Nonlinear Fourier Series

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Oberwolfach, Nov 2021

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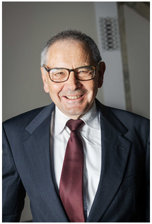
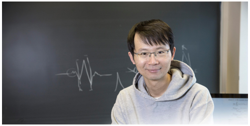
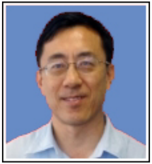


Raphy Coifman



Hau-tieng Wu



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We will work with periodic functions $f : [0, 2\pi] \to \mathbb{R}$.

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as a curve in the complex plane. It may also be helpful to think of

$$F : \mathbb{C} \to \mathbb{C}$$

as a polynomial.

Fourier series. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and

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This is the same as classical Fourier series because

$$e^{it} = \cos t + i \sin t$$

Recursively.

$$f(z) = f(0) + (f(z) - f(0))$$

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$$\begin{aligned} f(z) &= f(0) + zg(z) \\ &= f(0) + z(g(0) + (g(z) - g(0))) \\ &= f(0) + zg(0) + z^2 h(z) \\ &= f(0) + zg(0) + z^2 h(0) + \dots \end{aligned}$$

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We can factor out and get $f(z) - f(0) = z \cdot g(z)$.

Idea. Factor *all* the roots inside the unit disk. There is a canonical way of doing this.

Blaschke products

$$z^m \prod_{i=1}^{k} \frac{z - a_i}{1 - \overline{a_i} z}$$

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Theorem

Any holomorphic $F : \mathbb{C} \to \mathbb{C}$ can be written as

$$F = B \cdot G,$$

where B is a Blaschke product and G has no roots inside $\mathbb{D}$.

Blaschke products – moving roots around

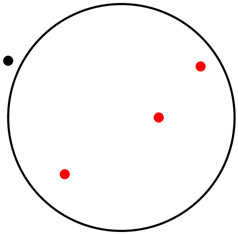

Figure: Roots of $F = BG$

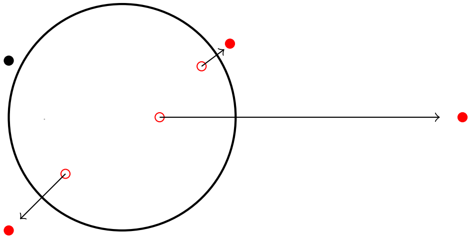

Figure: Roots of G

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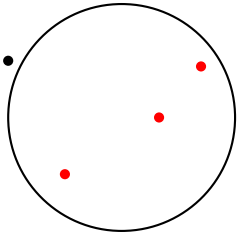

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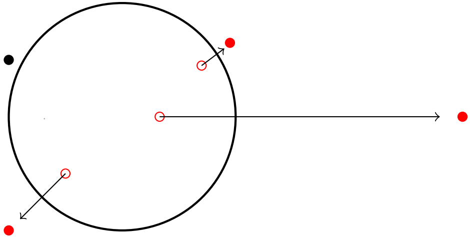

Figure: Roots of G

$$\underbrace{\left(z - \frac{1}{3}\right)(z-3)}_{F} = \underbrace{\frac{z - \frac{1}{3}}{1 - \frac{z}{3}}}_{B} \cdot \underbrace{\left(1 - \frac{z}{3}\right)(z-3)}_{G}$$

If the roots are all close to the boundary, not much is happening.

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Note that

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Note that

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In particular,

$$B(e^{it}) = e^{i\phi(t)},$$

and ϕ is monotonically increasing.

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On the boundary of the unit disk $|B| = 1$.

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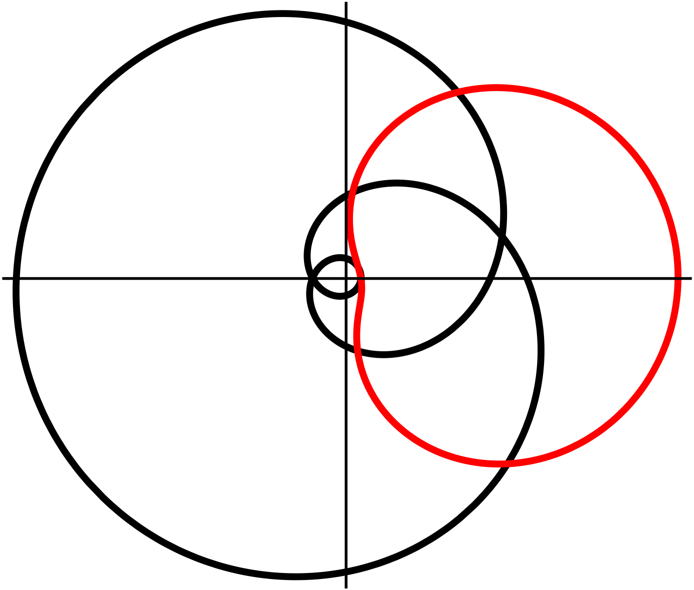


Figure: $F(\partial\mathbb{D})$ (black) and $G(\partial\mathbb{D})$ (red)

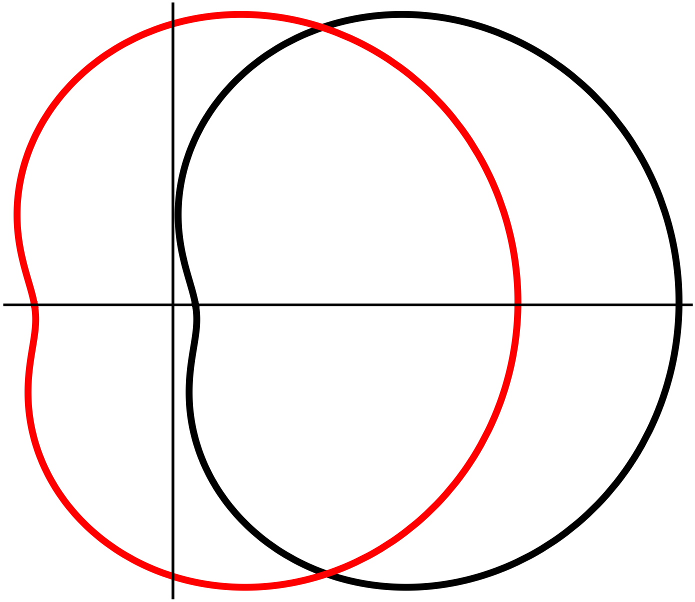


Figure: $G(\partial\mathbb{D})$ (black) and $(G - G(0))(\partial\mathbb{D})$ (red)

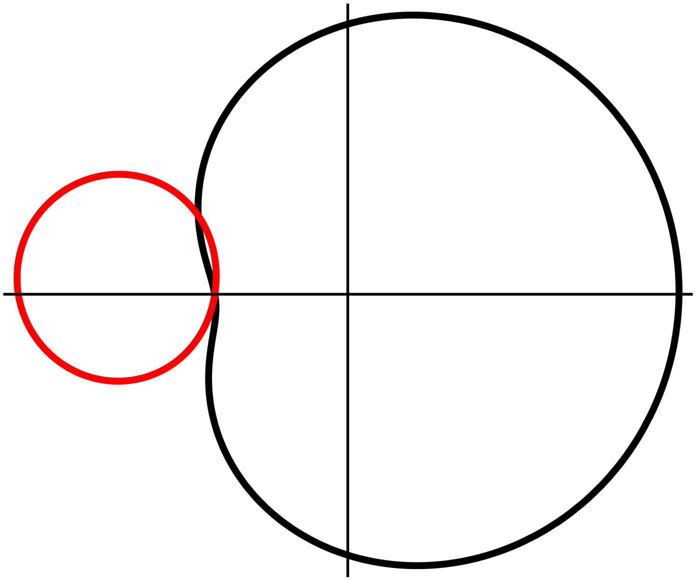
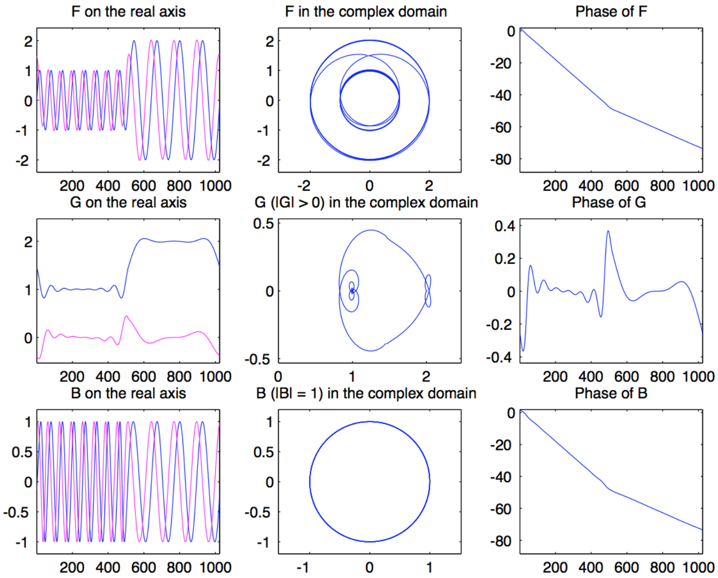
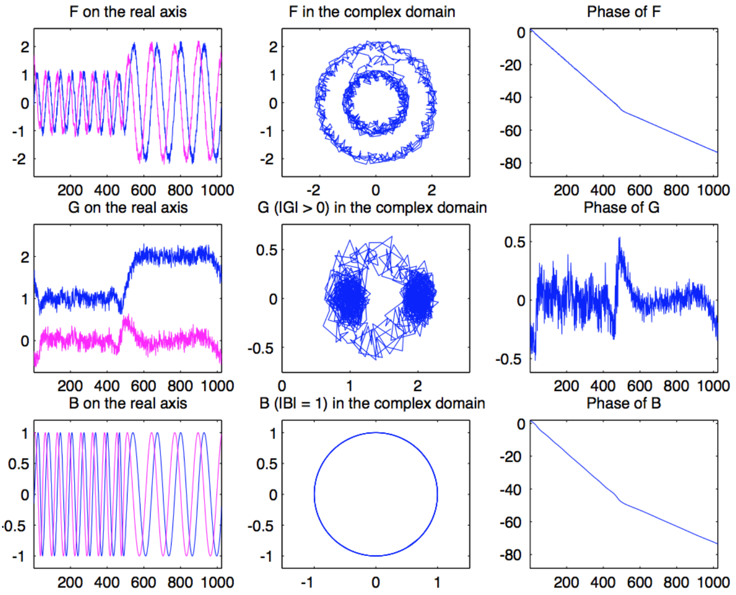


Figure: $(G - G(0))(\partial\mathbb{D})$ and its outer function



(Michel Nahon, Thesis)



(Robustness to noise, Michel Nahon, Thesis)

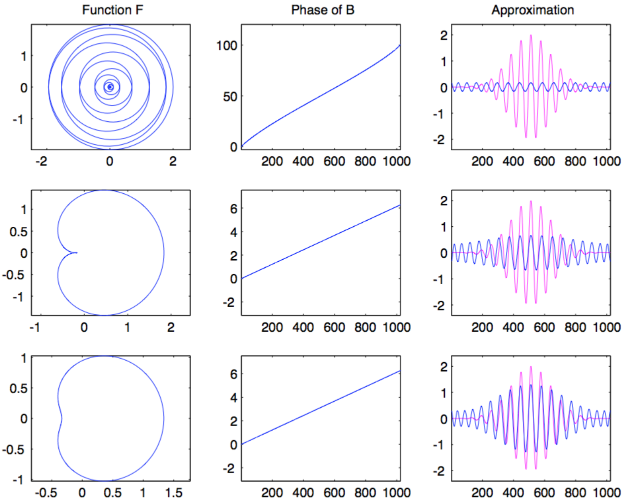


Figure 1.20: *Orthogonal decomposition of the modulated Gaussian signal* $F : \theta \mapsto e^{-(\theta-\theta_0)^2} \cdot e^{in\theta}$ *in three steps. The right column shows the evolution of the approximation.*

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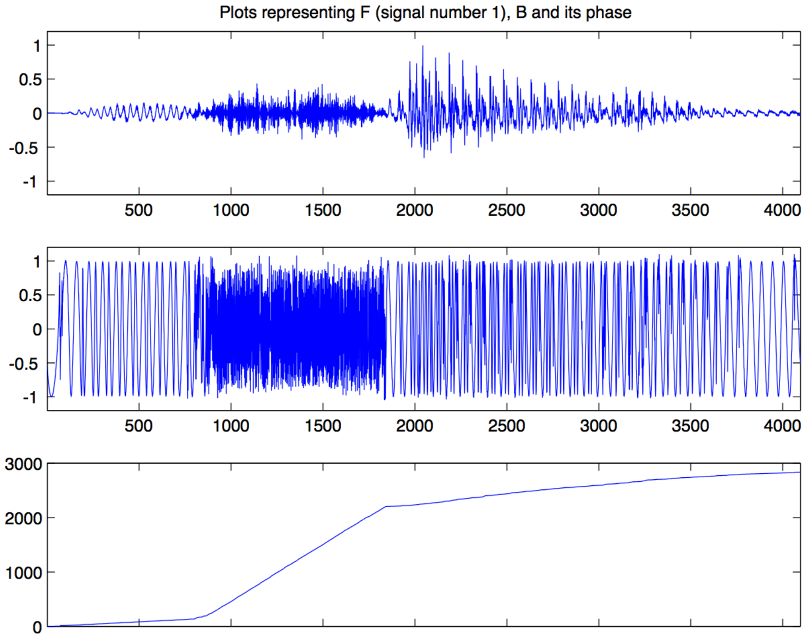

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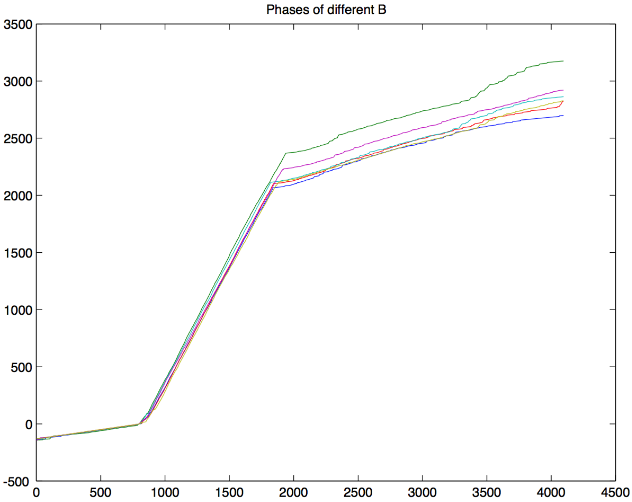
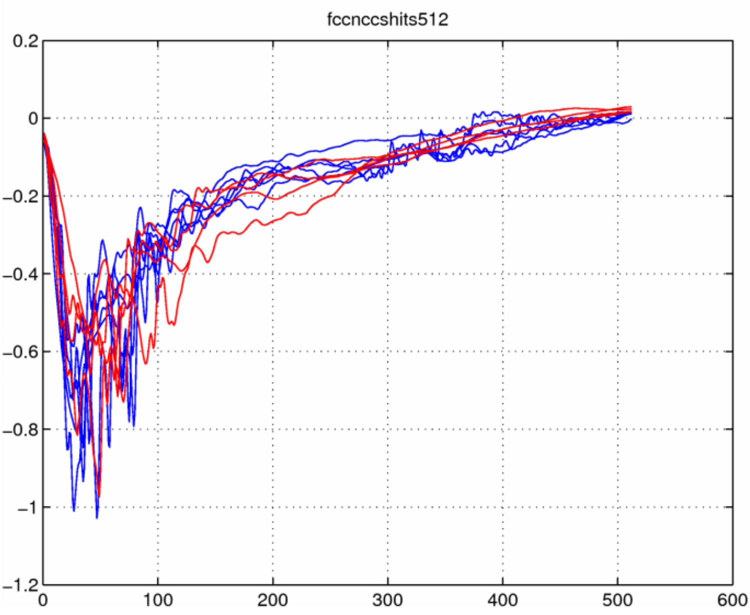
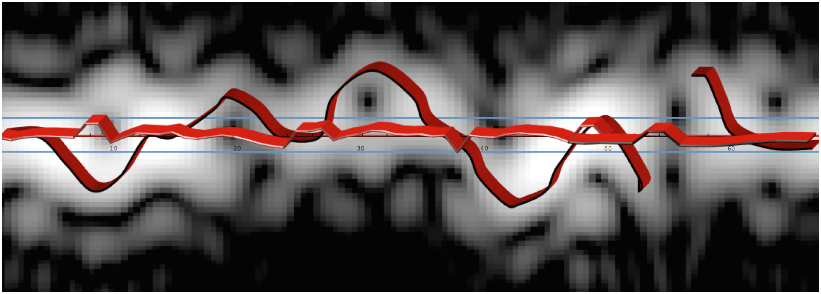


Acoustic Underwater Scattering (Letelier & Saito, 2009)



Doppler Effect (Healy, 2009)



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A DERIVATION OF THE MAIN RESULTS OF THE THEORY OF H^p SPACES

by MARY and GUIDO WEISS
(Facultad de' Ciencias Exactas y Naturales, Universidad de Buenos Aires)

§ 1. *Introduction.*

The space H^p, $p > 0$, is the vector space of all analytic functions $F(z)$ defined in the open unit disc, $|z| = r < 1$, such that

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta < \infty. \qquad [1]$$

The basic result in the theory of these spaces is the following theorem:

(Thanks to José Luis Romero.)

Theorem (Tao Qian, 2012)

The sequence converges in $\mathcal{H}^2$ for initial data in $\mathcal{H}^2$. The convergence is at least as fast as that of Fourier series.

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Proof.

Write $F(z) - F(0) = z \cdot B \cdot G$.

$$F = a_0 + a_1 z B_1 + a_2 z^2 B_1 B_2 + \cdots + a_n z^n B_1 \cdot \ldots B_n G.$$

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Observation: All these terms are mutually orthogonal in $L^2(\partial\mathbb{D})$. The last term is additionally orthogonal to

$$1, z, z^2, \ldots, z^{n-1}.$$

$\square$

Define two spaces X, Y for monotonically increasing weights γ_n

$$\left\| \sum_{n \geq 0} a_n z^n \right\|_X^2 := \sum_{n \geq 0} \gamma_n |a_n|^2 < \infty.$$

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(Basic Observation)

Suppose $F(z) = F(0) + z \cdot G(z)$. Then

$$\|G\|_X^2 \leq \|F\|_X^2 - \|F\|_Y^2.$$

(Theorem, Coifman and S)

If holomorphic F has a Blaschke factorization $F = B \cdot G$, then

$$\|G(e^{i\cdot})\|_X \leq \|F(e^{i\cdot})\|_X.$$

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Moreover, if $F(\alpha) = 0$ for some $\alpha \in \mathbb{D}$, we even have

$$\|G(e^{i\cdot})\|_X^2 \leq \|F(e^{i\cdot})\|_X^2 - (1 - |\alpha|^2) \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha} z} \right\|_Y^2.$$

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Corollary. Initial data in $X \implies$ convergence in Y

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Interesting special case: $X = \mathcal{D}$ (the Dirichlet space) and $Y = L^2$.

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(Carleson's formula for Blaschke product, 1960)

Assume F is holomorphic with roots $\{\alpha_i : i \in I\}$ in $\mathbb{D}$ and $F = B \cdot G$, then

$$\int_{\mathbb{D}} |F'(z)|^2 dz = \int_{\mathbb{D}} |G'(z)|^2 dz + \frac{1}{2}\int_{\partial\mathbb{D}} |G|^2 \sum_{i \in I} \frac{1-|a_i|^2}{|z-\alpha_i|^2}.$$

Numerical stability

Holomorphic function inside the unit disk is given by

$$f(x) = \int_{|z|=1} P_x(e^{it}) f(e^{it}) dt.$$

This implies drastic levels of stability under additive noise.

So what's the point?

We have a natural analogue of Fourier series

$$\begin{aligned}
F &= BG \\
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An amazing observation

In reality, convergence seems to happen at an **exponential** rate.

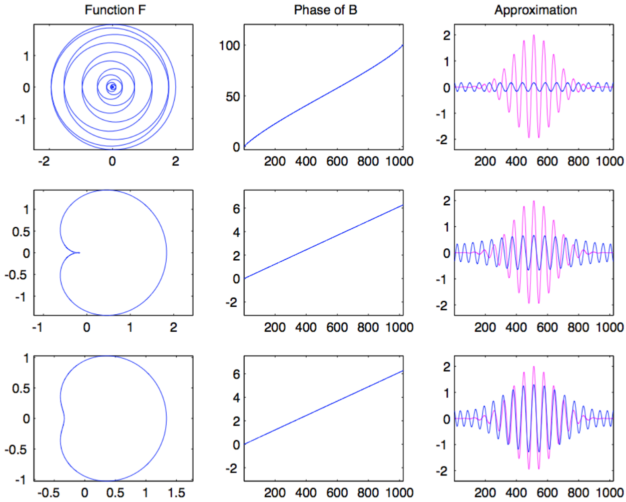
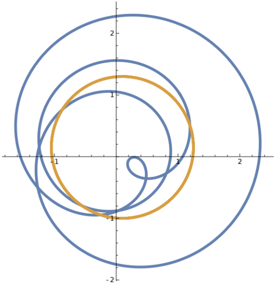
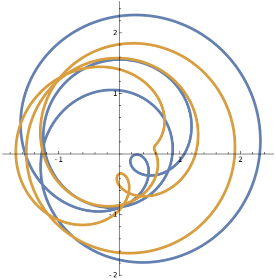


Figure 1.20: *Orthogonal decomposition of the modulated Gaussian signal* $F : \theta \mapsto e^{-(\theta-\theta_0)^2} \cdot e^{in\theta}$ *in three steps. The right column shows the evolution of the approximation.*

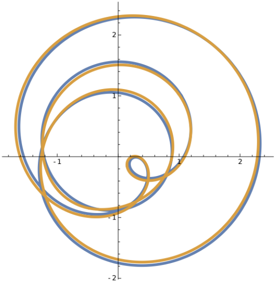
(Three steps of the algorithm, Michel Nahon, PhD Thesis)



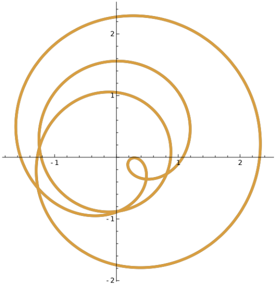
$F(0) + a_0 B_0$



$F(0) + a_0 B_0 + a_1 B_0 B_1$



$F(0) + a_0 B_0 + a_1 B_0 B_1 +$
$a_2 B_0 B_1 B_2$



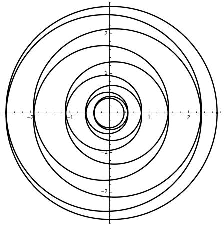
$F(0) + a_0 B_0 + a_1 B_0 B_1 +$
$a_2 B_0 B_1 B_2 + a_3 B_0 B_1 B_2 B_3$

(Lukianchikov, Nazarchuk and Xue, 2019)

The big open question

In reality, convergence seems to happen much, much faster.

$$\int_{\mathbb{D}} |F'(z)|^2 dz = \int_{\mathbb{D}} |G'(z)|^2 dz + \frac{1}{2} \int_{\partial\mathbb{D}} |G|^2 \sum_{i \in I} \frac{1 - |a_i|^2}{|z - \alpha_i|^2}$$



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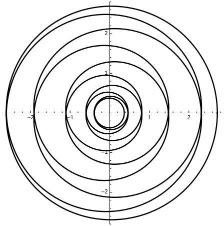

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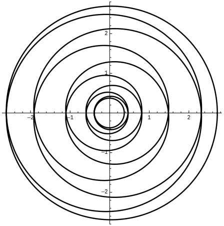

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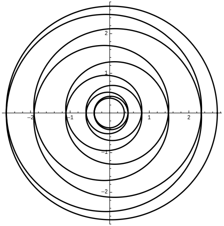

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$$(\text{winding number}) \int_{\partial\mathbb{D}} |G|^2 \sim \int_{\mathbb{D}} |F'(z)|^2 dz$$

$\implies$ exponential convergence(?)

Respiratory signal (Hau-tieng Wu)

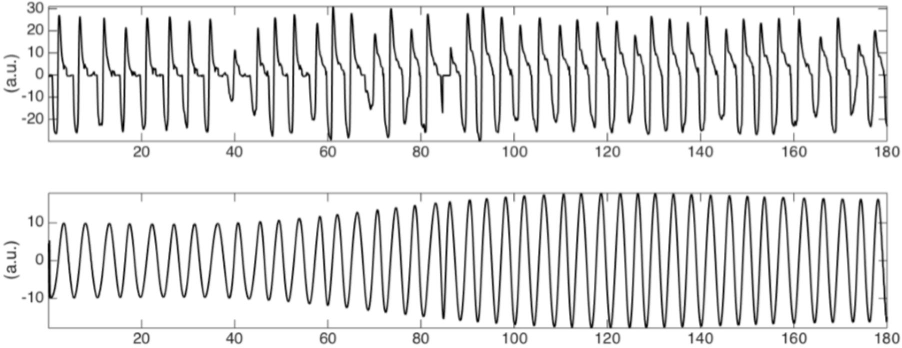

Figure: Respiratory signal and first Blaschke

Respiratory signal (Hau-tieng Wu,)

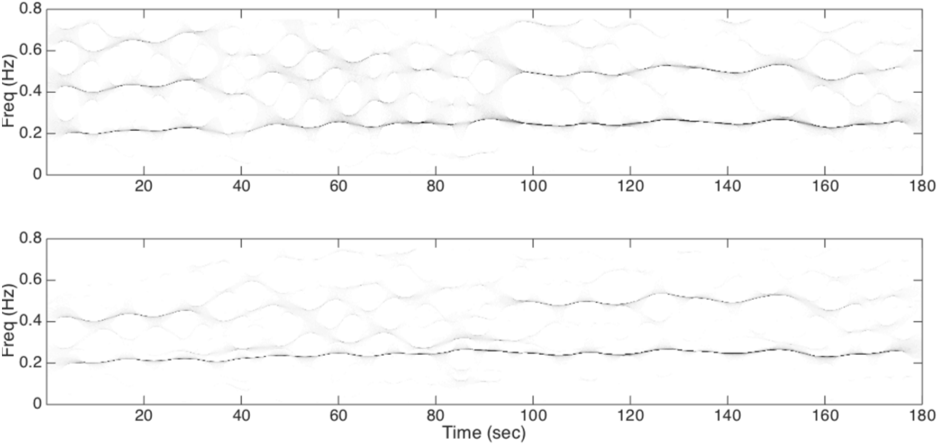

Figure: Synchrosqueezing vs. Blaschke-Synchrosqueezing

Gravity Wave (Hau-tieng Wu)

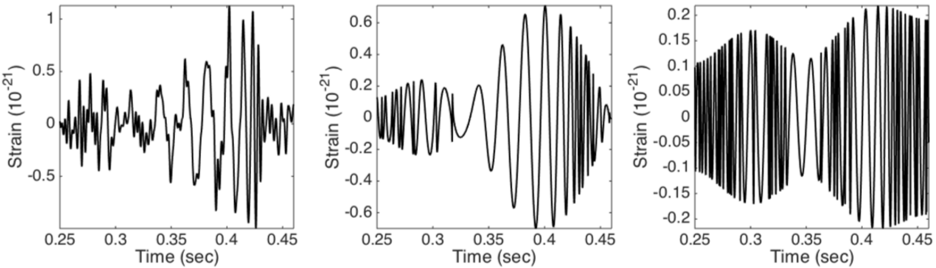

Figure: Gravity wave and the first two Blaschkes

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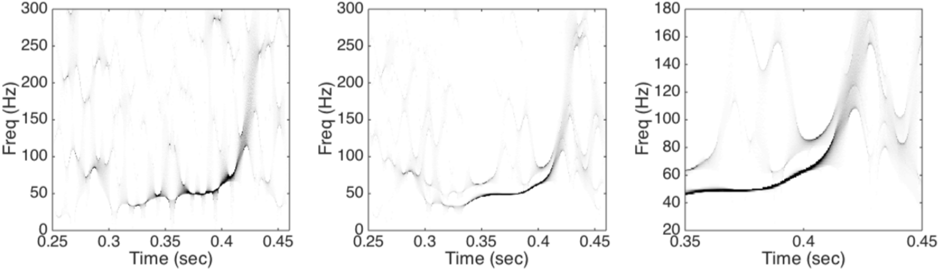

Figure: Synchrosqueezing, Blaschke-synchrosqueezing and zoom

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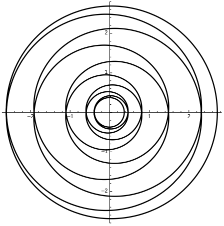

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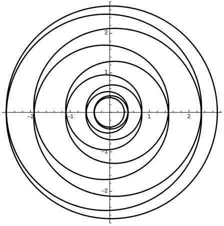

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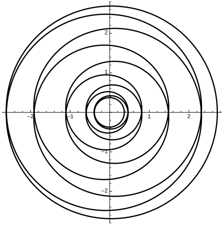

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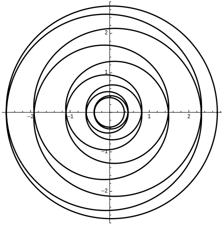

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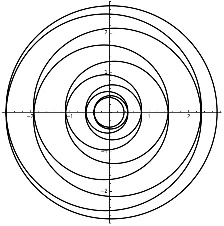

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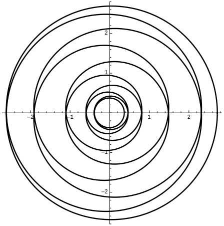

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Example 1: μ is standard Gaussian in $\mathbb{C}$

$z_1, \ldots, z_n \sim \mu$ (iid) and

$$p_n(z) = \prod_{k=1}^{n} (z - z_k)$$

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Then the roots of $p_n'(z)$ are also distributed according to μ.

Problem

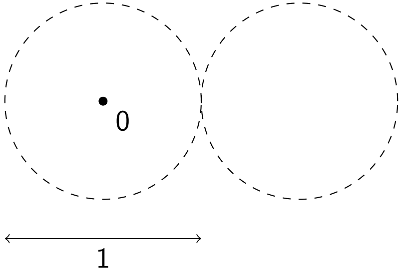
Let μ be a probability distribution in $\mathbb{C}$, let $z_1, \ldots, z_n$ i.i.d. random variables and consider the random polynomial

$$p_n(z) = \prod_{k=1}^{n} (z - z_k)$$

Where are the n roots of $p_n(z) - p_n(0)$?

Example 2: μ is the union of two circles

Pick roots uniformly at random from



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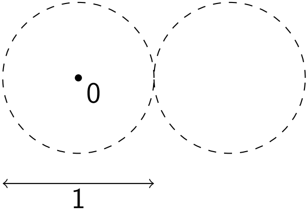

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Theorem (Hau-tieng Wu and S., IMRN 2021)

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In some regions of $\mathbb{C}$, the solutions of $p_n(z) - p_n(0) = 0$ are distributed exactly as μ (see Example 1).

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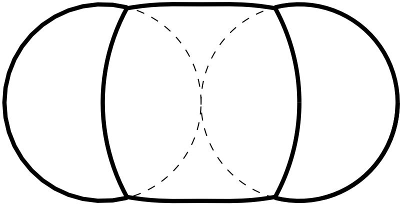
Remark. If μ is radial around 0, then only the first case appears. This is Example 1 (the Gaussian).

Example 2: μ is the union of two circles



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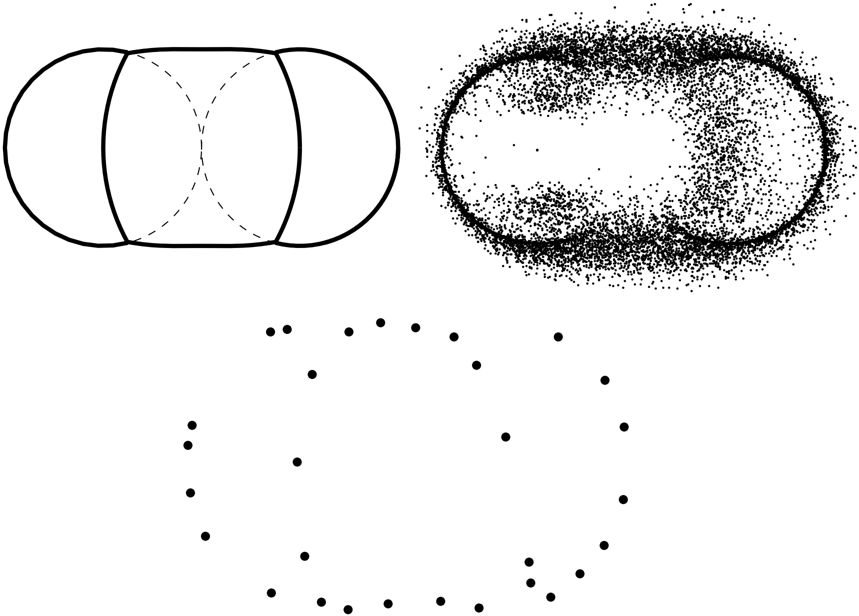

Other recent work

Theorem (Coifman and Peyriere)

Convergence in H^p.

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ON ALMOST-EVERYWHERE CONVERGENCE OF MALMQUIST-TAKENAKA SERIES

GEVORG MNATSAKANYAN

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ON ALMOST-EVERYWHERE CONVERGENCE OF MALMQUIST-TAKENAKA SERIES

GEVORG MNATSAKANYAN

We are interested in almost everywhere convergence of the MT series (1.3). By standard techniques, almost everywhere convergence can be deduced from estimates of the maximal partial sum operator. Denote

$$Tf(e^{ix}) := T^{(a_n)}f(e^{ix}) := \sup_n \Big|\sum_{n=0}^{N} \langle f, \phi_n\rangle \phi_n(e^{ix})\Big|. \tag{1.5}$$

Question. *Is the maximal partial sum operator (1.5) bounded on L^p?*

If $a_n \equiv 0$, then the MT series reduces to the classical Fourier series and the operator (1.5) reduces to the Carleson operator. In this case the positive answer to the above question is given by the Carleson-Hunt theorem [Car66, Hun68].

Deep Blaschke?

Coifman and Peyriere (2021)

One can do all these things just as easily on $\mathbb{R}$. The proper analogue is

$$B(x) = \prod_{k \geq 1} \frac{x - a_k}{x - \overline{a_k}}.$$

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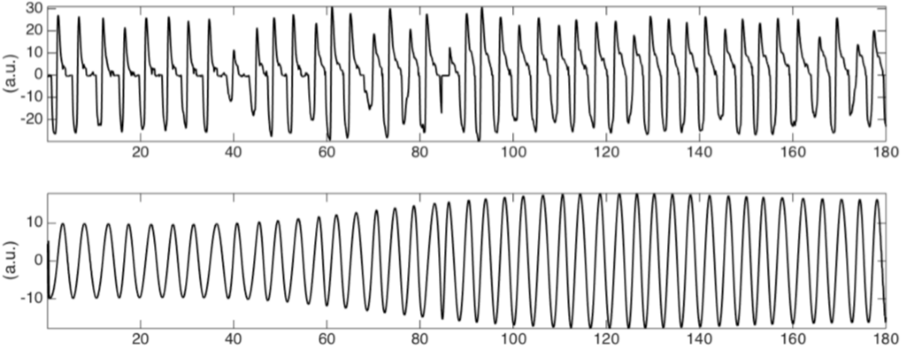
We have $B(x) = e^{i\theta(x)}$ where

$$\theta(x) = \sum_{k\geq 0} \sigma\left(\frac{x - \mathrm{Re}\ a_k}{\mathrm{Im}\ a_k}\right)$$

and

$$\sigma(x) = \frac{\pi}{2} + \arctan x.$$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THANK YOU!