Rooted Staggered Fermions: Good, Bad or Ugly?

Status report on the validity of the procedure of representing the determinant for a single fermion by the fourth root of the staggered fermion determinant.

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The Possibilities

- **GOOD**: Correct continuum limit.
- **BAD**: Wrong continuum limit.
- **UGLY**: Correct continuum limit, but unphysical contributions present for $a \neq 0$, requiring theoretical understanding and complicated fits.
Significant progress in the last year

Issues have been clarified and in some cases resolved.
I will focus on the following, mainly analytic, papers (in this order):


- Y. Shamir, “Renormalization-group analysis of the validity of staggered-fermion QCD with the fourth-root recipe,” hep-lat/0607007.


- M. Creutz, “Flavor extrapolations and staggered fermions,” hep-lat/0603020.


Other progress

Work I will not cover (my apologies):

- A. Hart, “Improved staggered eigenvalues and epsilon regime universality in $SU(2)$”, talk Tue.
- C. DeTar, “Taste breaking effects in scalar meson correlators”, talk Thur.
- E. Gregory, “Pseudoscalar flavor-singlets and staggered fermions”, talk Thur.

Previous discussions:

Outline

- What is rooting and what are its potential problems?
  - Non-locality, lack of unitarity [BGS]

- Why is it being used, and what is the status of results?
  - What are the stakes? [MILC, FNAL, HPQCD]

- What do we learn about non-local theories from statistical mechanics?

- Can we tame the non-locality?
  - Perturbation theory
  - Renormalization Group (RG) analysis [Shamir]
  - Effective field theory for rooted staggered fermions [Bernard]

- Do rooted staggered fermions have too many symmetries? [Creutz, BGSS, DH]

- Problems with rooting at non-zero chemical potential [GSS]

- Conclusions
What I will assume

- Violations of reflection positivity by improved actions are cut-off phenomena which do not effect long-distance physics
  - Improved Wilson, staggered, …

- Lattice QCD with Wilson, GW, … fermions ("anything but staggered") has the correct continuum limit

- Lattice QCD with “unrooted” staggered fermions has the correct continuum limit (with four degenerate “tastes”)
  - Includes assumption of perturbative renormalizability with correct $\beta$–function
  - Step towards proof provided by power-counting theorem of [Giedt]
Staggered fermions

Simple action: [Susskind]

\[ \bar{\chi} D_{\text{stag}} \chi = \sum_n \bar{\chi}_n \left[ \sum_\mu \frac{\eta_{n,\mu}}{2} \left( U_{n,\mu} \chi_{n+\mu} - U_{n-\mu,\mu}^\dagger \chi_{n-\mu} \right) + m_0 \chi_n \right] \]

- In practice use improved form (smeared \( U \), Naik term, \ldots)

\( 2^4 \) doublers = 4 Dirac components \( \times 4 \) tastes in classical continuum limit

- Hypercube basis: [Gliozzi, Kluberg-Stern et al.] \( Q_{\beta,b}(y) = \frac{1}{8} \sum_B \gamma_B \chi_{y+B} \)
- Free theory action in this basis: \( (\xi_\mu = \gamma^*_\mu) \)

\[ \sum_{p_y} \bar{Q}(p_y) \left\{ \left[ \sum_\mu i (\gamma_\mu \otimes 1) \sin p_{y,\mu} + (1 \otimes 1) m_0 \right] + \sum_\mu (\gamma_5 \otimes \xi_\mu \xi_5) (1 - \cos p_{y,\mu}) \right\} Q(p_y) \]

\( O(a) \) \( O(a^2) \)

More on free staggered action

\[ \sum_{p_y} \overline{Q}(p_y) \left\{ \left[ \sum_{\mu} i(\gamma_\mu \otimes 1) \sin p_y,\mu + (1 \otimes 1)m_0 \right] + \sum_{\mu} (\gamma_5 \otimes \xi_\mu \xi_5)(1 - \cos p_y,\mu) \right\} Q(p_y) \]

- Wilson-like term removes doublers and breaks taste symmetry
- Critical mass \((m_0 = 0)\) requires no tuning due to \(U(1)_\epsilon\) symmetry:
  \[ Q \rightarrow \exp[i\alpha(\gamma_5 \otimes \xi_5)], \quad \overline{Q} \rightarrow \overline{Q} \exp[i\alpha(\gamma_5 \otimes \xi_5)]. \]
- Wilson-like term is irrelevant \(\Rightarrow SU(4)\) taste restored in continuum limit
  - This naive analysis supported in presence of gauge fields by absence of additional relevant terms due to lattice symmetries [Golterman & Smit]
  - Further supported by RG framework of [Shamir]
Rooted staggered fermions

To obtain QCD (with 3 or 4 flavors) can:

1. Use tastes as flavors but make non-degenerate
   - Lose lattice symmetries, complicated, $D_{stag}$ non-hermitian

2. Use one staggered fermion per flavor and take fourth-root of determinant:

$$Z_{QCD}^{\text{root}} = \int DU e^{S_g} \left( \det[D_{stag}(m_u)] \det[D_{stag}(m_d)] \det[D_{stag}(m_s)] \right)^{1/4}$$

- $\det[D_{stag}(m)]$ is positive definite for $m \neq 0$: take positive root

Reasons for rooting: fast to simulate, and have $U(1)_c$ symmetry

Rationale: rooting legitimate in continuum limit, since taste-breaking vanishes

Question: does taking fourth-root commute with sending $a \to 0$?
Non-locality for $\alpha \neq 0$ [BGS]

Rooted staggered fermions cannot be described by a local theory with a single taste per flavor

- Assume that, for any gauge configuration [Adams]
  
  $$\left(\det[D_{stag}]\right)^{1/4} = \det[D_1] \exp(-\delta S_{\text{eff},g})$$

  with $D_1$ a local 1-taste operator and $S_{\text{eff},g}$ a local gauge action

- Then, for normal (= “unrooted”) staggered fermions
  
  $$\det[D_{stag}] = (\det[D_1])^4 \exp(-4\delta S_{\text{eff},g}) = \det(D_1 \otimes 1) \exp(-4\delta S_{\text{eff},g})$$

- Compare fermionic contributions to gluonic correlators (e.g. $\langle F^2(x)F^2(0)\rangle$)

- Those on LHS do not have $SU(4)$ taste symmetry, while those on RHS do

- Difference particularly striking at long distances: LHS has 1 PGB, RHS has 15

- **Contradiction** $\Rightarrow \delta S_{\text{eff},g}$ non-local
  
  $\Rightarrow$ No contradiction in perturbation theory (for $\alpha \to 0$)
Implications of non-locality

A reasonable person at this stage might say:

“Locality of the action guarantees universality, i.e. that we obtain the correct continuum limit. Non-local theories are unphysical (lacking unitarity, ...). The lore is that non-locality violates universality. I don’t want to use rooted staggered fermions.”

Another reasonable person might say:

“Rooted staggered fermions are so attractive numerically, that I am going to try and understand and “tame” the non-locality. If I can argue plausibly that the non-locality does not change the universality class, i.e. that the effects of non-locality vanish in the continuum limit, then the extensive numerical results based on the MILC ensemble will be physical.”

NOTE: If rooted staggered fermions have the wrong continuum limit, then results using them are WRONG, not approximations to QCD.

Most of the remainder of this talk will concern attempts to tame the non-locality.

But first, I want to recall why the stakes are high.
Outline

- What is rooting and what are its potential problems?
  - Non-locality, lack of unitarity [BGS]

- Why is it being used, and what is the status of results?
  - What are the stakes? [MILC, FNAL, HPQCD]

- What do we learn about non-local theories from statistical mechanics?

- Can we tame the non-locality?
  - Perturbation theory
  - Renormalization Group (RG) analysis [Shamir]
  - Effective field theory for rooted staggered fermions [Bernard]

- Do rooted staggered fermions have too many symmetries? [Creutz, BGSS, DH]

- Problems with rooting at non-zero chemical potential [GSS]

- Conclusions
Update on rooted staggered simulations [Sugar]

MILC ensemble now includes coarser and “super-fine” lattices:

<table>
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<tr>
<th>$a$ (fm)</th>
<th>$a\tilde{m}' / am'_s$</th>
<th>$10/g^2$</th>
<th>dims.</th>
<th># lats.</th>
<th>$m_{\pi}/m_{\rho}$</th>
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<tr>
<td>$\approx 0.15$</td>
<td>0.0290 / 0.0484</td>
<td>6.600</td>
<td>$16^3 \times 48$</td>
<td>600</td>
<td>0.522(2)</td>
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<td>600</td>
<td>0.256(5)</td>
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<td>0.03 / 0.05</td>
<td>6.81</td>
<td>$20^3 \times 64$</td>
<td>564</td>
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<td>200</td>
<td>0.474(5)</td>
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Widely used [talks by Lee, Onogi, Orginos]
MILC results for $m_{\pi}^2/(m_x + m_y)$ [Sugar]

- Results for $m_{\pi}^2/(m_x + m_y)$
- Part of global fit to PGB properties
- Super-fine lattice results agree with predictions
- Partially quenched staggered chiral perturbation theory describes data well

Updated MILC Results: Masses

Accurate results for PGB properties + staggered chiral perturbation theory lead to successful comparisons with data \((f_\pi, f_K \text{ [talk by Lee]})\) and determinations of quark masses.

Update: Super-fine (and coarser) lattices lead to very small changes:

\[
\begin{align*}
  m_s^{\text{MS}} (2 \text{ GeV}) &= 90(0)(5)(4)(0) \text{ MeV} \quad [87(0)(4)(4)(0) \text{ MeV}] \\
  m_s/\hat{m} &= 27.2(0)(4)(0)(0) \quad [27.4(1)(4)(0)(1)] \\
  m_u/m_d &= 0.42(0)(1)(0)(4) \quad [0.43(0)(1)(0)(8)]
\end{align*}
\]

- Errors are from statistics, simulation, perturbation theory, and EM effects.
Predictions using rooted staggered fermions

Having checked that “gold-plated” PGB, nucleon, $B$, $\psi$ and $\Upsilon$ properties agree with experiment, FNAL/MILC/HPQCD have made successful predictions for:

- $D \rightarrow K\ell\nu$ form factor (shape and normalization)
- $f_D$
- $B_c$ mass

Summarized in [Kronfeld, hep-lat/0607011]

This is exactly how we hoped lattice QCD would be used.

Successes are impressive, but do not provide definitive demonstration that rooted staggered fermions are correct.

- Systematics are complicated—we would want checks even if there were no theoretical issue.
- It could be a fluke—the wrong theory happens to give results close to experiment for some quantities but not others.
- There is a serious theoretical issue (non-locality) and it must be understood.
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Non-local interactions in statistical mechanics

- Considerable literature studying power-law interactions, e.g. Ising model

\[ H = J \sum_{\bar{x}, \bar{y}} s_{\bar{x}}^{\bar{x}} \frac{1}{|x - y|^{d+\sigma}} s_{\bar{y}} \]


\[ G(k)^{-1} \sim m + j_\sigma k^\sigma + j_2 k^2 + \ldots \]

- Differ from QCD with rooted staggered fermions most notably because:
  - More non-local than QCD, which has dominant local gauge interaction
  - Power-law scaling (compared to logarithmic for QCD in $d = 4$)

- Nevertheless, perhaps can “demystify” non-locality
Example from statistical mechanics

\[ \eta = 2 - \sigma \]

\[ \eta = 1/4 \]

*d=2 Ising model, 1/r \( d+\sigma \) interaction*

\[ G(k) \sim k^{-2+\eta} \]

- Mean-field regime with dimensions dependent on \( \sigma \)
- Non-trivial fixed point
- Exponents depend on \( \sigma \)
- Details resolved by simulations
- Interaction effectively short ranged
- Non-trivial exponents, independent of \( \sigma \)
- Non-local interactions determine finite-size corrections
Lessons from statistical mechanics

- Non-local interactions need not change the universality class
- Can analyze using RG equations (= perturbation theory)
- Check using numerics—does $\alpha_s$ run as predicted?
- Beware of enhanced finite size corrections.
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What do we learn from perturbation theory?

- Assume renormalizability for unrooted staggered fermions
- Holds for any integer number of “replicas”, \( n_r \), of unrooted staggered fermions
- Since amplitudes and counterterms are polynomial in \( n_r \) (which counts fermion loops), renormalizability extends to any \( n_r \) [Bernard & Golterman]
- An arbitrary \( n_r \) in PT is obtained by using \( \det^{n_r}(D_{stag}) \) in \( Z \)
- Thus PT for rooted staggered fermions (with \( n_r = 1/4 \)) is renormalizable, and get \( \beta \)-function and anomalous dimensions for \( N_f = 4n_r = 1 \) flavors
  ⇒? Rooting leads to the desired physical theory in perturbation theory (for \( a \to 0 \)) Really need more work to establish this? [Kennedy]
  - Why? Because it does not change form of propagator or vertices
  - Discretization errors scale to zero as \( a^2 \) (up to logs)
  - NOTE: renormalizability holds for any \( n_r \), e.g. \( n_r = \pi/4 \Rightarrow N_f = \pi \)
    - But only for integral \( N_f \) is resulting perturbative theory unitary

Important to extend renormalizability proof to staggered fermions, and complete argument on transition to \( n_r = 1/4 \)

Conclusions from perturbation theory

If we accept that rooting gives the correct perturbative fixed point, are we done?

No!

1. Non-perturbative effects at short-distances could invalidate the analysis
   - For Yang-Mills theory (and QCD with Wilson fermions) we checked this by studying scaling of $\alpha_s$ non-perturbatively

2. We know from [BGS] that the theory is non-local and thus unphysical for any $a \neq 0$, but the perturbative fixed point is physical.
   - Inconsistency? Not if non-locality vanishes as $a \to 0$.
   - We need to understand the non-locality in more detail in order to make such a possibility plausible.

3. Because we do not have a 1-taste construction, we do not have a standard RG framework to classify the (ir)relevance of operators
   - We need an appropriately extended framework
Scaling of $\alpha_s$ [Mason et al., hep-lat/0503005]

Compare Wilson loops to two loop perturbation theory to extract $\alpha_V(\mu)$

Scaling matches four-loop running with $N_f = 3$
Bounding non-locality using RG analysis [Shamir]

- Standard RG set up, except first step goes from single-component to taste basis
- Gauge invariance and other lattice symmetries maintained
- Use Gaussian blocking kernels with local map $Q^{(k)}$ and $\alpha_k \sim a_k^{-1}$:

$$\int D\bar{\psi}^{(k-1)} D\psi^{(k-1)} e^{-\alpha_k [\bar{\psi}^{(k)} - \bar{\psi}^{(k-1)}] + [\psi^{(k)} - Q^{(k)} \psi^{(k-1)}]}$$

- Integrate out all fermions except $\psi^{(n)}$ while keeping all levels of gauge fields

$$\det[D_{stag}] \propto \prod_{k=0}^{n} \left[ \det[G_{k}^{-1}] \det[D_{n}] \exp(-4S_{eff}^{(k)}) \right]$$

RG analysis: locality of $S_{\text{eff}}^{(k)}$

- For unrooted staggered fermions, expect locality:

\[
\begin{align*}
\det[D_{\text{stag}}] & \propto \prod_{k=0}^{n} \exp[-4S_{\text{eff}}^{(k)}] \det[D_{n}] \\
\text{local at scale } a_f & \text{ local at scale } a_k \text{ local at scale } a_c
\end{align*}
\]

- Can understand locality of effective gauge actions $S_{\text{eff}}^{(k)} = \text{tr ln } G_k/4$ because $G_k^{-1}$ are like Wilson-Dirac operators with a large negative mass, e.g.

\[
G_0^{-1} = D_{\text{stag}} + \alpha_0 Q^{(0)}Q^{(0)}
\]

- $G_k^{-1}$ can have small eigenvalues (below the "mobility edge") on rough gauge configurations, but eigenvectors are localized so $G_k$ remains local at $a_k$

\[
D_k = \alpha_k - \alpha_k^2 Q^{(k)}G_kQ^{(k)}\dagger \text{ is local at } a_k
\]

- Extends understanding of locality of $S_{\text{eff}}^{(k)}$ beyond perturbation theory

\[
\Rightarrow \text{ Plausible that } S_{\text{eff}}^{(k)}, \text{ and thus } D_n, \text{ remain local on rooted ensemble}
\]

\[
\begin{align*}
\det[D_{\text{stag}}]^{1/4} & \propto \prod_{k=0}^{n} \left( \det[G_k^{-1}]^{1/4} \right) \exp(-S_{\text{eff}}^{(k)}) \det[D_n]^{1/4} \\
\text{non-local 1-taste theory due to root} & \text{ non-locality here, in IR}
\end{align*}
\]

RG analysis: bounding non-localities

- Decompose blocked Dirac operator (local on both ensembles)

\[ D_n = \begin{array}{c} \text{taste invariant} \\ \tilde{D}_{\text{inv},n} \otimes 1 \end{array} + \begin{array}{c} \text{taste-breaking} \\ \Delta_n \end{array} \]

- Expect standard RG scaling of "dimension-5" \( \Delta_n \sim \partial^2 \) between \( a_f \) and \( a_c \) in both ensembles since interactions are local in that range ("true" in PT)

\[ ||\Delta_n|| \leq \frac{a_f}{a_c^2} (1 + \text{logs}) \]

\( \Rightarrow \) Non-localities are bounded on rooted ensemble! (Using \( ||\tilde{D}_{\text{inv},n}^{-1}|| \lesssim 1/m(a_c) \))

\[ \det[D_n]^{1/4} = \det[\tilde{D}_{\text{inv},n} \otimes 1]^{1/4} \det[1 + \Delta_n (\tilde{D}_{\text{inv},n} \otimes 1)^{-1}]^{1/4} \]

\[ = \det[\tilde{D}_{\text{inv},n}] \exp\{(1/4)\text{tr} \ln[1 + \Delta_n (\tilde{D}_{\text{inv},n} \otimes 1)^{-1}]\} \]

local 1-taste theory non-locality vanishes as \( a_f \to 0 \)
Based on two plausible (and to some extent testable) assumptions:

1. Locality of $G_k$ on rooted ensemble
2. Scaling of $\Delta_n$ on rooted ensemble

Can show that, when $a_f \to 0$, rooted staggered QCD is equivalent to a theory which is local at scale $a_c$ and manifestly in same universality class as QCD

- Can now send $a_c \to 0$ and obtain continuum QCD
- Alternatively, for $a_f \neq 0$ define a 1-taste/flavor theory, local at scale $a_c$ ("rewighted theory")

In same universality class as QCD, and becomes equivalent to rooted staggered fermions when $a_f \to 0$

However, for any $a_f \neq 0$, rooted staggered fermions will have taste-breaking non-localities in the IR, because $\Delta_n \neq 0$ (consistent with [BGS])
RG analysis: how does it work?

- After one blocking in free theory \((\bar{p}_\mu = \frac{\sin ap_\mu}{a}, \hat{p}_\mu = \frac{2\sin ap_\mu/2}{a})\) [BGS]

\[
D_{\text{inv},0} = \frac{\sum_\mu i(\gamma_\mu \otimes 1)\bar{p}_\mu + (1 \otimes 1)[m + \alpha_0^{-1}(\hat{p}^2 + m^2)]}{1 + 2m\alpha_0^{-1} + \alpha_0^{-2}(\hat{p}^2 + m^2)}
\]

\[
\Delta_0 = \frac{af \sum_\mu (\gamma_5 \otimes \xi_\mu \xi_5)\hat{p}_\mu^2}{1 + 2m\alpha_0^{-1} + \alpha_0^{-2}(\hat{p}^2 + m^2)}
\]

- Separated removal of doublers from taste-breaking:
  - \(D_{\text{inv},0}\) is Wilson-like, with no doublers
  - \(\Delta_0\) breaks taste, but not needed to remove doublers

- Under further blocking \(\Delta_n\) gets smaller, and \(D_{\text{inv},n}\) approaches a GW operator
If partially quenched (PQ) staggered chiral perturbation theory (S\(\chi\)PT) is a valid effective field theory (EFT) for unrooted but partially quenched staggered fermions, (and accepting some plausible technical assumptions,) then:

1. The correct EFT for rooted staggered fermions is “S\(\chi\)PT with the replica trick” (rS\(\chi\)PT);
2. This theory is unphysical for \(a \neq 0\) (as required by [BGS]) but contains a subsector which becomes \(\chi\)PT for QCD when \(a = 0\).

- This argument provides alternate to RG approach for taming non-localities
  - Advantage: gives explicit formulae (essential for fitting)
  - Disadvantages:
    1. Useful only where \(\chi\)PT is useful (mainly the PGB sector)
    2. Relies on PQ\(\chi\)PT, whose theoretical foundations are not as strong as for \(\chi\)PT [SS & Shoresh]

- Surprising result: “rooted S\(\chi\)PT” involves the replica trick and seems somewhat ad hoc

What is (rooted) staggered $\chi$PT?

- EFT for single flavor of (four-taste) staggered fermions, including discretization errors, is $S\chi$PT of [Lee & SS]

- EFT for $n_r$ copies of $D_{stag}(m_u)$, $n_r$ copies of $D_{stag}(m_d)$, etc. (a local lattice theory if $n_r$ integral, with $N_c$ adjusted so asymptotically free) is $S\chi$PT of [Aubin & Bernard]
  - EFT results are polynomial in $n_r$ at any finite order.

- $rS\chi$PT is defined by setting $n_r = 1/4$
  (and is usually called just $S\chi$PT) [Aubin & Bernard]
  - EFT analog of rooting done at quark level
  - Low energy constants are taken to be those of presumed continuum theory
    (here $N_f = 2 + 1$ QCD)
Consider four flavors of rooted staggered fermions:

\[
Z^{\text{root}}(m_1, m_2, m_3, m_4) = \sqrt[4]{\int d\mu_g \left\{ \prod_{i=1}^{4} \det[D_{\text{stag}}(m_i)] \right\}}
\]

Related to theories of interest:

\[
Z^{\text{stag}}(m) = \sqrt[4]{\int d\mu_g \det[D_{\text{stag}}(m)]} = Z^{\text{root}}(m, m, m, m)
\]

\[
Z^{\text{MILC}}(m_\ell, m_s) = Z^{\text{root}}(m_\ell, m_\ell, m_s, 1/a)
\]

Idea is to go (within the EFTs)

- from \(Z^{\text{stag}}\) (a physical theory with known EFT)
- to \(Z^{\text{MILC}}\) (the rooted theory being simulated)

by calculating all derivatives w.r.t. \(m_i\) and summing the series

Assumes no non-analyticities (plausible since \(m_i\) are positive)

One is constructing EFT for \(Z^{\text{MILC}}\)
Key steps in EFT argument (1)

- Derivatives of $Z^{\text{root}}$ evaluated for $m_i$ equal are “unrooted”:

$$Z^{\text{root}}(m_1, m_2, m_3, m_4) = \int d\mu_g \left\{ \prod_{i=1}^{4} \exp \left[ \frac{1}{4} \text{tr} \ln D_{\text{stag}}(m_i) \right] \right\}$$

$$C(x, y) = \frac{\partial^2 \ln Z^{\text{root}}}{\partial m_4(x) \partial m_4(y)} \bigg|_{m_i \text{ equal}} = \left\langle \frac{1}{4^2} x \circlearrowright \ y - \frac{1}{4} x \circlearrowright \ y \right\rangle_{Z^{\text{stag}}(m)}$$

- Rooting gives rise to factors of $(1/4)^{\text{#loops}}$
- Correlators cannot be obtained by applying $\partial/\partial m$ on $Z^{\text{stag}}(m)$
- Although evaluated on physical ensemble, correlators are partially quenched and thus unphysical
- We “know” EFT for PQ correlators (PQS\chiPT) so, in principle, we know all derivatives and can sum
- Result is rS\chiPT for four flavors

Traded rooting for PQing!

Explicit form of intermediate PQ theory

- Need to consider PQ but unrooted staggered theory

\[ Z^{\text{PQstag}}(m, M_V, \tilde{M}_V) = \int d\mu_g \det[D_{\text{stag}}(m)] \times \]

\[ \times \prod_{i=1}^{N_V} D\bar{\chi}_V_i D\chi_V_i D\bar{\chi}_V_i^\dagger D\chi_V_i \exp[-\bar{\chi}_V_i D_{\text{stag}}(M_V,ij)\chi_V_j - \bar{\chi}_V_i^\dagger D_{\text{stag}}(\tilde{M}_V,ij)\tilde{\chi}_V_j] \]

- Derivatives of \( Z^{\text{root}} \) wrt \( m_i \) related to those of \( Z^{\text{PQstag}} \) wrt \( M_V \), e.g.

\[ C(x, y) = \left< \frac{1}{4} x \circ y \right>_Z^{\text{stag}}(m) \]

\[ = \frac{1}{4^2} \frac{\partial^2 \ln Z^{\text{PQstag}}}{\partial M_{V,11}(x) \partial M_{V,22}(y)} \bigg|_{M_V = \tilde{M}_V = m} + \frac{1}{4} \frac{\partial^2 \ln Z^{\text{PQstag}}}{\partial M_{V,12}(x) \partial M_{V,21}(y)} \bigg|_{M_V = \tilde{M}_V = m} \]
Key steps in EFT argument (2)

- At this stage, have EFT for $Z^{\text{root}}(m_i)$ if all $m_i \ll \Lambda_{\text{QCD}}$:
  - four-flavor rS$\chi$PT
- Send $m_4 \to 2m_s$ (edge of validity of $\chi$PT):
  - EFT for 3 light flavors is three-flavor rS$\chi$PT (decoupling within EFT)
- Send $m_4 \to 1/a$: EFT should not change form, although LECs change
  $\Rightarrow Z^{\text{MILC}}$ described by three-flavor rS$\chi$PT
  - Unphysical for $a \neq 0$ but becomes QCD $\chi$PT for $a = 0$
  - Must rely on RG argument to show that, since get correct continuum limit, LECs must be those of QCD
- Can extend to 2 and 1 flavors: again, correct continuum limit is built in
  - Resolves 1 flavor “paradox” of having unwanted pions—they decouple
Outline

- What is rooting and what are its potential problems?
  - Non-locality, lack of unitarity [BGS]

- Why is it being used, and what is the status of results?
  - What are the stakes? [MILC, FNAL, HPQCD]

- What do we learn about non-local theories from statistical mechanics?

- Can we tame the non-locality?
  - Perturbation theory
  - Renormalization Group (RG) analysis [Shamir]
  - Effective field theory for rooted staggered fermions [Bernard]

- Do rooted staggered fermions have too many symmetries? [Creutz, BGSS, DH]

- Problems with rooting at non-zero chemical potential [GSS]

- Conclusions
Rooted theory maintains $U(1)_\epsilon$ symmetry of four-taste (unrooted) theory:

\[
(det[D_{stag}(m)])^{N_f/4} = \left(det[D_{stag}(me^{i\epsilon(n)\theta_{\epsilon}})]\right)^{N_f/4} = (det[D_{stag}(-m)])^{N_f/4} (\theta_{\epsilon} = \pi)
\]

$\epsilon(n) = \pm 1$ for even/odd sites

$\Rightarrow$ Rooted theory is an even function of $m$ for any $N_f$

In finite volume and for $a \neq 0$ there can be no non-analyticities for real $m$

$\Rightarrow$ Rooted theory in finite volume is a function of $m^2$ (not $|m|$)

Continuum theory (or lattice theory with overlap fermions) with odd $N_f$

has no $m \rightarrow -m$ symmetry due to $U(1)_A$ anomaly

$\Rightarrow$ Continuum theory in finite volume is a general function of $m$

Inconsistency? Disease? **No! A limitation, or ugly feature**

- Rooted staggered fermions are only claimed to give physical behavior in continuum limit
- Can show how odd powers of $m$ occur naturally

---

How rooted staggered fermions can work for $N_f = 1$

- Not proof; demonstration that there is no inconsistency
- Odd powers of $m$ arise from zero modes of $D_{\text{cont}}$, e.g.

$$\det[D_{\text{cont}}(m)] = m F_{\text{cont}}(m^2) \quad (\nu = 1)$$

$$F_{\text{cont}}(m^2) \sim \prod_{\lambda > 0} (i\lambda + m)(-i\lambda + m) > 0$$

- Staggered fermions have no index theorem, so would-be zero modes become a quartet of two pairs connected by $U(1)_{\epsilon}$

$$\{\det[D_{\text{stag}}(m)]\}^{1/4} = \left\{ [\lambda_{1\text{stag}}^2 + m^2][\lambda_{2\text{stag}}^2 + m^2] F_{\text{stag}}(m^2) \right\}^{1/4}$$

$$\lambda_{1,2} \propto a \text{ (or } a^2?) \text{ and } F_{\text{stag}} > 0$$

- Manifestly a function of $m^2$

- Recover expected form when take continuum limit

$$\left\{ [\lambda_{1\text{stag}}^2 + m^2][\lambda_{2\text{stag}}^2 + m^2] \right\}^{1/4} \xrightarrow{a \to 0} |m|$$

- Rooting (with positive root) gives continuum form, but with positive
continuum mass, regardless of the sign of the staggered fermion mass
Non-commutativity of $a \to 0$ and $m \to 0$ limits?

- Expect limits to commute for physical quantities in rooted staggered QCD (with $m_q > 0$) [Bernard]
- Expect non-commutativity if one quark mass vanishes [Durr, BGSS, DH]
- Example: condensate for $N_f = 1$

\[
\langle \bar{\psi}\psi \rangle_{\text{cont}} = -\frac{1}{Z_{\text{cont}}V} \frac{\partial Z_{\text{cont}}(m)}{\partial m} \to_{m \to 0} \text{non-zero constant}
\]

non-zero at $m = 0$ (even in finite volume) due to $\nu = 1$ zero mode

- With rooted staggered fermions condensate is odd function of $m$

\[
\langle \bar{\psi}\psi \rangle_{\text{stag}} \propto \frac{m[(\lambda^{\text{stag}}_1)^2 + (\lambda^{\text{stag}}_2)^2 + 2m^2]}{\left\{[(\lambda^{\text{stag}}_1)^2 + m^2][(\lambda^{\text{stag}}_2)^2 + m^2]\right\}^{3/4}}
\]

\[
\to_{m \to 0} \quad 0 \quad \text{WRONG ANSWER}
\]

\[
\to_{a \to 0} \quad 2\text{sign}(m) \quad \text{CORRECT for } m > 0
\]

Numerical study in Schwinger model [DH]

Analogous issues arise from square-rooting of two-taste staggered determinant
Compare condensate in rooted staggered to one-flavor overlap

Study $m < 0$ using rooted staggered with mass $|m|$ plus $\theta = \pi$ term!

Consistent with explanation of BGSS
No similar issue when $N_f = 2$—analog of $2 + 1$ QCD
More on “extra” symmetries

$U(1)\epsilon$ symmetry leads to Ward identities in rooted theory. Are these inconsistent with expected properties of continuum theory?

- No! Can show that it is one of many extra symmetries due to rooting that have no impact on continuum theory

Two examples show this:

1. rS\(\chi\)PT: it is $U(1)\epsilon$ symmetric and yet has physical QCD subsector
2. Extended continuum theory with exact taste symmetry introduced by hand (expected continuum limit of rooted staggered theory)

\[
det^{1/4} [(D_{\text{cont}}(M) \otimes 1) + J] = \det[D_{\text{cont}}(M)] \exp \left\{ \frac{1}{4} \text{tr} \ln [1 + J(D_{\text{cont}}(M) \otimes 1)^{-1}] \right\}
\]

- Can derive exact (but unphysical) Ward identities using taste non-singlet $J$
- Includes $U(1)\epsilon$ Ward identities (since $U(1)\epsilon$ has taste $\xi_5$)
- If set $J \rightarrow (\tilde{J} \otimes 1)$ then generate correlation functions of QCD
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- Conclusions
Eigenvalues of staggered $D(U) + m$

$\mu = 0$

$\mu \gtrsim m_\pi/2$

(Akemann et al. hep-th/0411030)
Fourth root of $\Delta[U] = \text{Det} [D(U) + m]$

Solution ("ideal prescription"):
Define phases of $\eta_i \in \text{quartet}$ to be close to each other; then

$$\arg \left( \prod_{1}^{4} \eta_i \right)^{1/4} \equiv \frac{1}{4} \sum_{1}^{4} \arg \eta_i$$

$\Rightarrow$ NO jumps in phase when any $\eta_i$ crosses axis.

Smooth replacement of four tastes by one quark per flavor
Problematic configurations:

- NO clear definition of phase of root!
  ⇒ Such configurations should be dropped.

Probability of prob. config:

\[\mathcal{O}(a^2 V \Lambda^6)\] quenched
\[\mathcal{O}[(a \sqrt{V} \Lambda^3)^3]\] reweighted

= systematic error of algorithm

Note:
Volume required is fixed by physics (e.g., \(m_\pi L \gtrsim 3\)).
We must take \(a \to 0\) before \(V \to \infty\).

- similar to requirement \(a \to 0\) before \(m \to 0\)

(Bernard hep-lat/0412030)
Conclusions on $\mu \neq 0$

- Rooting leads to unavoidable ambiguities when $\mu \gtrsim m_\pi/2$ (independent of issues discussed by [Splittorff])
- Systematic error grows with volume and for present simulations is $\gtrsim 200\%$; need much smaller lattice spacings
- Criticism does not apply to Taylor expansions about $\mu = 0$ or use of imaginary $\mu$. 

Conclusions

BAD or UGLY?
My conclusion: Ugly

- Plausible theoretical arguments are now added to the numerical evidence that rooted staggered fermions have the correct continuum limit.

- Picture that emerges: Non-locality/non-unitarity is present, but is pushed into the IR, is bounded, and vanishes when $a \to 0$. Furthermore, we have a plausible understanding of the far IR using rSχPT.
  - Not pretty, and systematic errors due to non-locality can be significant
  - Limits utility of purely staggered simulations—mixed-action simulations may be preferable for many quantities

- Plausibility is in the eye of the beholder—need further work to study assumptions of arguments
  - Prove renormalizability of unrooted staggered fermions
  - Provide better basis for PQχPT
  - Test assumptions of RG argument

- As always, we need to cross-check all numerical results with other discretizations—true irrespective of rooting issue.

Fire away!

Perturbation theory for staggered fermions

[Sharatchandra, Thun & Weisz, Kawamoto & Smit, Golterman & Smit, ...]

- PT for unrooted staggered fermions involves \(2^4\) poles in Brillouin zone (BZ)
- Repackage calculation using reduced BZ \((-\pi/2 < p'_\mu \leq \pi/2\) plus hypercube vector labeling reduced zones
  - Propagator is taste symmetric: \((A, B\) hypercube vectors\)

\[
G^{-1}(q' + B\pi, p' + A\pi) = \bar{\delta}(q' + p') \left[ \sum_\mu i \sin q'_\mu \bar{\gamma}_\mu \otimes \bar{1}_{BA} + m(1 \otimes 1)_{BA} \right]
\]

- Vertices break taste symmetry in general, e.g. \(\bar{q}qg\) vertex \((C_\mu = 0)\)

\[
V_\mu(q' + \pi B, p' + \pi A, k' + \pi C) = -ig\bar{\delta}(p' + q' + k') \cos (q'_\mu + k'_\mu/2) \bar{\gamma}_\mu \bar{C} \otimes \bar{\xi}_C \otimes 1_{BA}
\]

  but not when gluon is nearly physical \((C = 0)\)

- Need to show that:
  - continuum-like parts (all momenta nearly physical), which are taste symmetric, have same logarithmic divergences as in continuum
  - gluons with \(k \sim O(1)\) coupling to nearly on-shell fermions give short-distance artifacts (and thus finite corrections) which are taste symmetric due to momentum conservation

Other criticisms from [Creutz]

- Is it plausible that IR and UV “cutoffs” (i.e. $m \to 0$ and $a \to 0$) do not commute in theory with no physical massless particles?
  - Yes. It can be understood as an “Aoki-phase” phenomenon in the extended theory.