Rooted Staggered Fermions: Good, Bad or Ugly?

Status report on the validity of the procedure of representing the determinant for a single fermion by the fourth root of the staggered fermion determinant.

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The Possibilities

- GOOD: Correct continuum limit.
- BAD: Wrong continuum limit.
- UGLY: Correct continuum limit, but unphysical contributions present for $a \neq 0$, requiring theoretical understanding and complicated fits.

Significant progress in the last year

Issues have been clarified and in some cases resolved.
I will focus on the following, mainly analytic, papers (in this order):

- [BGS] C. Bernard, M. Golterman and Y. Shamir, "Observations On Staggered Fermions At Non-Zero Lattice Spacing," Phys. Rev. D **73**, 114511 (2006), hep-lat/0604017.
- Y. Shamir, "Renormalization-group analysis of the validity of staggered-fermion QCD with the fourth-root recipe," hep-lat/0607007.
- C. Bernard, "Staggered chiral perturbation theory and the fourth-root trick," Phys. Rev. D **73**, 114503 (2006), hep-lat/0603011.
- M. Creutz, "Flavor extrapolations and staggered fermions," hep-lat/0603020.
- [BGSS] C. Bernard, M. Golterman, Y. Shamir and S. R. Sharpe, "Comment on 'Flavor extrapolations and staggered fermions'," hep-lat/0603027.
- [DH] S. Dürr and C. Hoelbling, "Lattice fermions with complex mass," hep-lat/0604005.
- [GSS] M. Golterman, Y. Shamir and B. Svetitsky, "Failure Of Staggered Fermions At Nonzero Chemical Potential," hep-lat/0602026.

Other progress

Work I will not cover (my apologies):

- J. Giedt, "Power-Counting Theorem For Staggered Fermions," hep-lat/0606003.
- A. Hasenfratz, "Universality and Quark Masses of the Staggered Fermion Action", hep-lat/0511021.
- A. Hasenfratz & R. Hoffmann, "Validity of the rooted staggered determinant in the continuum limit", hep-lat/0604010, and talks Tues.
- A. Hart, "Improved staggered eigenvalues and epsilon regime universality in $SU(2)$", talk Tue.
- C. DeTar, "Taste breaking effects in scalar meson correlators", talk Thur.
- E. Gregory, "Pseudoscalar flavor-singlets and staggered fermions", talk Thur.

Previous discussions:

- K. Jansen, "Actions for dynamical fermion simulations: Are we ready to go?," Nucl. Phys. Proc. Suppl. **129**, 3 (2004), hep-lat/0311039.
- S. Dürr, "Theoretical issues with staggered fermion simulations," PoS **LAT2005**, 021 (2006), hep-lat/0509026.

Outline

- What is rooting and what are its potential problems?
  - Non-locality, lack of unitarity [BGS]
- Why is it being used, and what is the status of results?
  - What are the stakes? [MILC, FNAL, HPQCD]
- What do we learn about non-local theories from statistical mechanics?
- Can we tame the non-locality?
  - Perturbation theory
  - Renormalization Group (RG) analysis [Shamir]
  - Effective field theory for rooted staggered fermions [Bernard]
- Do rooted staggered fermions have too many symmetries? [Creutz, BGSS, DH]
- Problems with rooting at non-zero chemical potential [GSS]
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What I will assume

- Violations of reflection positivity by improved actions are cut-off phenomena which do not effect long-distance physics
  - Improved Wilson, staggered, . . .
- Lattice QCD with Wilson, GW, . . . fermions ("anything but staggered") has the correct continuum limit
- Lattice QCD with "unrooted" staggered fermions has the correct continuum limit (with four degenerate "tastes")
  - Includes assumption of perturbative renormalizability with correct $\beta-$function
  - Step towards proof provided by power-counting theorem of [Giedt]

Staggered fermions

Simple action: [Susskind]

$$\bar\chi D_{\rm stag}\chi \quad = \quad \sum_n \bar\chi_n \left[\sum_\mu \frac{\eta_{n,\mu}}{2}\left(U_{n,\mu}\chi_{n+\mu} - U^\dagger_{n-\mu,\mu}\chi_{n-\mu}\right) + m_0\chi_n\right]$$

- In practice use improved form (smeared U, Naik term, ...)

2^4 doublers = 4 Dirac components × 4 tastes in classical continuum limit

- Hypercube basis: [Gliozzi, Kluberg-Stern *et al.*] $Q_{\beta,b}(y) = \frac{1}{8}\sum_B \gamma_B \chi_{y+B}$
- Free theory action in this basis: ($\xi_\mu = \gamma_\mu^*$)

$$\sum_{p_y} \overline{Q}(p_y)\left\{\underbrace{\left[\sum_\mu i(\gamma_\mu\otimes\mathbf{1})\sin p_{y,\mu} + (\mathbf{1}\otimes\mathbf{1})m_0\right]}_{O(a)} + \underbrace{\sum_\mu (\gamma_5\otimes\xi_\mu\xi_5)(1-\cos p_{y,\mu})}_{O(a^2)}\right\} Q(p_y)$$

More on free staggered action

$$\sum_{p_y} \overline{Q}(p_y) \left\{ \underbrace{\left[\sum_\mu i(\gamma_\mu \otimes \mathbf{1}) \sin p_{y,\mu} + (\mathbf{1} \otimes \mathbf{1}) m_0 \right]}_{O(a)} + \underbrace{\sum_\mu (\gamma_5 \otimes \xi_\mu \xi_5)(1 - \cos p_{y,\mu})}_{O(a^2)} \right\} Q(p_y)$$

- Wilson-like term removes doublers **and** breaks taste symmetry
- Critical mass ($m_0 = 0$) requires no tuning due to $U(1)_\epsilon$ symmetry:

$$Q \to \exp[i\alpha(\gamma_5 \otimes \xi_5)] , \qquad \overline{Q} \to \overline{Q} \exp[i\alpha(\gamma_5 \otimes \xi_5)] .$$

- Wilson-like term is irrelevant $\Rightarrow SU(4)$ taste restored in continuum limit
  - This naive analysis supported in presence of gauge fields by absence of additional relevant terms due to lattice symmetries [Golterman & Smit]
  - Further supported by RG framework of [Shamir]

Rooted staggered fermions

To obtain QCD (with 3 or 4 flavors) can:

1. Use tastes as flavors but make non-degenerate
   - Lose lattice symmetries, complicated, $D_{\rm stag}$ non-hermitian
2. Use one staggered fermion per flavor and take fourth-root of determinant:

$$Z_{\rm QCD}^{\rm root} = \int DU e^{S_g} \left(\det[D_{\rm stag}(m_u)] \det[D_{\rm stag}(m_d)] \det[D_{\rm stag}(m_s)]\right)^{1/4}$$

   - $\det[D_{\rm stag}(m)]$ is positive definite for $m \neq 0$: take positive root

Reasons for rooting: fast to simulate, and have $U(1)_\epsilon$ symmetry

Rationale: rooting legitimate in continuum limit, since taste-breaking vanishes

$\boxed{\textbf{Question: does taking fourth-root commute with sending } \mathbf{a \to 0}\textbf{?}}$

Non-locality for $a \neq 0$ [BGS]

Rooted staggered fermions cannot be described by a local theory with a single taste per flavor

- Assume that, for any gauge configuration [Adams]

$$(\det[D_{\rm stag}])^{1/4} = \det[D_1] \exp(-\delta S_{\rm eff,g})$$

with D_1 a local 1-taste operator and $S_{\rm eff,g}$ a local gauge action

- Then, for normal (= "unrooted") staggered fermions

$$\det[D_{\rm stag}] = (\det[D_1])^4 \exp(-4\delta S_{\rm eff,g}) = \det(D_1 \otimes \mathbf{1}) \exp(-4\delta S_{\rm eff,g})$$

- Compare fermionic contributions to gluonic correlators (e.g. $\langle F^2(x) F^2(0)\rangle$)
- Those on LHS do not have $SU(4)$ taste symmetry, while those on RHS do
- Difference particularly striking at long distances: LHS has 1 PGB, RHS has 15
- **Contradiction** $\Rightarrow \delta S_{\rm eff,g}$ non-local
  - No contradiction in perturbation theory (for $a \to 0$)

Implications of non-locality

A reasonable person at this stage might say:

> **"Locality of the action guarantees universality, i.e. that we obtain the correct continuum limit. Non-local theories are unphysical (lacking unitarity, . . .). The lore is that non-locality violates universality. I don't want to use rooted staggered fermions."**

Another reasonable person might say:

> "Rooted staggered fermions are so attractive numerically, that I am going to try and understand and "tame" the non-locality. If I can argue plausibly that the non-locality does not change the universality class, i.e. that the effects of non-locality vanish in the continuum limit, then the extensive numerical results based on the MILC ensemble will be physical."

NOTE: If rooted staggered fermions have the wrong continuum limit, then results using them are **WRONG**, not approximations to QCD.

Most of the remainder of this talk will concern attempts to tame the non-locality.

But first, I want to recall why the stakes are high.

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Update on rooted staggered simulations [Sugar]

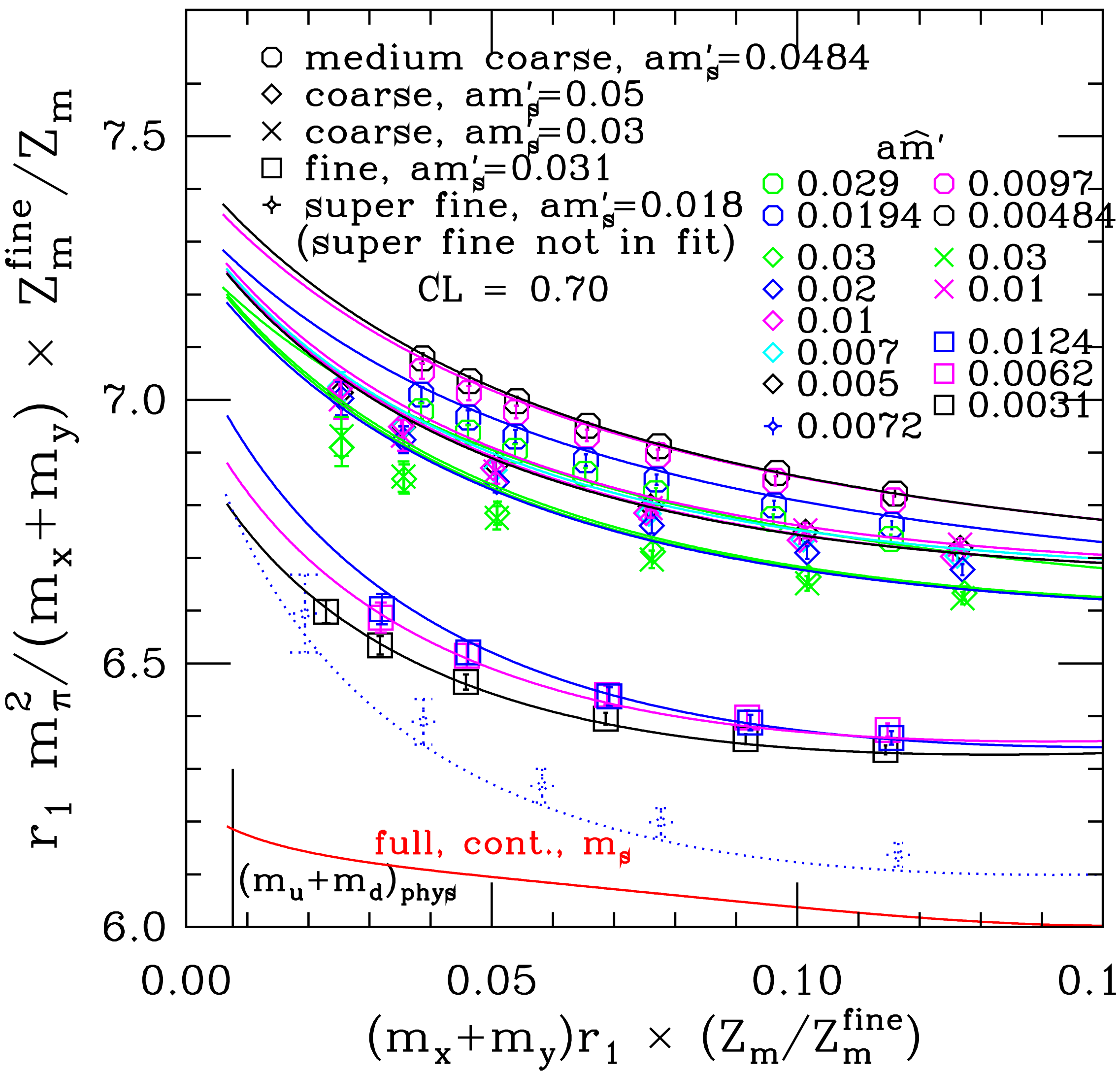
MILC ensemble now includes coarser and "super-fine" lattices:

a (fm)	$a\hat{m}'$ / am_s'	$10/g^2$	dims.	# lats.	m_π/m_ρ
≈ 0.15	0.0290 / 0.0484	6.600	$16^3 \times 48$	600	0.522(2)
≈ 0.15	0.0194 / 0.0484	6.586	$16^3 \times 48$	600	0.454(3)
≈ 0.15	0.0097 / 0.0484	6.572	$16^3 \times 48$	600	0.348(4)
≈ 0.15	0.00484 / 0.0484	6.566	$20^3 \times 48$	600	0.256(5)
≈ 0.12	0.03 / 0.05	6.81	$20^3 \times 64$	564	0.582(1)
≈ 0.12	0.02 / 0.05	6.79	$20^3 \times 64$	484	0.509(2)
≈ 0.12	0.01 / 0.05	6.76	$20^3 \times 64$	658	0.394(3)
≈ 0.12	0.01 / 0.05	6.76	$28^3 \times 64$	241	0.395(2)
≈ 0.12	0.007 / 0.05	6.76	$20^3 \times 64$	493	0.342(3)
≈ 0.12	0.005 / 0.05	6.76	$24^3 \times 64$	527	0.299(4)
≈ 0.12	0.03 / 0.03	6.81	$20^3 \times 64$	350	0.590(1)
≈ 0.12	0.01 / 0.03	6.76	$20^3 \times 64$	349	0.398(4)
≈ 0.09	0.0124 / 0.031	7.11	$28^3 \times 96$	531	0.495(2)
≈ 0.09	0.0062 / 0.031	7.09	$28^3 \times 96$	583	0.380(3)
≈ 0.09	0.0031 / 0.031	7.08	$40^3 \times 96$	473	0.297(3)
≈ 0.06	0.0072 / 0.018	7.48	$48^3 \times 144$	200	0.474(5)

Widely used [talks by Lee, Onogi, Orginos]

MILC results for $m_\pi^2/(m_x+m_y)$ [Sugar]

- Results for $m_\pi^2/(m_x+m_y)$
- Part of global fit to PGB properties
- Super-fine lattice results agree with predictions
- Partially quenched staggered chiral perturbation theory describes data well



Updated MILC Results: Masses

Accurate results for PGB properties + staggered chiral perturbation theory lead to successful comparisons with data (f_π, f_K [talk by Lee]) and determinations of quark masses.

Update: Super-fine (and coarser) lattices lead to very small changes:

$$
\begin{aligned}
m_s^{\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) &= 90(0)(5)(4)(0)\ \mathrm{MeV} \qquad && [87(0)(4)(4)(0)\ \mathrm{MeV}] \\
m_s/\hat{m} &= 27.2(0)(4)(0)(0) && [27.4(1)(4)(0)(1)] \\
m_u/m_d &= 0.42(0)(1)(0)(4) && [0.43(0)(1)(0)(8)]
\end{aligned}
$$

- Old results in red are from HPQCD, MILC & UKQCD (Aubin et al.b), Phys. Rev. D 70(R), 031504 (2004), MILC (Aubin et al.), PRD 70 (2004) 114501, and HPQCD (Mason et al.), PRD 73 (2006) 114501.
- Errors are from statistics, simulation, perturbation theory, and EM effects.

Predictions using rooted staggered fermions

Having checked that "gold-plated" PGB, nucleon, B, ψ and Υ properties agree with experiment, FNAL/MILC/HPQCD have made successful predictions for:

- $D \to K\ell\nu$ form factor (shape and normalization)
- f_D
- B_c mass

Summarized in [Kronfeld, hep-lat/0607011]

This is exactly how we hoped lattice QCD would be used.

Successes are impressive, but do not provide definitive demonstration that rooted staggered fermions are correct.

- Systematics are complicated—we would want checks even if there were no theoretical issue.
- It could be a fluke—the wrong theory happens to give results results close to experiment for some quantities but not others.
- There is a serious theoretical issue (non-locality) and it must be understood.

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Non-local interactions in statistical mechanics

- Considerable literature studying power-law interactions, e.g. Ising model

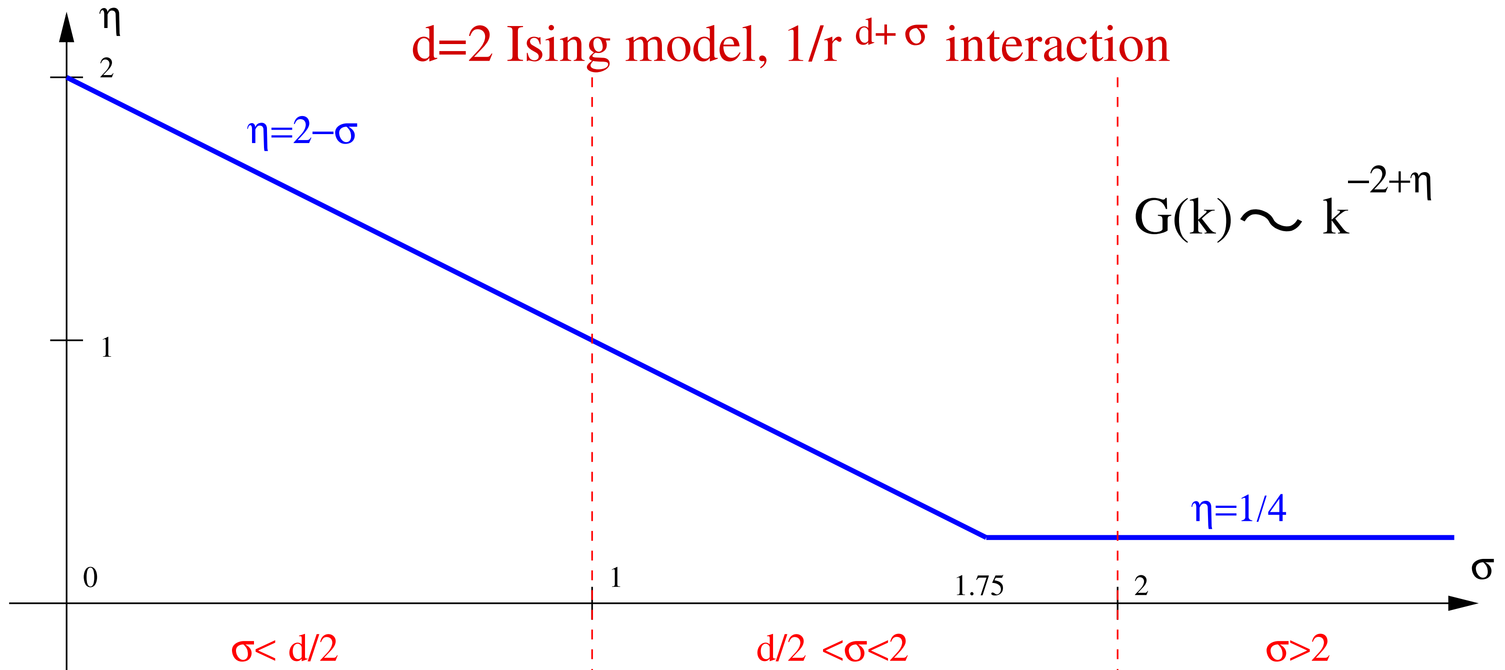
$$H = \mathcal{J} \sum_{\vec{x},\vec{y}} s_{\vec{x}} \text{``}\frac{1}{|x-y|^{d+\sigma}}\text{''} s_{\vec{y}}$$

- Studied using RG methods + ϵ expansion [Fisher,Ma & Nickel (1972), ..., Dantchev & Rudnick (2001)] , exact methods [Aizenman & Fernandez (1988), ...] , and numerical simulations [Luijten & Blöte (1997,2002), ...]

$$G(k)^{-1} \sim m + j_\sigma k^\sigma + j_2 k^2 + \dots$$

- Differ from QCD with rooted staggered fermions most notably because:
  - More non-local than QCD, which has dominant local gauge interaction
  - Power-law scaling (compared to logarithmic for QCD in $d = 4$)
- Nevertheless, perhaps can "demystify" non-locality

Example from statistical mechanics



$\sigma < d/2$
- Mean–field regime with dimensions dependent on σ

$d/2 < \sigma < 2$
- Non–trivial fixed point
- Exponents depend on σ
- Details resolved by simulations

$\sigma > 2$
- Interaction effectively short ranged
- Non–trivial exponents, independent of σ
- Non–local interactions determine finite–size corrections

Lessons from statistical mechanics

- Non-local interactions need not change the universality class
- Can analyze using RG equations ($=$ perturbation theory)
- Check using numerics—does α_s run as predicted?
- Beware of enhanced finite size corrections.

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What do we learn from perturbation theory?

- Assume renormalizability for unrooted staggered fermions
- Holds for any integer number of "replicas", n_r, of unrooted staggered fermions
- Since amplitudes and counterterms are polynomial in n_r (which counts fermion loops), renormalizability extends to any n_r [Bernard & Golterman]
- An arbitrary n_r in PT is obtained by using $\det^{n_r}(D_{\rm stag})$ in Z
- Thus PT for rooted staggered fermions (with $n_r = 1/4$) is renormalizable, and get $\beta-$function and anomalous dimensions for $N_f = 4n_r = 1$ flavors

$\Rightarrow$? Rooting leads to the desired physical theory in perturbation theory (for $a \to 0$) **Really need more work to establish this?**[Kennedy]

  - Why? Because it does not change form of propagator or vertices

▷ Discretization errors scale to zero as a^2 (up to logs)

▷ NOTE: renormalizability holds for any n_r, e.g. $n_r = \pi/4 \Rightarrow N_f = \pi$

  - But only for integral N_f is resulting perturbative theory unitary

Important to extend renormalizability proof to staggered fermions, and complete argument on transition to $\mathbf{n_r = 1/4}$

Conclusions from perturbation theory

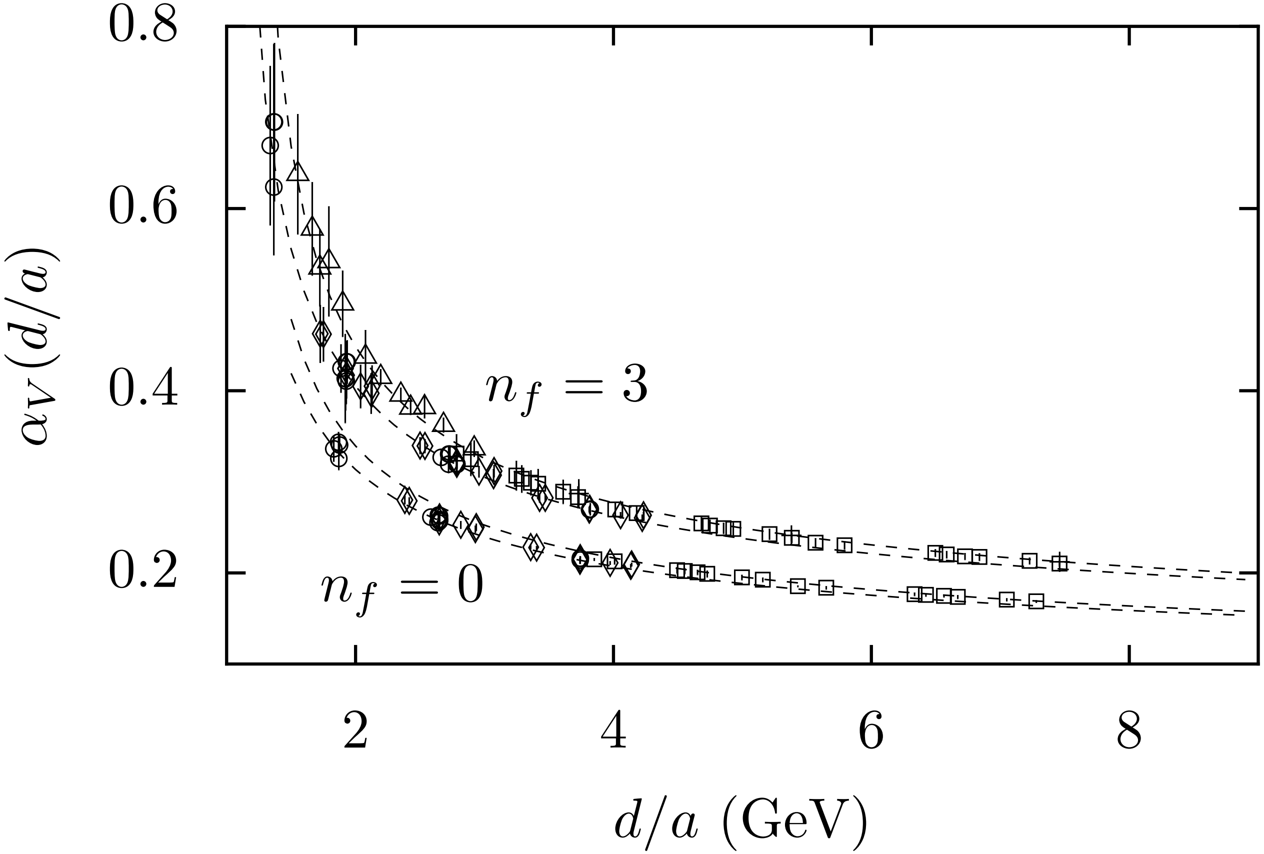
If we accept that rooting gives the correct perturbative fixed point, are we done?

No!

1. Non-perturbative effects at short-distances could invalidate the analysis
   - For Yang-Mills theory (and QCD with Wilson fermions) we checked this by studying scaling of α_s non-perturbatively
2. We know from [BGS] that the theory is non-local and thus unphysical for any $a \neq 0$, but the perturbative fixed point is physical.
   - Inconsistency? Not if non-locality vanishes as $a \to 0$.
   - We need to understand the non-locality in more detail in order to make such a possibility plausible.
3. Because we do not have a 1-taste construction, we do not have a standard RG framework to classify the (ir)relevance of operators
   - We need an appropriately extended framework

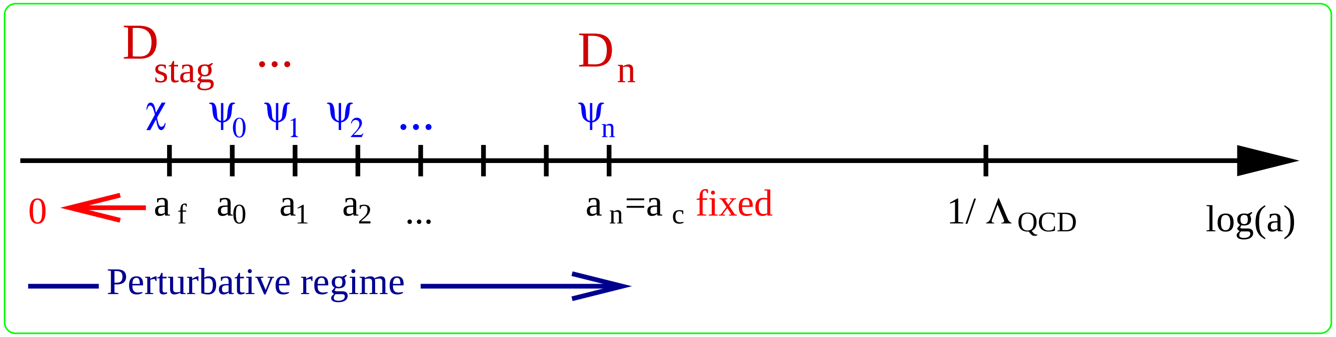
Scaling of α_s [Mason *et al.*, hep-lat/0503005]

Compare Wilson loops to two loop perturbation theory to extract $\alpha_V(\mu)$



Scaling matches four-loop running with $N_f = 3$

Bounding non-locality using RG analysis [Shamir]



- Standard RG set up, except first step goes from single-component to taste basis
- Gauge invariance and other lattice symmetries maintained
- Use Gaussian blocking kernels with local map $Q^{(k)}$ and $\alpha_k \sim a_k^{-1}$:

$$\int D\bar{\psi}^{(k-1)} D\psi^{(k-1)} e^{-\alpha_k [\bar{\psi}^{(k)\dagger} - \bar{\psi}^{(k-1)\dagger} Q^{(k)\dagger}][\psi^{(k)} - Q^{(k)} \psi^{(k-1)}]}$$

- Integrate out all fermions except $\psi^{(n)}$ while keeping *all* levels of gauge fields

$$\det[D_{\text{stag}}] \propto \prod_{k=0}^{n} \underbrace{\det[G_k^{-1}]}_{\exp(-4S_{\text{eff}}^{(k)})} \quad \det[D_n]$$

RG analysis: locality of $S_{\rm eff}^{(k)}$

- For unrooted staggered fermions, expect locality:

$$\underbrace{\det[D_{\rm stag}]}_{\text{local at scale } a_f} \propto \prod_{k=0}^{n} \underbrace{\exp[-4S_{\rm eff}^{(k)}]}_{\text{local at scale } a_k} \quad \underbrace{\det[\,D_n\,]}_{\text{local at scale } a_c}$$

- Can understand locality of effective gauge actions $S_{\rm eff}^{(k)} = \mathrm{tr}\ln G_k/4$ because G_k^{-1} are like Wilson-Dirac operators with a large negative mass, e.g.

$$G_0^{-1} = D_{\rm stag} + \alpha_0 Q^{(0)\dagger} Q^{(0)}$$

- G_k^{-1} can have small eigenvalues (below the "mobility edge") on rough gauge configurations, but eigenvectors are localized so G_k remains local at a_k

  $\Rightarrow$ $D_k = \alpha_k - \alpha_k^2 Q^{(k)} G_k Q^{(k)\dagger}$ is local at a_k

- Extends understanding of locality of $S_{\rm eff}^{(k)}$ beyond perturbation theory

$\Rightarrow$ **Plausible that $\mathbf{S_{eff}^{(k)}}$, and thus $\mathbf{D_n}$, remain local on rooted ensemble**

$$\underbrace{\det[D_{\rm stag}]^{1/4}}_{\text{non-local 1-taste theory due to root}} \propto \prod_{k=0}^{n} \underbrace{\det[G_k^{-1}]^{1/4}}_{\exp(-S_{\rm eff}^{(k)})} \quad \underbrace{\det[D_n]^{1/4}}_{\text{non-locality here, in IR}}$$

RG analysis: bounding non-localities

- Decompose blocked Dirac operator (local on both ensembles)

$$D_n = \underbrace{\widetilde{D}_{\mathrm{inv},n} \otimes \mathbf{1}}_{\text{taste invariant}} + \underbrace{\Delta_n}_{\text{taste-breaking}}$$

- Expect standard RG scaling of "dimension-5" $\Delta_n \sim \partial^2$ between a_f and a_c in **both** ensembles since interactions are local in that range ("true" in PT)

$$||\Delta_n|| \le \frac{a_f}{a_c^2}(1 + \mathrm{logs})$$

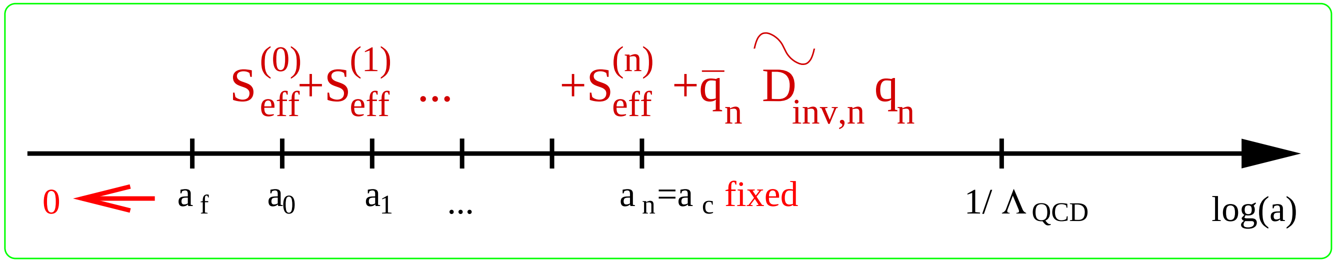
$\Rightarrow$ **Non-localities are bounded on rooted ensemble!** (Using $||\widetilde{D}^{-1}_{\mathrm{inv},n}|| \lesssim 1/m(a_c)$)

$$\begin{aligned}
\det[D_n]^{1/4} &= \det[\widetilde{D}_{\mathrm{inv},n} \otimes \mathbf{1}]^{1/4} \det[1 + \Delta_n(\widetilde{D}_{\mathrm{inv},n} \otimes \mathbf{1})^{-1}]^{1/4} \\
&= \underbrace{\det[\widetilde{D}_{\mathrm{inv},n}]}_{\text{local 1-taste theory}} \quad \exp\{\underbrace{(1/4)\mathrm{tr}\ln[1 + \Delta_n(\widetilde{D}_{\mathrm{inv},n} \otimes \mathbf{1})^{-1}]}_{\text{non-locality vanishes as } a_f \to 0}\}
\end{aligned}$$

RG analysis: conclusion

- Based on two plausible (and to some extent testable) assumptions:
  1. Locality of G_k on rooted ensemble
  2. Scaling of Δ_n on rooted ensemble

  can show that, when $a_f \to 0$, rooted staggered QCD is equivalent to a theory which is local at scale a_c and manifestly in same universality class as QCD
- **Can now send $\mathbf{a_c \to 0}$ and obtain continuum QCD**
- Alternatively, for $a_f \neq 0$ define a 1-taste/flavor theory, local at scale a_c ("reweighted theory")



  In same universality class as QCD, and becomes equivalent to rooted staggered fermions when $a_f \to 0$
- However, for any $a_f \neq 0$, rooted staggered fermions **will have** taste-breaking non-localities in the IR, because $\Delta_n \neq 0$ (consistent with [BGS])

RG analysis: how does it work?

- After one blocking in free theory ($\bar{p}_\mu = \frac{\sin ap_\mu}{a}$, $\hat{p}_\mu = \frac{2\sin ap_\mu/2}{a}$) [BGS]

$$D_{\mathrm{inv},0} = \frac{\sum_\mu i(\gamma_\mu \otimes \mathbf{1})\bar{p}_\mu + (\mathbf{1}\otimes\mathbf{1})[m + \alpha_0^{-1}(\hat{p}^2 + m^2)]}{1 + 2m\alpha_0^{-1} + \alpha_0^{-2}(\hat{p}^2 + m^2)}$$

$$\Delta_0 = \frac{a_f \sum_\mu (\gamma_5 \otimes \xi_\mu \xi_5)\hat{p}_\mu^2}{1 + 2m\alpha_0^{-1} + \alpha_0^{-2}(\hat{p}^2 + m^2)}$$

- Separated removal of doublers from taste-breaking:
  - $D_{\mathrm{inv},0}$ is Wilson-like, with no doublers
  - Δ_0 breaks taste, but not needed to remove doublers
- Under further blocking Δ_n gets smaller, and $D_{\mathrm{inv},n}$ approaches a GW operator

Understanding non-locality using EFT [Bernard]

If partially quenched (PQ) staggered chiral perturbation theory (SχPT) is a valid effective field theory (EFT) for unrooted but partially quenched staggered fermions, (and accepting some plausible technical assumptions,) then:

1. **The correct EFT for rooted staggered fermions is "SχPTwith the replica trick" (rSχPT);**
2. **This theory is unphysical for $a \neq 0$ (as required by [BGS]) but contains a subsector which becomes χPT for QCD when $a = 0$.**

- This argument provides alternate to RG approach for taming non-localities
  - Advantage: gives explicit formulae (essential for fitting)
  - Disadvantages:
    1. Useful only where χPT is useful (mainly the PGB sector)
    2. Relies on PQχPT, whose theoretical foundations are not as strong as for χPT [SS & Shoresh]
- Surprising result: "rooted SχPT" involves the replica trick and seems somewhat *ad hoc*

What is (rooted) staggered χPT?

- EFT for single flavor of (four-taste) staggered fermions, including discretization errors, is SχPT of [Lee & SS]

- EFT for n_r copies of $D_{\rm stag}(m_u)$, n_r copies of $D_{\rm stag}(m_d)$, *etc.* (a local lattice theory if n_r integral, with N_c adjusted so asymptotically free) is SχPT of [Aubin & Bernard]
  - EFT results are polynomial in n_r at any finite order.

- rSχPT is defined by setting $n_r = 1/4$ (and is usually called just SχPT) [Aubin & Bernard]
  - EFT analog of rooting done at quark level
  - Low energy constants are taken to be those of presumed continuum theory (here $N_f = 2+1$ QCD)

Essence of argument

- Consider four flavors of rooted staggered fermions:

$$Z^{\rm root}(m_1, m_2, m_3, m_4) \quad = \quad \int d\mu_g \left\{ \prod_{i=1}^{4} \det[D_{\rm stag}(m_i)] \right\}^{1/4}$$

- Related to theories of interest:

$$\begin{aligned} Z^{\rm stag}(m) &= \int d\mu_g \det[D_{\rm stag}(m)] = Z^{\rm root}(m, m, m, m) \\ Z^{\rm MILC}(m_\ell, m_s) &= Z^{\rm root}(m_\ell, m_\ell, m_s, 1/a) \end{aligned}$$

- Idea is to go (within the EFTs)
  - from $Z^{\rm stag}$ (a physical theory with known EFT)
  - to $Z^{\rm MILC}$ (the rooted theory being simulated)

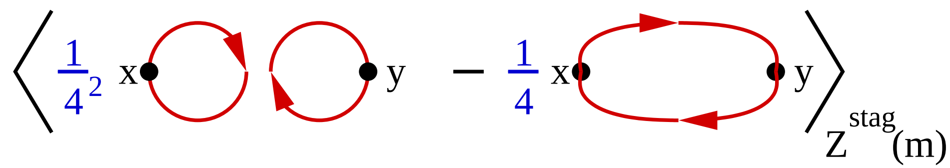
  by calculating all derivatives w.r.t. m_i and summing the series
- Assumes no non-analyticities (plausible since m_i are positive)
- One is **constructing** EFT for $Z^{\rm MILC}$

Key steps in EFT argument (1)

- Derivatives of Z^{root} evaluated for m_i equal are "unrooted":

$$Z^{\text{root}}(m_1, m_2, m_3, m_4) \quad = \quad \int d\mu_g \left\{ \prod_{i=1}^{4} \exp[\frac{1}{4} \text{tr} \ln D_{\text{stag}}(m_i)] \right\}$$

$$C(x,y) = \frac{\partial^2 \ln Z^{\text{root}}}{\partial m_4(x) \partial m_4(y)} \bigg|_{m_i \ \text{equal}} =$$



  - Rooting gives rise to factors of $(1/4)^{\# \text{loops}}$
  - Correlators **cannot** be obtained by applying $\partial/\partial m$ on $Z^{\text{stag}}(m)$
  - Although evaluated on physical ensemble, correlators are partially quenched and thus unphysical
  - We "know" EFT for PQ correlators (PQSχPT) so, in principle, we know all derivatives and can sum
  - Result is rSχPT for four flavors

Traded rooting for PQing!

Explicit form of intermediate PQ theory

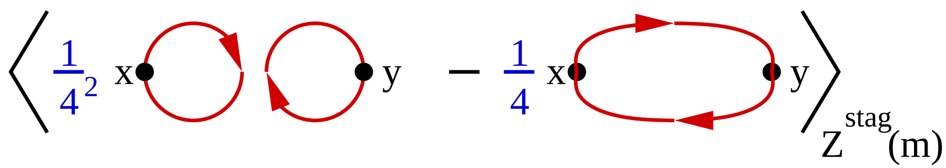
- Need to consider PQ but unrooted staggered theory

$$Z^{\rm PQstag}(m, M_V, \widetilde{M}_V) = \int d\mu_g \det[D_{\rm stag}(m)]\times$$

$$\times \prod_{i=1}^{N_V} D\bar\chi_{V_i} D\chi_{V_i} D\widetilde\chi^\dagger_{V_i} D\widetilde\chi_{V_i} \;\exp[-\underbrace{\bar\chi_{V_i} D_{\rm stag}(M_{V,ij})\chi_{V_j}}_{\text{valence quarks}} - \underbrace{\widetilde\chi^\dagger_{V_i} D_{\rm stag}(\widetilde{M}_{V,ij})\widetilde\chi_{V_j}}_{\text{ghost quarks}}]$$

- Derivatives of $Z^{\rm root}$ wrt m_i related to those of $Z^{\rm PQstag}$ wrt M_V, e.g.

$C(x,y) =$



$$= \frac{1}{4^2}\frac{\partial^2 \ln Z^{\rm PQstag}}{\partial M_{V,11}(x)\partial M_{V,22}(y)}\Big|_{M_V=\widetilde{M}_V=m} + \frac{1}{4}\frac{\partial^2 \ln Z^{\rm PQstag}}{\partial M_{V,12}(x)\partial M_{V,21}(y)}\Big|_{M_V=\widetilde{M}_V=m}$$

Key steps in EFT argument (2)

- At this stage, have EFT for $Z^{\rm root}(m_i)$ if all $m_i \ll \Lambda_{\rm QCD}$:
  - four-flavor rSχPT
- Send $m_4 \to 2m_s$ (edge of validity of χPT):
  - EFT for 3 light flavors is three-flavor rSχPT (decoupling within EFT)
- Send $m_4 \to 1/a$: EFT should not change form, although LECs change

  $\Rightarrow$ $Z^{\rm MILC}$ described by three-flavor rSχPT
  - Unphysical for $a \neq 0$ but becomes QCD χPT for $a = 0$
  - Must rely on RG argument to show that, since get correct continuum limit, LECs must be those of QCD
- Can extend to 2 and 1 flavors: again, correct continuum limit is built in
  - Resolves 1 flavor "paradox" of having unwanted pions—they decouple

Outline

- What is rooting and what are its potential problems?
  - Non-locality, lack of unitarity [BGS]
- Why is it being used, and what is the status of results?
  - What are the stakes? [MILC, FNAL, HPQCD]
- What do we learn about non-local theories from statistical mechanics?
- Can we tame the non-locality?
  - Perturbation theory
  - Renormalization Group (RG) analysis [Shamir]
  - Effective field theory for rooted staggered fermions [Bernard]
- Do rooted staggered fermions have too many symmetries? [Creutz, BGSS, DH]
- Problems with rooting at non-zero chemical potential [GSS]
- Conclusions

Diseases of rooted staggered fermions?[Creutz,BGSS]

- Rooted theory maintains $U(1)_\epsilon$ symmetry of four-taste (unrooted) theory:

$$\begin{aligned}(\det[D_{\rm stag}(m)])^{N_f/4} &= \left(\det[D_{\rm stag}(me^{i\epsilon(n)\theta_\epsilon})]\right)^{N_f/4} \\ &= (\det[D_{\rm stag}(-m)])^{N_f/4} \quad (\theta_\epsilon = \pi)\end{aligned}$$

  - $\epsilon(n) = \pm 1$ for even/odd sites

  $\Rightarrow$ **Rooted theory is an even function of m for any $\mathrm{N_f}$**

- In finite volume and for $a \neq 0$ there can be no non-analyticities for real m

  $\Rightarrow$ **Rooted theory in finite volume is a function of $\mathrm{m^2}$ (not $|\mathrm{m}|$)**

- Continuum theory (or lattice theory with overlap fermions) with odd N_f has no $m \to -m$ symmetry due to $U(1)_A$ anomaly

  $\Rightarrow$ **Continuum theory in finite volume is a general function of m**

- Inconsistency? Disease? **No! A limitation, or ugly feature**
  - Rooted staggered fermions are only claimed to give physical behavior in continuum limit
  - Can show how odd powers of m occur naturally

How rooted staggered fermions can work for $N_f = 1$

- Not proof; demonstration that there is no inconsistency
- Odd powers of m arise from zero modes of $D_{\rm cont}$, e.g.

$$\det[D_{\rm cont}(m)] = m\ F_{\rm cont}(m^2) \quad (\nu = 1)$$

  - ▷ $F_{\rm cont}(m^2) \sim \prod_{\lambda>0}(i\lambda + m)(-i\lambda + m) > 0$
- Staggered fermions have no index theorem, so would-be zero modes become a quartet of two pairs connected by $U(1)_\epsilon$

$$\{\det[D_{\rm stag}(m)]\}^{1/4} = \left\{[(\lambda_1^{\rm stag})^2 + m^2][(\lambda_2^{\rm stag})^2 + m^2] F_{\rm stag}(m^2)\right\}^{1/4}$$

  - ▷ $\lambda_{1,2}^{\rm stag} \propto a$ (or a^2?) and $F_{\rm stag} > 0$
  - ▷ Manifestly a function of m^2
- Recover expected form when take continuum limit

$$\left\{[(\lambda_1^{\rm stag})^2 + m^2][(\lambda_2^{\rm stag})^2 + m^2]\right\}^{1/4} \underset{a\to 0}{\longrightarrow} |m|$$

  - ▷ Rooting (with positive root) gives continuum form, but with **positive** continuum mass, regardless of the sign of the staggered fermion mass

Non-commutativity of $a \to 0$ and $m \to 0$ limits?

- Expect limits to commute for physical quantities in rooted staggered QCD (with $m_q > 0$) [Bernard]
- Expect non-commutativity if one quark mass vanishes [Durr, BGSS, DH]
- Example: condensate for $N_f = 1$

$$\langle \bar\psi \psi \rangle_{\rm cont} = -\frac{1}{Z_{\rm cont} V} \frac{\partial Z_{\rm cont}(m)}{\partial m} \underset{m \to 0}{\longrightarrow} \text{non-zero constant}$$

non-zero at $m = 0$ (even in finite volume) due to $\nu = 1$ zero mode

- With rooted staggered fermions condensate is odd function of m

$$\langle \bar\psi \psi \rangle_{\rm stag} \propto \frac{m[(\lambda_1^{\rm stag})^2 + (\lambda_2^{\rm stag})^2 + 2m^2]}{\left\{[(\lambda_1^{\rm stag})^2 + m^2][(\lambda_2^{\rm stag})^2 + m^2]\right\}^{3/4}}$$

$$\underset{m \to 0}{\longrightarrow} 0 \qquad \underline{\textbf{WRONG ANSWER}}$$

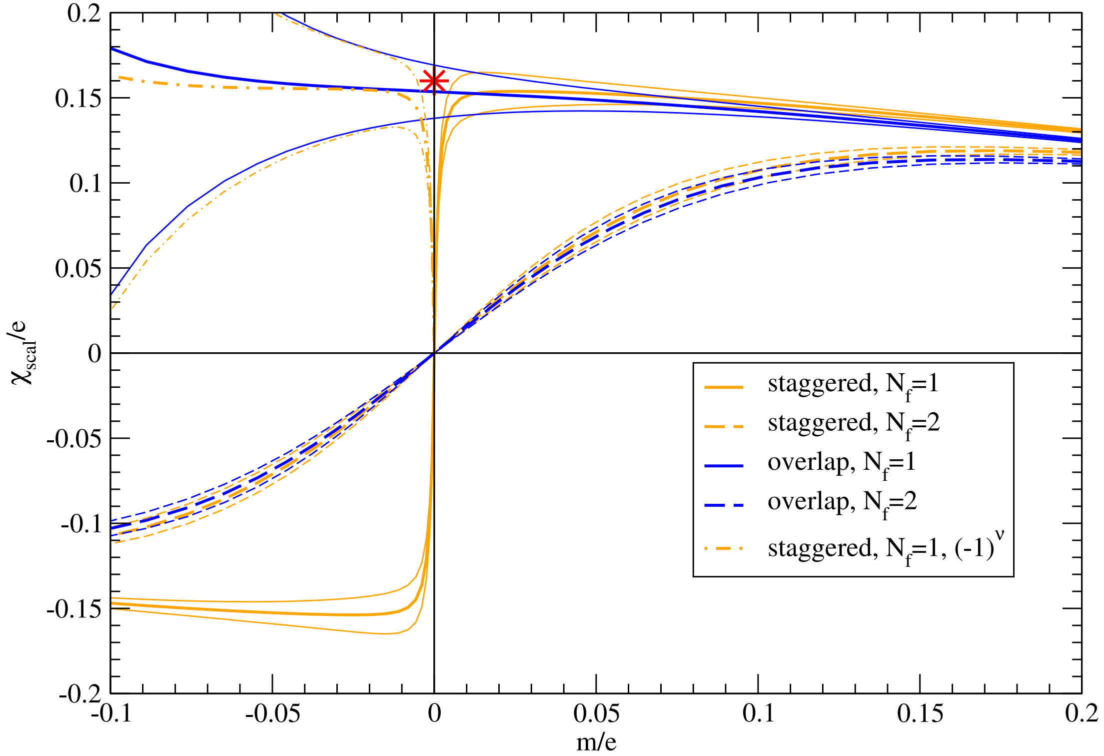
$$\underset{a \to 0}{\longrightarrow} 2\,\mathrm{sign}(m) \qquad \underline{\textbf{CORRECT for } m > 0}$$

- Disease? **No.** Ugliness? **Yes—need care with limits.**

Numerical study in Schwinger model [DH]

Analogous issues arise from square-rooting of two-taste staggered determinant

Compare condensate in rooted staggered to one-flavor overlap



Study $m < 0$ using rooted staggered with mass $|m|$ plus $\theta = \pi$ term!

Consistent with explanation of BGSS

No similar issue when $N_f = 2$—analog of $2 + 1$ QCD

More on “extra” symmetries

$U(1)_\epsilon$ symmetry leads to Ward identities in rooted theory. Are these inconsistent with expected properties of continuum theory?

- **No! Can show that it is one of many extra symmetries due to rooting that have no impact on continuum theory**

Two examples show this:

1. rSχPT: it is $U(1)\epsilon$ symmetric and yet has physical QCD subsector
2. Extended continuum theory with exact taste symmetry introduced by hand (expected continuum limit of rooted staggered theory)

$$\det{}^{1/4}\left[(D_{\rm cont}(M)\otimes \mathbf{1}) + J\right] = \det[D_{\rm cont}(M)]\exp\left\{\frac{1}{4}\mathrm{tr}\ln[1 + J(D_{\rm cont}(M)\otimes\mathbf{1})^{-1}]\right\}$$

- Can derive exact (but unphysical) Ward identities using taste non-singlet J
- Includes $U(1)_\epsilon$ Ward identities (since $U(1)_\epsilon$ has taste ξ_5)
- If set $J \to (\widetilde{J}\otimes\mathbf{1})$ then generate correlation functions of QCD

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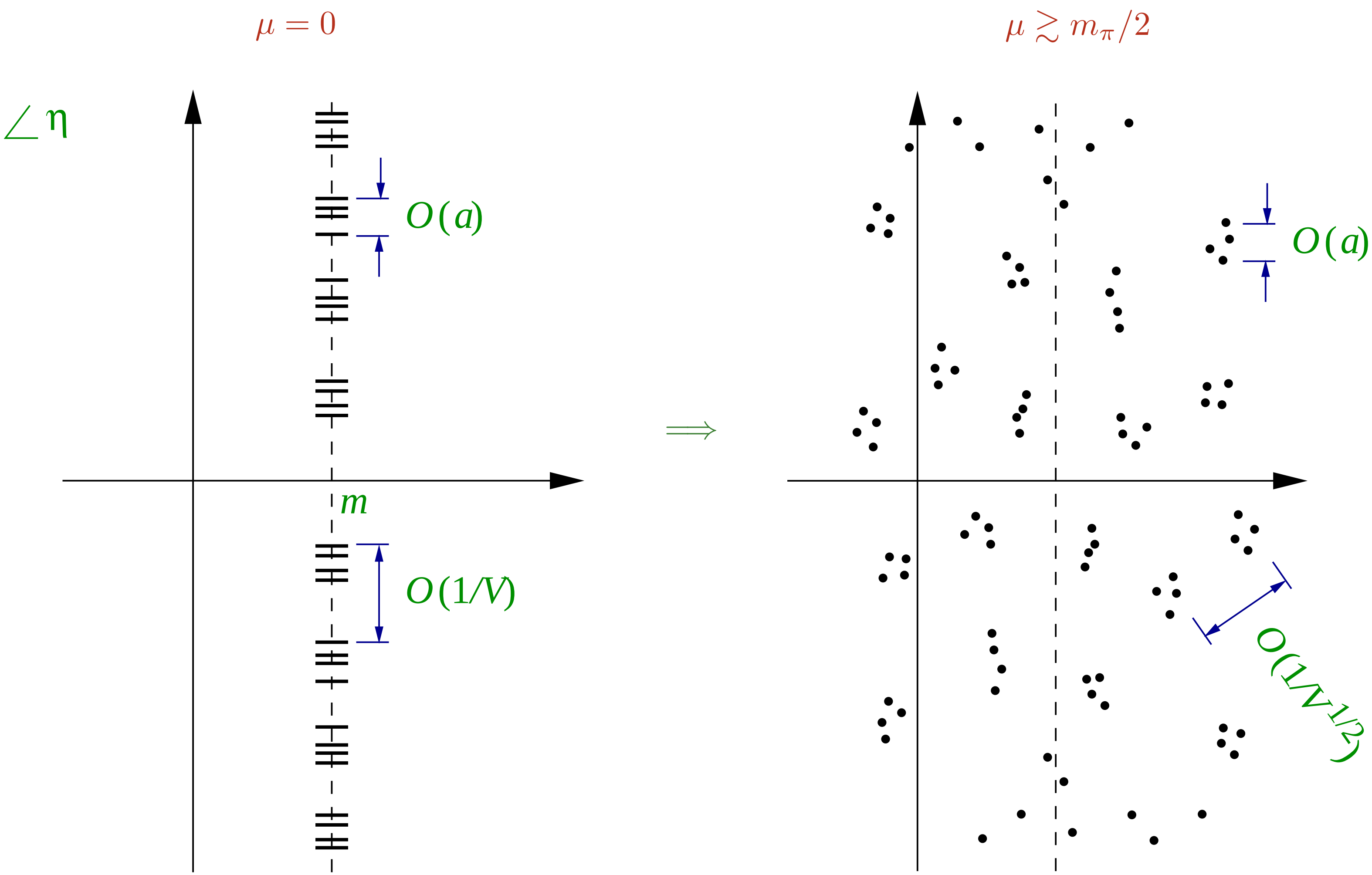
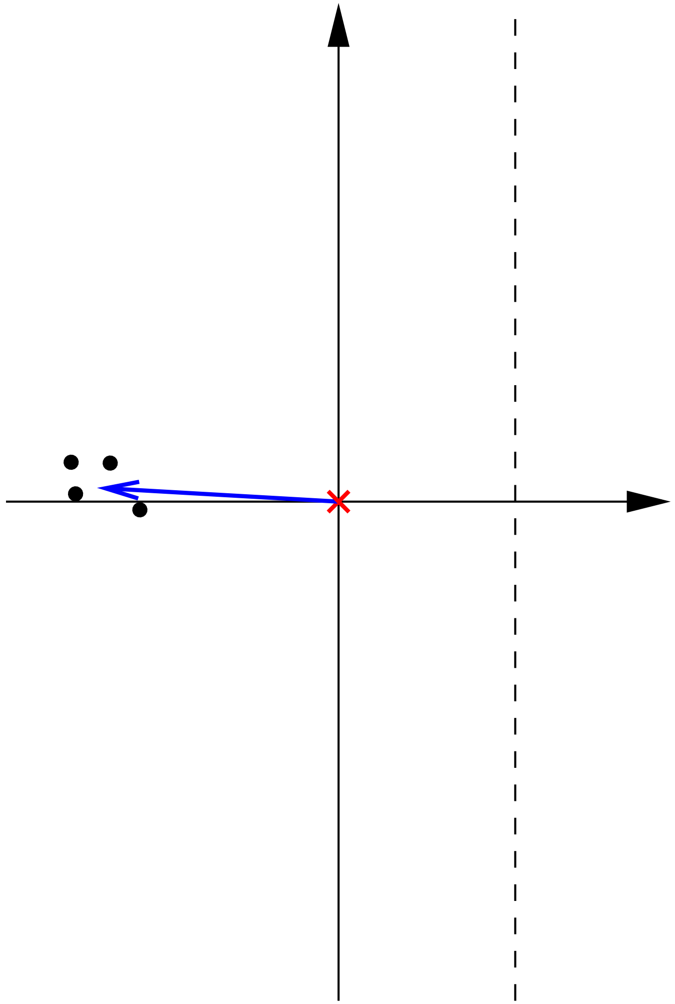
Eigenvalues of staggered $D(U) + m$

(Akemann et al. hep-th/0411030)



Fourth root of $\Delta[U] = \mathrm{Det}\,[D(U)+m]$



Solution ("ideal prescription"):

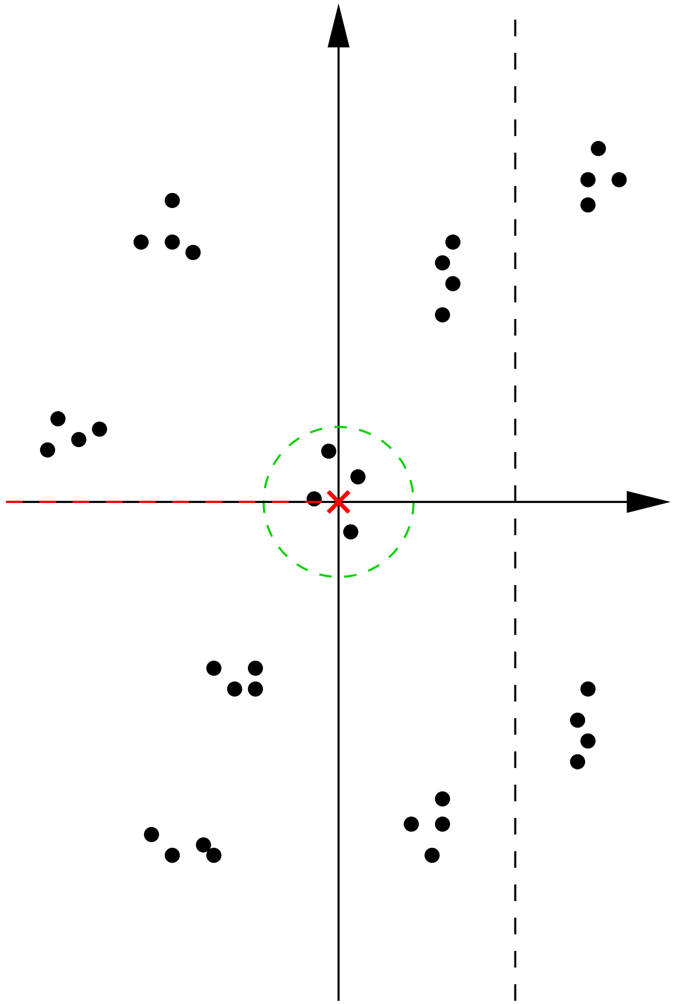
Define phases of $\eta_i \in \mathrm{quartet}$ to be close to each other; then

$$\arg\left(\prod_1^4 \eta_i\right)^{1/4} \equiv \frac{1}{4}\sum_1^4 \arg\eta_i$$

$\Rightarrow$ NO jumps in phase when any η_i crosses axis.

Smooth replacement of four tastes by one quark per flavor

Problematic configurations:



NO clear definition of phase of root!
$\Rightarrow$ Such configurations should be dropped.

Probability of prob. config:

$O(a^2 V \Lambda^6)$	quenched
$O[(a\sqrt{V}\Lambda^3)^3]$	reweighted

$=$ systematic error of algorithm

Note:
Volume required is fixed by physics (e.g., $m_\pi L \gtrsim 3$).
We must take $a \to 0$ before $V \to \infty$.

- similar to requirement $a \to 0$ before $m \to 0$ (`Bernard hep-lat/0412030`)

Conclusions on $\mu \neq 0$

- Rooting leads to unavoidable ambiguities when $\mu \gtrsim m_\pi/2$ (independent of issues discussed by [Splittorff])
- Systematic error grows with volume and for present simulations is $\gtrsim 200\%$; need much smaller lattice spacings
- Criticism does not apply to Taylor expansions about $\mu = 0$ or use of imaginary μ.

Conclusions

BAD or UGLY?

My conclusion: Ugly

- Plausible theoretical arguments are now added to the numerical evidence that rooted staggered fermions have the correct continuum limit.
- Picture that emerges: Non-locality/non-unitarity is present, but is pushed into the IR, is bounded, and vanishes when $a \to 0$. Furthermore, we have a plausible understanding of the far IR using rSχPT.
  - Not pretty, and systematic errors due to non-locality can be significant
  - Limits utility of purely staggered simulations—mixed-action simulations may be preferable for many quantities
- Plausibility is in the eye of the beholder—need further work to study assumptions of arguments
  - Prove renormalizability of unrooted staggered fermions
  - Provide better basis for PQχPT
  - Test assumptions of RG argument
- As always, we need to cross-check all numerical results with other discretizations—true irrespective of rooting issue.

Fire away!

Perturbation theory for staggered fermions

[Sharatchandra, Thun & Weisz, Kawamoto & Smit, Golterman & Smit, ...]

- PT for unrooted staggered fermions involves 2^4 poles in Brillouin zone (BZ)
- Repackage calculation using reduced BZ ($-\pi/2 < p'_\mu \le \pi/2$) plus hypercube vector labeling reduced zones
  - Propagator is taste symmetric: (A, B hypercube vectors)

$$G^{-1}(q' + B\pi, p' + A\pi) = \bar{\delta}(q' + p') \left[\sum_\mu i \sin q'_\mu \overline{\overline{(\gamma_\mu \otimes \mathbf{1})}}_{BA} + m \overline{\overline{(\mathbf{1} \otimes \mathbf{1})}}_{BA} \right]$$

  - Vertices break taste symmetry in general, e.g. $\bar{q}qg$ vertex ($C_\mu = 0$)

$$V_\mu(\underbrace{q' + \pi B}_{q}, \underbrace{p' + \pi A}_{\bar{q}}, \underbrace{k' + \pi C}_{g}) = -ig\bar{\delta}(p' + q' + k') \cos(q'_\mu + k'_\mu/2) \overline{\overline{(\gamma_{\mu\widetilde{C}} \otimes \xi_{\widetilde{C}})}}_{AB}$$

    but not when gluon is nearly physical ($C = 0$)

- Need to show that:
  - continuum-like parts (all momenta nearly physical), which are taste symmetric, have same logarithmic divergences as in continuum
  - gluons with $k \sim O(1)$ coupling to nearly on-shell fermions give short-distance artifacts (and thus finite corrections) which are taste symmetric due to momentum conservation

Other criticisms from [Creutz]

- Is it plausible that IR and UV "cutoffs" (i.e. $m \to 0$ and $a \to 0$) do not commute in theory with no physical massless particles?
  - Yes. It can be understood as an "Aoki-phase" phenomenon in the extended theory.