New Interactions between Analysis and Number Theory

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Karl Popper, Conjectures and Refutations, 1963

[...] in the natural sciences [...] what we look for is truth *which has a high degree of explanatory power* [...]

This talk: truth with *low* degree of explanatory power.

Outline.

1. Poincaré inequalities on the torus $\mathbb{T}^d$
2. Number Theory in the Hardy-Littlewood maximal function
3. A Mystery hiding in Ulam's integer sequence [$200]

(1) Poincaré inequalities on the torus $\mathbb{T}^d$

The Poincaré inequality

General setting: $\Omega \subset \mathbb{R}^n$ bounded and nice enough. Then

$$\int_\Omega f(x)dx = 0 \implies \int_\Omega |\nabla f(x)|^p dx \geq c_{p,\Omega} \int_\Omega |f(x)|^p dx.$$

'If a function has large values, it has to have large growth.'

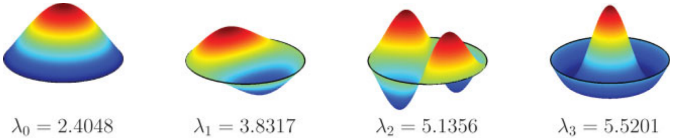

Figure : The best functions on a disk with Dirichlet condition.

Everything is easy on the torus!

Particularly simple on $\mathbb{T}^d$ and $p = 2$. Then, if f has mean value 0,

$$\int_{\mathbb{T}^d} |\nabla f(x)|^2 dx \geq \int_{\mathbb{T}^d} |f(x)|^2 dx$$

and this is the sharp result.

Proof. Convexity!

$$f(x) = \sum_{\mathbf{k} \neq \mathbf{0}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot x}$$

$$\nabla f(x) = \sum_{\mathbf{k} \neq \mathbf{0}} \mathbf{k} a_{\mathbf{k}} e^{i\mathbf{k}\cdot x}$$

$$\|f\|^2_{L^2(\mathbb{T}^2)} = \sum_{\mathbf{k} \neq \mathbf{0}} |a_{\mathbf{k}}|^2 \leq \sum_{\mathbf{k} \neq \mathbf{0}} |\mathbf{k}|^2 |a_{\mathbf{k}}|^2 \leq \|\nabla f\|^2_{L^2(\mathbb{T}^2)}$$

Main result

Theorem (S., special case $d = 2$)

There exist $\alpha \in \mathbb{T}^2$ and $c_\alpha > 0$ so that for all functions with mean value 0

$$\|\nabla f\|_{L^2(\mathbb{T}^2)} \| \langle \nabla f, \alpha \rangle \|_{L^2(\mathbb{T}^2)} \geq c_\alpha \|f\|^2_{L^2(\mathbb{T}^2)}$$

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$$\|\nabla f\|_{L^2(\mathbb{T}^2)} \| \langle \nabla f, \alpha \rangle \|_{L^2(\mathbb{T}^2)} \geq c_\alpha \|f\|^2_{L^2(\mathbb{T}^2)}$$

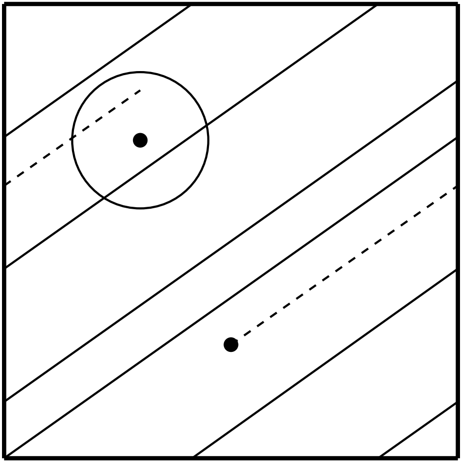
Clearly $\alpha = (1, 0)$ does not work because that would give

$$\|\nabla f\|_{L^2(\mathbb{T}^2)} \|\partial_x f\|_{L^2(\mathbb{T}^2)} \geq c_\alpha \|f\|^2_{L^2(\mathbb{T}^2)}$$

and the function might vary along the $y-$direction. Clearly $\alpha = (m, n) \in \mathbb{Z}^2$ does not work either: $\sin(nx - my)$.

Non-closed geodesics

$$\|\nabla f\|_{L^2(\mathbb{T}^2)} \| \langle \nabla f, \alpha \rangle \|_{L^2(\mathbb{T}^2)} \geq c_\alpha \|f\|^2_{L^2(\mathbb{T}^2)}$$



Bad non-periodic geodesics

$$\alpha = \left(1, \sum_{n=1}^{\infty} \frac{1}{10^{n!}}\right) \sim (1, 0.110001\dots)$$

where the number, Liouville's constant, is known to be irrational.

$$f_N(x,y) = \sin\left(10^{N!}\left(\sum_{n=1}^{N} \frac{x}{10^{n!}} - y\right)\right),$$

then

$$\|f_N\|_{L^2(\mathbb{T}^2)}^2 = 2\pi^2 \qquad \text{and} \qquad \|\nabla f_N\|_{L^2(\mathbb{T}^2)} \le 6 \cdot 10^{N!}$$

while

$$\|\langle \nabla f_N, \alpha\rangle\|_{L^2(\mathbb{T}^2)} = \sqrt{2\pi^2}\left(\sum_{n=N+1}^{\infty} \frac{10^{N!}}{10^{n!}}\right) \ll 10^{-2\cdot N!} \qquad \text{for } N \ge 3.$$

Theorem (special case $d = 2$)

$$\|\nabla f\|_{L^2(\mathbb{T}^2)} \| \langle \nabla f, \alpha \rangle \|_{L^2(\mathbb{T}^2)} \geq c_\alpha \|f\|^2_{L^2(\mathbb{T}^2)}$$

Characterization (special case $d = 2$)

$\alpha = (\alpha_1, \alpha_2) \in \mathbb{T}^2$ is admissible if and only if α_2/α_1 has a bounded continued fraction expansion.

$\alpha = (1, \sqrt{2})$ is admissible.
$\alpha = (1, e)$ is *not* admissible.
$\alpha = (1, \pi)$ is *conjectured to* not be admissible.

Main idea of the proof

$$
\begin{aligned}
\| \langle \nabla f, \alpha \rangle \|_{L^2(\mathbb{T}^d)}^2 &= \left\| \left\langle \sum_{k \in \mathbb{Z}^d} a_k k e^{ik\cdot x}, \alpha \right\rangle \right\|_{L^2(\mathbb{T}^d)}^2 \\
&= \left\| \sum_{k \in \mathbb{Z}^d} a_k \langle k, \alpha \rangle e^{ik\cdot x} \right\|_{L^2(\mathbb{T}^d)}^2 \\
&= (2\pi)^d \sum_{k \in \mathbb{Z}^d} |a_k|^2 |\langle k, \alpha \rangle|^2.
\end{aligned}
$$

The term $|\langle k, \alpha \rangle|$ is not bounded away from 0.

Main idea of the proof

Theorem (Dirichlet's approximation theorem)

There exist infinitely many $k \in \mathbb{Z}^d$ with

$$|\langle k, \alpha \rangle| \lesssim \frac{1}{|k|^d}.$$

Theorem (Perron)

There exist $\alpha \in \mathbb{T}^d$ such that for all $k \in \mathbb{Z}^d$

$$|\langle k, \alpha \rangle| \gtrsim \frac{1}{|k|^d}.$$

Main idea of the proof

Theorem (Perron)

There exist $\alpha \in \mathbb{T}^d$ such that for all $k \in \mathbb{Z}^d$

$$| \langle k, \alpha \rangle | \gtrsim \frac{1}{|k|^d}.$$

We want

$$\|\nabla f\|_{L^2(\mathbb{T}^2)} \| \langle \nabla f, \alpha \rangle \|_{L^2(\mathbb{T}^2)} \geq c_\alpha \|f\|^2_{L^2(\mathbb{T}^2)}.$$

If $\|\nabla f\|_{L^2(\mathbb{T}^2)}$ is of a certain size, then some of the L^2-norm is on low frequencies and we may employ Perron's result.

More results

Theorem

$$\|\nabla f\|_{L^2(\mathbb{T}^d)}^{d-1} \| \langle \nabla f, \alpha \rangle \|_{L^2(\mathbb{T}^d)} \geq c_\alpha \|f\|_{L^2(\mathbb{T}^2)}^{d}$$

More results

Theorem

$$\|\nabla f\|_{L^2(\mathbb{T}^d)}^{d-1} \| \langle \nabla f, \alpha \rangle \|_{L^2(\mathbb{T}^d)} \geq c_\alpha \|f\|_{L^2(\mathbb{T}^2)}^{d}$$

Theorem (Coifman)

$$\left\| \left\langle D^d f, \alpha \right\rangle \right\|_{L^2(\mathbb{T}^d)} \geq c_\alpha \|f\|_{L^2(\mathbb{T}^2)}$$

More results

Theorem

$$\|\nabla f\|_{L^2(\mathbb{T}^d)}^{d-1}\| \langle \nabla f, \alpha\rangle \|_{L^2(\mathbb{T}^d)} \geq c_\alpha \|f\|_{L^2(\mathbb{T}^2)}^{d}$$

Theorem (Coifman)

$$\left\|\left\langle D^d f, \alpha\right\rangle\right\|_{L^2(\mathbb{T}^d)} \geq c_\alpha \|f\|_{L^2(\mathbb{T}^2)}$$

Irrationality measure of π

$$\|\nabla f\|_{L^2(\mathbb{T}^2)}^{7/8}\| \langle \nabla f, (1,\pi)\rangle \|_{L^2(\mathbb{T}^2)}^{1/8} \geq c\|f\|_{L^2(\mathbb{T}^2)}.$$

More results

Markov spectrum (special case due to Hurwitz)

Let $d = 2$. If

$$\|\nabla f\|_{L^2(\mathbb{T}^2)} \| \langle \nabla f, \alpha \rangle \|_{L^2(\mathbb{T}^2)} \geq c_\alpha \|f\|^2_{L^2(\mathbb{T}^2)}$$

then

$$c_\alpha \leq \frac{|\alpha|}{\sqrt{5}}.$$

'The best flow on the torus is given by the golden ratio.'

More results

Markov spectrum

$$\|\nabla f\|_{L^2(\mathbb{T}^2)} \| \langle \nabla f, \alpha \rangle \|_{L^2(\mathbb{T}^2)} \geq c_\alpha \|f\|^2_{L^2(\mathbb{T}^2)}$$

$$0 \leq c_\alpha \leq \frac{|\alpha|}{\sqrt{5}}.$$

Sharp: let $\alpha = \left(1, \frac{1+\sqrt{5}}{2}\right)$ and $f_n(x, y) = \sin\left(F_{n+1}x - F_n y\right)$, where F_n is the $n-$th Fibonacci number. One needs to prove

$$\lim_{n\to\infty} \left| \frac{F_{n+1}}{F_n} - \frac{1+\sqrt{5}}{2} \right| F_n^2 = \frac{1}{\sqrt{5}}.$$

Several directions

Theorem
Let $1 \leq \ell \leq d-1$. Then there exists a set $\mathcal{B}_\ell \in (\mathbb{T}^d)^\ell$ such that for every $(\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathcal{B}_\ell$ there is a $c_\alpha > 0$ with

$$\|\nabla f\|_{L^2(\mathbb{T}^d)}^{d-1} \left(\sum_{i=1}^{\ell} \| \langle \nabla f, \alpha_i \rangle \|_{L^2(\mathbb{T}^d)} \right)^{\ell} \geq c_\alpha \|f\|_{L^2(\mathbb{T}^d)}^{d-1+\ell}$$

for all $f \in H^1(\mathbb{T}^d)$ with mean 0.

There are also stronger versions conditional on very recent results.

Further directions

Khintchine

For every $\delta < 1/2$, the set of $\alpha \in \mathbb{T}^2$ for which

$$\|\nabla f\|_{L^2(\mathbb{T}^2)}^{1-\delta} \| \langle \nabla f, \alpha \rangle \|_{L^2(\mathbb{T}^2)}^{\delta} \geq c \|f\|_{L^2(\mathbb{T}^2)}$$

has full Lebesgue measure.

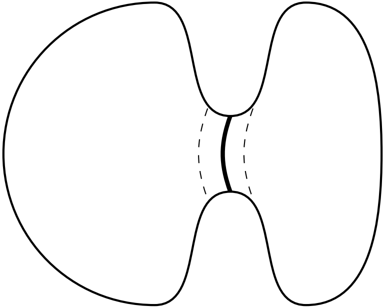
The general problem

General question: nice geometry, smooth vector field Y on that geometry

$$\|\nabla f\|_{L^2}^{1-\delta} \| \langle \nabla f, Y \rangle \|_{L^2}^{\delta} \geq c \|f\|_{L^2}$$

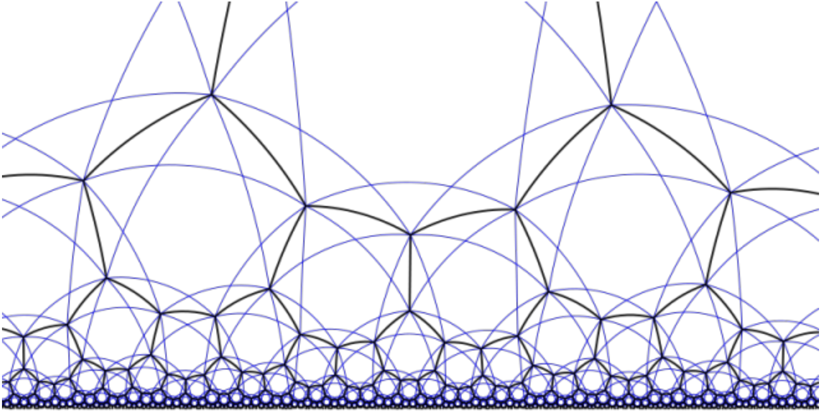
Further directions

$\mathbb{S}^2$ **Hairy ball theorem.** A continuous vector field on an even-dimensional sphere vanishes somehwere.
$\mathbb{S}^3$ **Seifert conjecture (false).** Every nonsingular, continuous vector field on the 3-sphere has a closed orbit.



Very daring conjecture. $\mathbb{T}^d$ is the best geometry (i.e. smallest δ).

Further directions (in progress)



(2) Number Theory in the Hardy-Littlewood maximal function

One version of the statement

Theorem (S. 2015)

Let $f \in C^{1/2+}$ be periodic. If, for all $x \in \mathbb{R}$,

$$\int_{x-1}^{x+1} f(z)dz = f(x-1) + f(x+1),$$

then

$$f(x) = a + b\sin(cx + d) \qquad \text{for some } a, b, c, d \in \mathbb{R}.$$

Why? Is it trivial? Also: why even think about this?

Lax (2007)

A CURIOUS FUNCTIONAL EQUATION

By

PETER D. LAX

For Israel Gohberg, outstanding analyst, with affection and admiration.

$$\frac{1}{x}\int_0^x f(y)dy = f(x/2). \tag{7}$$

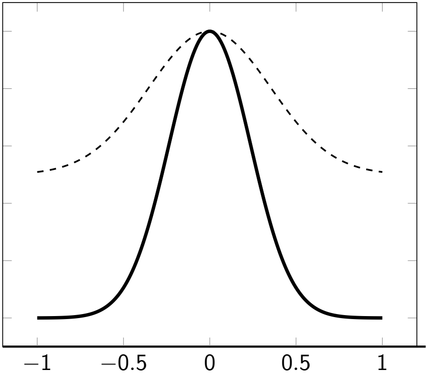
Theorem 3. *A solution f of* (7) *which is infinitely differentiable at $x = 0$ is of the form $f(x) = c + mx$.*

Hardy-Littlewood maximal function

Definition.

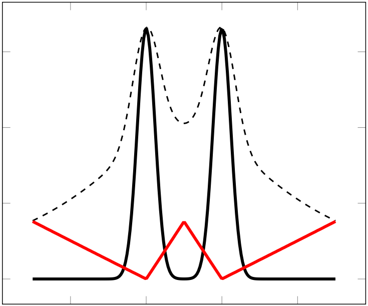
Let $f : \mathbb{R} \to \mathbb{R}_+$. We set

$$(\mathcal{M}f)(x) := \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} f(z)dz.$$



The computational question

How is the maximal function being computed?



Definition.

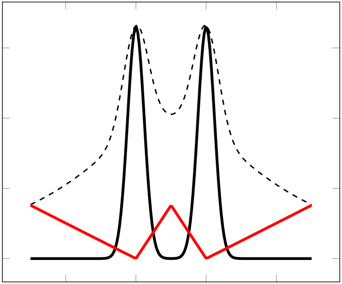
Given a function $f : \mathbb{R} \to \mathbb{R}_+$, the smallest optimal radius $r_f : \mathbb{R} \to \mathbb{R}$ is

$$r_f(x) = \inf \left\{ r > 0 : \frac{1}{2r} \int_{x-r}^{x+r} f(z)dz = (\mathcal{M}f)(x) \right\}.$$

The combinatorial question

Question.

$$r_f \quad \text{has certain properties} \Leftrightarrow f \text{ has certain properties.}$$



Trivial example. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic. If

$$\left| \bigcup_{x \in \mathbb{R}} \{r_f(x)\} \right| = 1, \qquad \text{then } f \text{ is constant.}$$

Simple functions are trigonometric

Theorem

Let $f \in C^{1/2+}$ be periodic. If

$$\left| \left(\bigcup_{x \in \mathbb{R}} \{r_f(x)\} \right) \cup \left(\bigcup_{x \in \mathbb{R}} \{r_{-f}(x)\} \right) \right| \leq 2,$$

then

$$f(x) = a + b \sin(cx + d) \qquad \text{for some } a, b, c, d \in \mathbb{R}.$$

Periodic solutions of a DDE are trigonometric

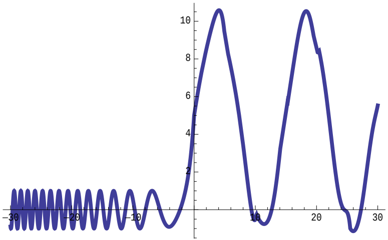
Theorem (equivalent)

Let $\alpha > 0$ be fixed and let $f \in C^1(\mathbb{R}, \mathbb{R})$ be a solution of the delay differential equation

$$f'(x+\alpha) - \frac{1}{\alpha} f(x+\alpha) = -f'(x-\alpha) - \frac{1}{\alpha} f(x-\alpha).$$

If f is periodic, then

$$f(x) = a + b\sin(cx+d) \qquad \text{for some } a, b, c, d \in \mathbb{R}.$$



Proof

After a standard application of Fourier series:

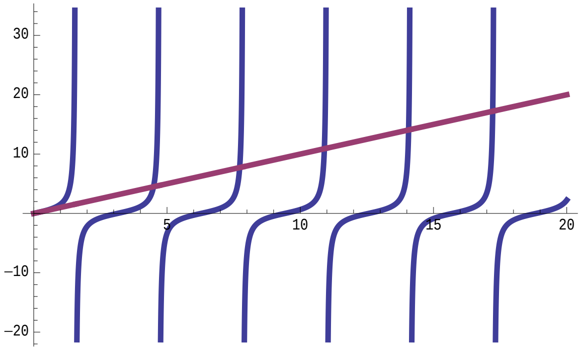
Theorem (again equivalent)

Let $(\alpha, m, n) \in \mathbb{R} \times \mathbb{N} \times \mathbb{N}$. If

$$\tan \alpha m = \alpha m$$
$$\tan \alpha n = \alpha n,$$

then $\alpha = 0$ or $m = n$.

Proof II



We need that any two elements in the set

$$\{x \in \mathbb{R}_{>0} : x = \tan x\} = \{4.49.., 7.72.., 10.90.., 14.06.., ...\}$$

are linearly independent over $\mathbb{Q}$.

Proof III

Use multiple angle formulas.

$$\tan\alpha = \alpha$$
$$\tan 3\alpha = 3\alpha$$

$$3\alpha \underbrace{=}_{\text{2nd eq}} \tan 3\alpha \underbrace{=}_{\text{trig identity}} \frac{((\tan\alpha)^2 - 3)\tan\alpha}{3(\tan\alpha)^2 - 1} \underbrace{=}_{\text{1st eq}} \frac{\alpha^2 - 3}{3\alpha^2 - 1}\alpha$$

Yields very complicated polynomials very quickly.

It would be *exceedingly* nice if we wouldn't have to deal with polynomials.

Proof IV - the miracle

Corollary of the Lindemann-Weierstrass theorem.

tan(nonzero algebraic number) is transcendental.

$$\boxed{\text{If} \quad \tan\beta = \beta \qquad \text{then} \qquad \beta \text{ is transcendental (or } \beta = 0).}$$

Proof V

$$\tan \alpha m = \alpha m$$
$$\tan \alpha n = \alpha n$$

implies

$$n \tan \alpha m - m \tan \alpha n = 0.$$

Rewriting these as polynomials of $\tan \alpha$, we get

$$0 = n \tan \alpha m - m \tan \alpha n = n \frac{p_m(\tan \alpha)}{q_m(\tan \alpha)} - m \frac{p_n(\tan \alpha)}{q_n(\tan \alpha)}$$

and therefore after multiplication with $q_m(\tan \alpha) q_n(\tan \alpha)$

$$0 = n q_n(\tan \alpha) p_m(\tan \alpha) - m q_m(\tan \alpha) p_n(\tan \alpha).$$

Proof VI

$$0 = nq_n(\tan\alpha)p_m(\tan\alpha) - mq_m(\tan\alpha)p_n(\tan\alpha).$$

This means that $\tan\alpha$ is algebraic. Algebraic numbers form a field (closed under sums, products and division). Since

$$\tan n\alpha = \frac{p_n(\tan\alpha)}{q_n(\tan\alpha)},$$

$\tan n\alpha$ is algebraic (and, little extra work, not 0). However, by assumption,

$$\tan n\alpha = n\alpha$$

and therefore

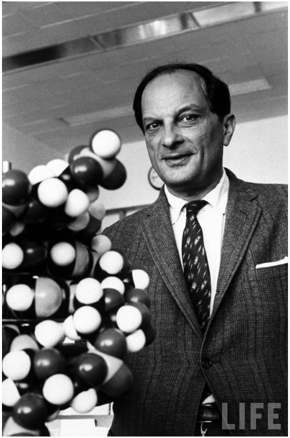
$$\tan \underbrace{\tan n\alpha}_{\text{algebraic}} = \underbrace{\tan n\alpha}_{\text{algebraic}} .$$

This means the tangent sends a nonzero algebraic number to an algebraic number. Contradiction. $\square$

A complete mystery in $\mathbb{N}$

Ulam (1964)

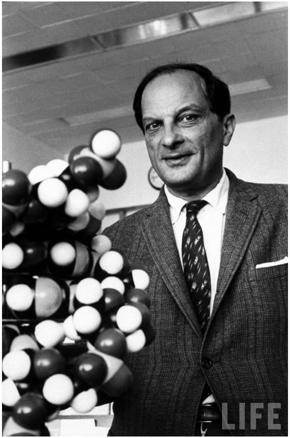
One can consider a rule for growth of patterns – in one dimension it would be merely a rule for obtaining successive integers. [...] In both cases simple questions that come to mind about the properties of a sequence of integers thus obtained are notoriously hard to answer.



1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53...

Ulam sequence

Start with 1,2. The next element is the smallest integer that can be *uniquely* written as the sum of two distinct earlier terms.



1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53...

The sequence grows at most exponentially. Nothing else is known.

1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53...

Roth's theorem

If the function has nontrivial density, then it has many arithmetic progressions of length 3.

Self-consistency

If $a \cdot n + b$ is in the sequence for $n = 1, 2, 3$, then a or $2a$ is not.

Fourier series detect correlation with linear phases, let's look at

$$\mathrm{Re} \sum_{n=1}^{N} e^{i a_n x} = \sum_{n=1}^{N} \cos\left(a_n x\right)$$

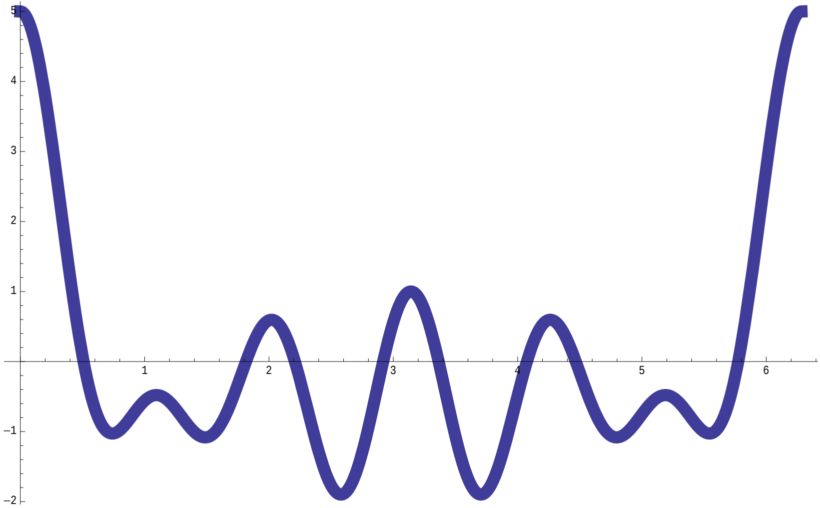


Figure : $N = 5$

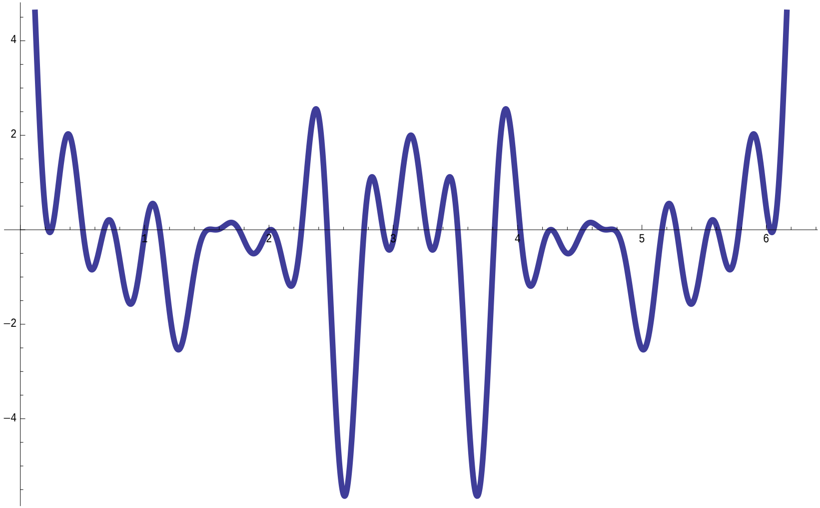


Figure : $N = 10$

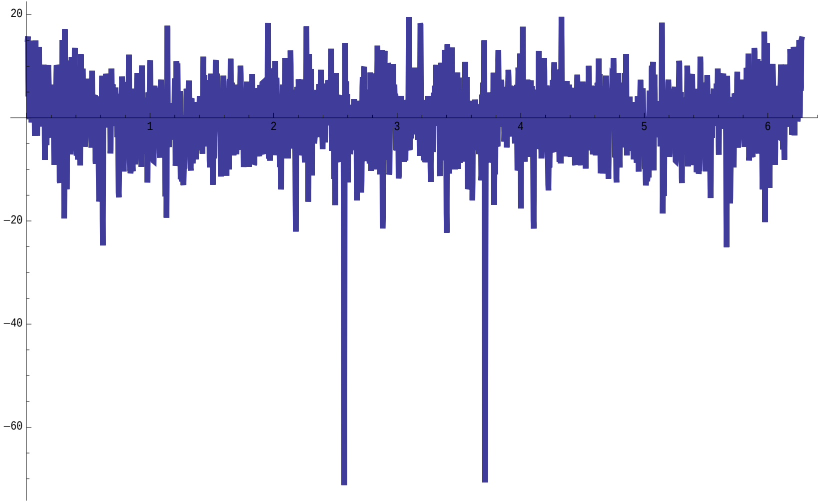
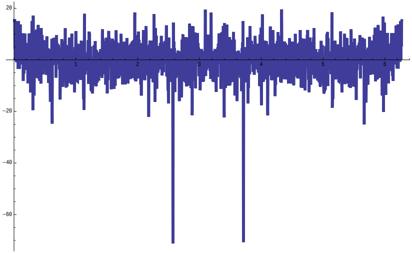


Figure : $N = 100$



Peak roughly at (thanks to data provided by Dan Strottman!)

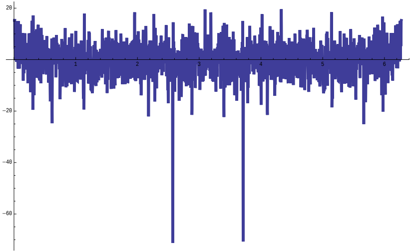
$$\alpha \sim 2.5714474\ldots$$

and of strength

$$\mathcal{R}\sum_{n=1}^{N} e^{i a_n x} = \sum_{n=1}^{N} \cos(a_n x) \sim -0.79N.$$

Indeed, we have (empirically, up to 10^{11})

$$\cos(\alpha a_n) < 0 \qquad \text{for all numbers except} \quad \{2, 3, 47, 69\}\,.$$



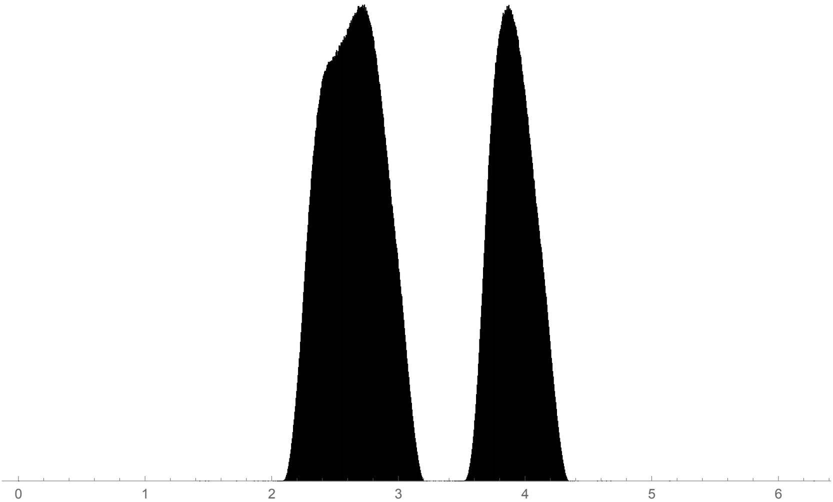
Indeed, we have (at least up to 10^{11})

$$\cos(\alpha a_n) < 0 \qquad \text{for all numbers except} \quad \{2, 3, 47, 69\}.$$

This means that the $\cos(\alpha a_n)$ terms have to line up.

The relevant set is $(\alpha a_n \bmod 2\pi)_{n=1}^{N}$.

The limiting distribution



Jordan Ellenberg @JSEllenberg · 18. Dez.

Why is there a spike in the Fourier transform of the Ulam sequence?!? arxiv.org/abs/1507.00267

6 15

An efficient method for computing Ulam numbers

Philip Gibbs

The Ulam numbers form an increasing sequence beginning 1,2 such that each subsequent number can be uniquely represented as the sum of two smaller Ulam numbers. An algorithm is described and implemented in Java to compute the first billion Ulam numbers.

Fast computation (Donald Knuth)

30. That index and link mechanism is somewhat tricky, so I'd better have a subroutine to check that it isn't messed up.

```
#define flag #80000000   /* flag temporarily placed into the next fields */
#define panic(m)
        {
          fprintf(stderr, "Oops,␣"O"s!␣(h="O"d,␣r="O"d,␣j="O"d,␣x="O"d)\n", m, h, r, j, x);
          return;
        }
⟨Subroutines 10⟩ +≡
  void sanity(void)
  {
    register int h, j, nextj, x, y, r, lastr;
    ullng u, lastu;
```

PhD thesis (Daniel Ross, in progress)

The Ulam sequence and related phenomena

Daniel Ross

Contents

Timothy Gowers +6

+1

My initial reaction, after having read nothing but the definition (which alone merits a +1 for this post), was to think that the density ought to be similar to that of the squares. The rough reason: if you have significantly greater than that density, then there should be lots of numbers expressible as a sum of two (distinct) terms of your sequence in at least two ways.

But I was assuming that the sequence would be fairly random, and the rest of your post makes it clear that that is very much not the case. Now that it occurs to me that the odd numbers have the property that no member of the sequence can be written as a sum of two earlier members of the sequence, I see that my heuristic argument basically misses the point completely.

What a weirdly interesting problem.

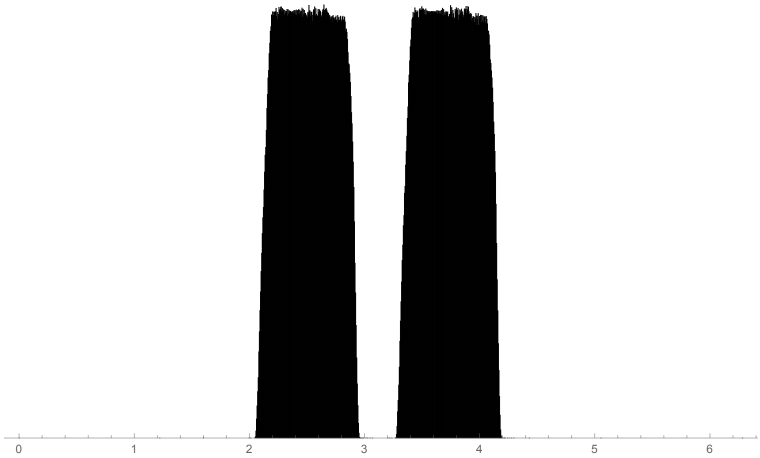


Figure : Initial values $(1, 3)$

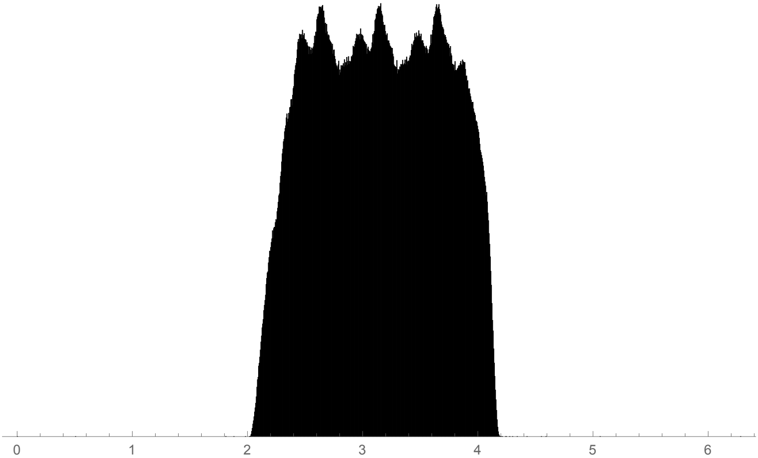


Figure : Initial values $(1, 4)$

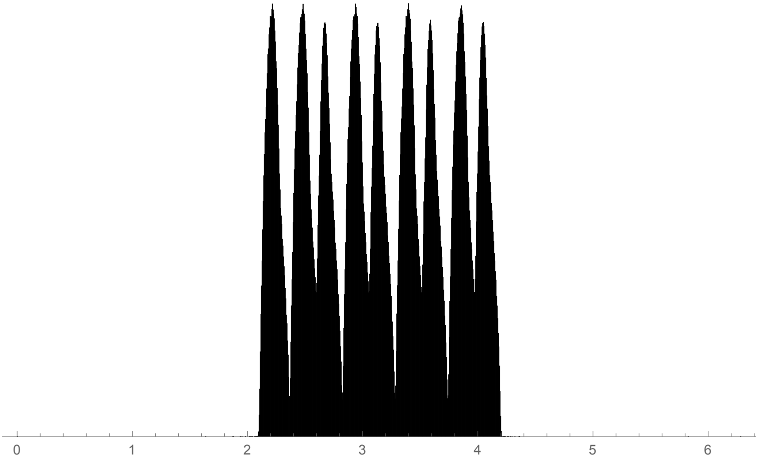
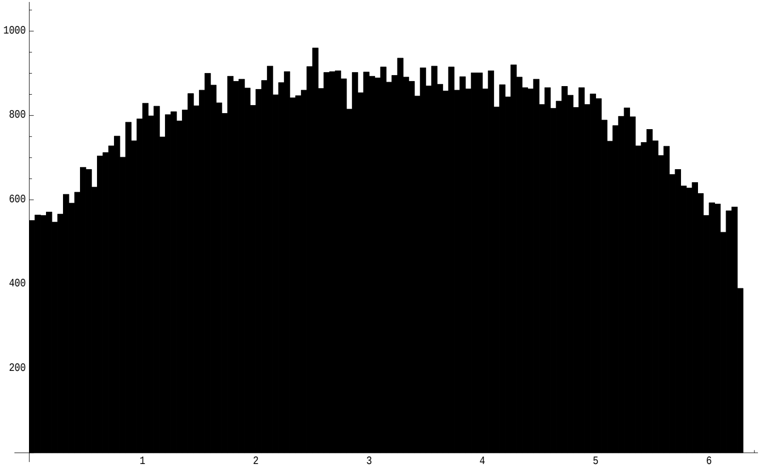


Figure : Initial values $(2, 3)$

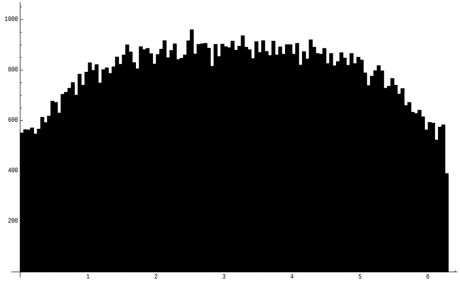
Zeroes of the Riemann ζ−function on the critical line

$$((\log 5)t_n \text{ mod } 2\pi)_{n=1}^{100.000}$$

where $\zeta(1/2 + it_n) = 0$.



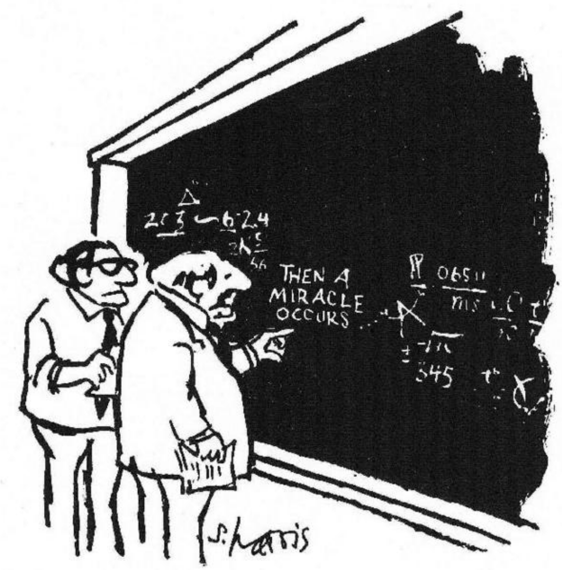
'Almost' another example



$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}.$$

$$\left| \frac{1}{1-p^{-\left(\frac{1}{2}+it_n\right)}} \right| \leq 1 \quad \text{related to} \quad \frac{\pi}{2} \leq [(\log p)t_n \bmod 2\pi] \leq \frac{3\pi}{2}.$$

Explained by Landau (1912) and Ford-Zaharescu (2005).



"I think you should be more explicit here in step two."

Thank you!