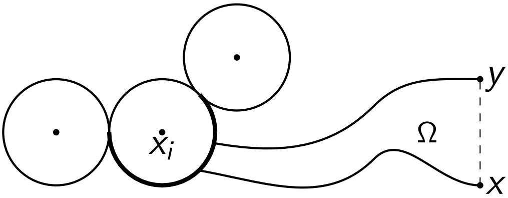


$f(x) = \sum_{i=1}^{n} \|x - x_i\|^{-\alpha}$

Subharmonic + Green $$0 \le \int_\Omega \Delta f(x) dx = \int_{\partial\Omega} \frac{\partial f}{\partial \nu} d\sigma.$$

Gradient flows implies $\partial f/\partial\nu = 0$ on the long gradient flow lines. On one endpoint there's an asymptotic expansion (we get the probability),

Proof of the Beurling-type estimate



$f(x) = \sum_{i=1}^{n} \|x - x_i\|^{-\alpha}$

Subharmonic + Green $\quad 0 \leq \int_{\Omega} \Delta f(x) dx = \int_{\partial\Omega} \frac{\partial f}{\partial \nu} d\sigma.$

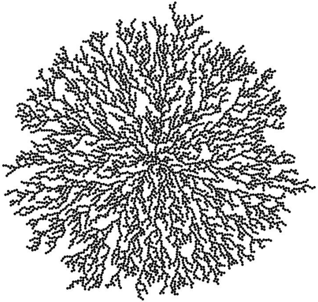
Gradient flows implies $\partial f / \partial \nu = 0$ on the long gradient flow lines. On one endpoint there's an asymptotic expansion (we get the probability), on the other endpoint concrete pointwise bound.

Beurling implies (via Kesten) a growth bound

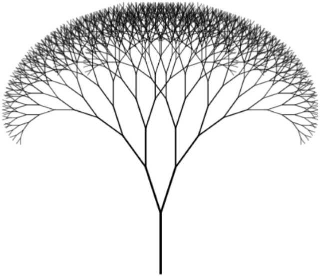
Kesten's method is really done on $\mathbb{Z}^2$ while we work on $\mathbb{R}^2$.

Beurling implies (via Kesten) a growth bound

Kesten's method is really done on $\mathbb{Z}^2$ while we work on $\mathbb{R}^2$.



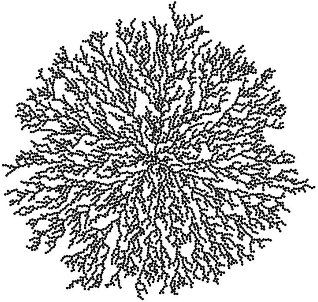
$\Longrightarrow$



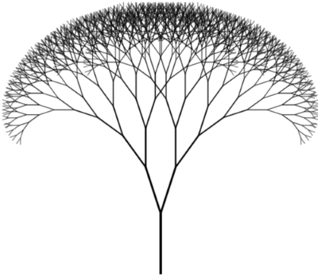
infinite $6-$regular tree

Beurling implies (via Kesten) a growth bound

Kesten's method is really done on $\mathbb{Z}^2$ while we work on $\mathbb{R}^2$.



$\Longrightarrow$



infinite $6-$regular tree

Then: arguments by Benjamini - Yadin (2017).

Ballistic vs. sub-ballistic growth

Question

Is there a phase transition $0 < \alpha_0 < \infty$ such that

$$\|x_n\| \leq c_\alpha n^{1-\varepsilon_\alpha} \qquad \text{for } \alpha < \alpha_0$$

and

$$\|x_n\| \geq c_\alpha n \qquad \text{for } \alpha > \alpha_0?$$

Ballistic vs. sub-ballistic growth

Question

Is there a phase transition $0 < \alpha_0 < \infty$ such that

$$\|x_n\| \leq c_\alpha n^{1-\varepsilon_\alpha} \qquad \text{for } \alpha < \alpha_0$$

and

$$\|x_n\| \geq c_\alpha n \qquad \text{for } \alpha > \alpha_0?$$

I do not know.

Ballistic vs. sub-ballistic growth

Question

Is there a phase transition $0 < \alpha_0 < \infty$ such that

$$\|x_n\| \leq c_\alpha n^{1-\varepsilon_\alpha} \qquad \text{for } \alpha < \alpha_0$$

and

$$\|x_n\| \geq c_\alpha n \qquad \text{for } \alpha > \alpha_0?$$

I do not know. If you have a gut feeling, let me know. Certainly $\alpha_0 \geq 1$. Maybe $\alpha_0 = \infty$?

The case $\alpha = \infty$

The case $\alpha = \infty$

That case is actually **very** nice.

The case $\alpha = \infty$

That case is actually **very** nice.

1. Generate a random point at ∞ (meaning random angle).

The case $\alpha = \infty$

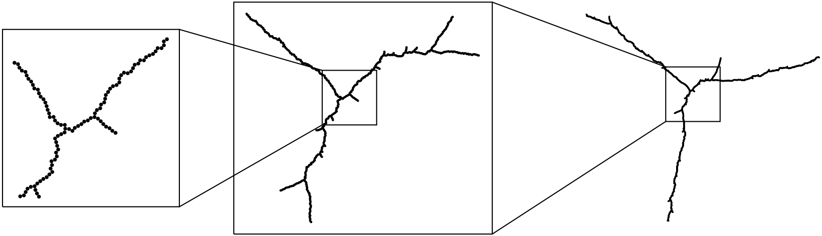
That case is actually **very** nice.

1. Generate a random point at ∞ (meaning random angle).
2. Project onto the closest point.

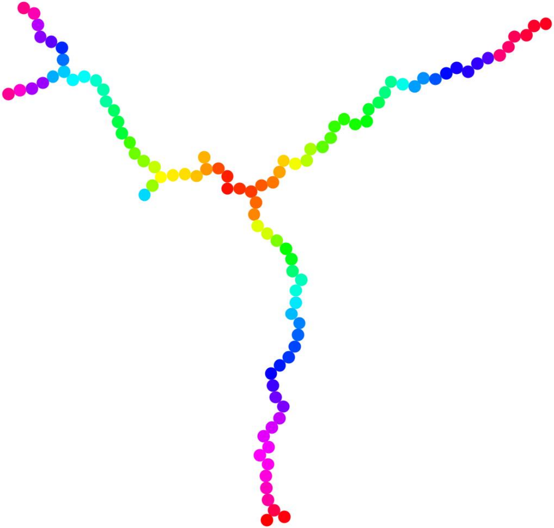
The case $\alpha = \infty$

That case is actually **very** nice.

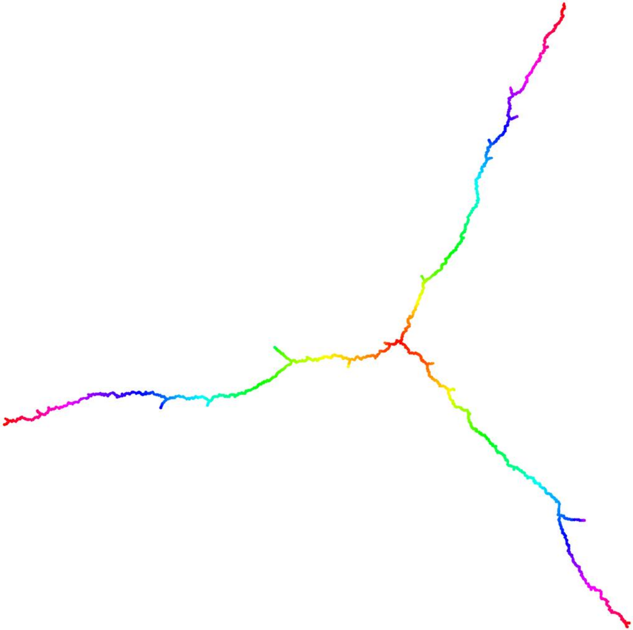
1. Generate a random point at ∞ (meaning random angle).
2. Project onto the closest point.



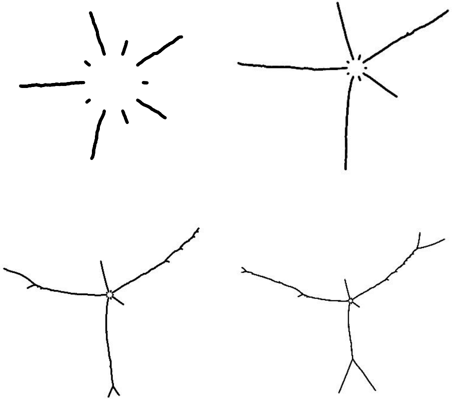
The case $\alpha = \infty$



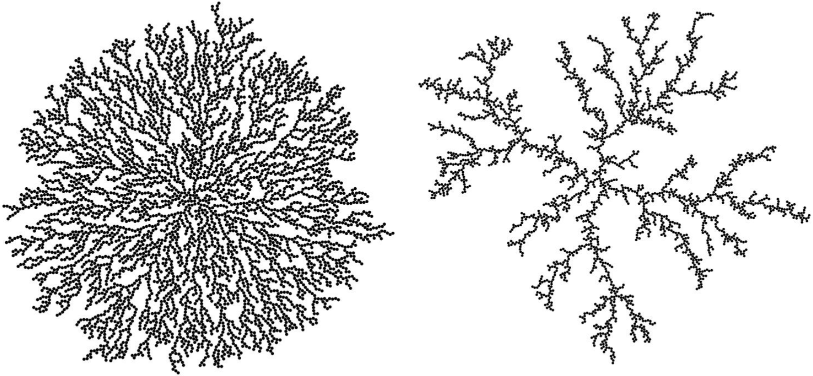
The case $\alpha = \infty$



$\alpha = \infty$, Initial Conditions do not seem to matter.



10 vertices of a regular polygon at distance 100 from the origin.
After 1.000, 5.000, 15.000 and 25.000 steps, respectively.



THANK YOU!