Can Eguchi-Kawai reduction provide a practical method for studying large-$N_c$ theories on the lattice?

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Outline

- Introduction & Motivation
- Short history of “volume reduction”
- Application to QCD with 1 & 2 adjoint fermions: adjoint Eguchi-Kawai [AEK] model
- Twisted adjoint Eguchi-Kawai [TAEK] model
- Outlook
Overview: beyond QCD

QCD
N=3; \(N_f = 2+1\)
Confinement
Chiral Sym. breaking

Add colors

\[ N \to \infty, \ N_f = 2+1 \]
‘t Hooft limit of QCD

Make (nearly) conformal:
N=3; \(N_f = 8-16\)
N=3, \(N_f = 2\) sextets
N=2, \(N_f = 2\) adjoints

S. Sharpe, “Large N reduction” 6/29/13 @ Cracow School of Theoretical Physics, Zakopane, Poland
Overview: beyond QCD

**QCD**
- $N = 3$; $N_f = 2+1$
- Confinement
- Chiral Sym. breaking

- Add colors

- $N \to \infty$, $N_f = 2+1$
  - 't Hooft limit of QCD

**Make (nearly) conformal:**
- $N = 3$; $N_f = 8-16$
- $N = 3$, $N_f = 2$ sextets
- $N = 2$, $N_f = 2$ adjoints

- Add both

- $N \to \infty$
  - $N_f$ two-index irreps
  - e.g. $N_f = 1, 2$ adjoints
  - Corrigan-Ramond limit
Why add colors?

- At first sight, this seems foolhardy!
  - Increasing the number of degrees of freedom while still studying a strongly coupled theory
- However, there are important theoretical and computational simplifications
  - Planarity
  - Gauge-gravity duality
  - Volume independence
Planarity ['t Hooft, Witten,...]

* Limit is $N \rightarrow \infty$ with $\lambda = g^2 N$ & $N_f$ fixed
* Only planar diagrams contribute in perturbation thy
* Mesons & glueballs are stable (widths $\sim 1/N$)
* Expectation values factorize: $\langle O_1 O_2 \rangle = \langle O_1 \rangle \langle O_2 \rangle$

→ Simplified theory sharing asymptotic freedom, confinement & Chiral SB with QCD

• Long-standing hope that analytic progress is possible
• Lattice calculations can help guide search for string-theory duals
Volume Independence [Eguchi & Kawai]

* Under non-trivial conditions, certain properties of gauge theories at large $N$ are independent of volume.

Does this reduction in degrees of freedom provide a practical method to access the theoretical simplicity of large $N$ theories? Are the conditions satisfied?
After a hiatus, much recent interest, e.g.


A. Gonzalez-Arroyo & M. Okawa, PL 120B (1983) 174 [TEK model]

......


B. Bringoltz & S.R. Sharpe, PRD 80 (2009) 065031 [massive Nf=1 AEK works]


A. Gonzalez-Arroyo & M. Okawa, JHEP 1007 (2010) 043 [TEK lives and thrives]


A. Gonzalez-Arroyo & M. Okawa, arXiv:1305.6253 [ATEK for N up to 29²=841]
I will not discuss:

- Novel simulations of single-site SUSY lattice theories aimed at testing AdS/CFT correspondence and learning about string theories & quantum gravity
  
  [J. Nishimura, M. Hanada, T. Wiseman, S. Catterall, ..........]

- Partial reduction of QCD in 't Hooft limit
  - If $L > L_c \approx 1$ fm then results independent of $L$ [Narayanan & Neuberger]

- Obtaining results for large $N$ by extrapolating from $N=3,4,5,6$ (useful for pure gauge theory) [Teper,...]

- Reduction of one dimension [Cossu & D’Elia]
History of large-N volume independence
First example

Lattice $SU(N)$ on $L^d$ $^N\equiv\infty$ Lattice $SU(N)$ on $1^d$

Now usually called "large-N volume independence"
Lattice $SU(N)$ on $L^d \stackrel{N \equiv \infty}{\cong} L^d$

<table>
<thead>
<tr>
<th>gauge theory</th>
<th>“reduced” or “matrix” model</th>
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Links all different
Lattice $SU(N)$ on $L^d \overset{N=\infty}{\Rightarrow}$ Lattice $SU(N)$ on $1^d$

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Links all different
### Lattice $SU(N)$ on $L^d$ \( N \equiv \infty \) vs. Lattice $SU(N)$ on 1$^d$

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<td>$W_C = \frac{1}{N} \text{tr} U_{x,\mu} U_{x+\mu,\nu} \cdots U_{x-\nu-\rho,\rho} U_{x-\nu,\hat{\nu}},$</td>
<td>$W_C^{\text{reduced}} = \frac{1}{N} \text{tr} U_{\mu} U_{\nu} \cdots U_{\rho} U_{\nu}.$</td>
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\[
\langle W_C \rangle_{\text{gauge theory}} = \langle W_C^{\text{reduced}} \rangle_{\text{reduced}} + O(1/N^2).
\]

S. Sharpe, “Large N reduction” 6/29/13 @ Cracow School of Theoretical Physics, Zakopane, Poland
Lattice $SU(N)$ on $L^d \equiv \infty$ Lattice $SU(N)$ on $1^d$

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**Gauge symmetry**

**“Center” symmetry**

$U_{[\bar{n},\tau),\mu]} \rightarrow U_{[\bar{n},\tau),\mu]} z_\mu$ ; $z_\mu \in Z_N$  

$U_\mu \rightarrow U_\mu z_\mu$ ; $z_\mu \in Z_N$
EK’s demonstration of vol. indep.

• Show equivalence of Dyson-Schwinger eqs for Wilson loops

\[
\text{gauge: } U_{n,\mu} \rightarrow U_{n\mu} (1 + i\epsilon t^a) \quad \text{reduced: } U_\mu \rightarrow U_\mu (1 + i\epsilon t^a)
\]

• Crucial difference

\[
\text{gauge: } \text{tr} \left( \cdots U_{n,\mu} U_{n+\mu,\nu} \cdots U_{m,\mu} U_{m-\mu,\rho} \cdots \right) \quad \text{reduced: } \text{tr} \left( \cdots U_\mu U_\nu \cdots U_\mu U_\rho \cdots \right)
\]

• Get extra terms on the reduced side: must vanish for reduction to hold

\[
\text{e.g. } \left\langle \text{tr} \left( \begin{array}{c} \vdots \end{array} \right) \text{tr} \left( \begin{array}{c} \vdots \end{array} \right) \right\rangle_{\text{reduced}} = 0
\]

• Extra terms correspond to “open loops” in gauge theory

\[
\text{e.g. } \left\langle \text{tr} \left( U_\mu U_\nu^\dagger \right) \text{tr} \left( U_\mu^\dagger U_\nu \right) \right\rangle_{\text{reduced}} = 0
\]
EK’s demonstration of volume independence

Reduction holds if
\[ \left\langle \text{tr}(\ldots) \text{tr}(\ldots) \right\rangle = 0 \]

- Valid if have large-N factorization

\[ \langle W_{C_1} W_{C_2} \rangle_{\text{reduced}} = \langle W_{C_1} \rangle_{\text{reduced}} \langle W_{C_2} \rangle_{\text{reduced}} + O(1/N^2), \]

- ... and if center symmetry is unbroken (\( Z_4^N : U_\mu \to U_\mu z_\mu \))

\[ \langle W_{\text{open}} \rangle_{\text{reduced}} = 0. \]

CONCLUSION: \( \text{tr}U_\mu, \text{tr}U_\mu U_\nu, \) etc.

must all vanish in the reduced model
Volume independence is an example of a larger class of equivalences: large-N orbifold equivalences [Kovtun, Unsal & Yaffe]

Restrict to zero-momentum fields

Orbifold w.r.t. combined gauge and center transformation

Orbifold equivalence holds if “orbifolding symmetries” (translation invariance and center symmetry) are unbroken
Reduction fails! [Bhanot, Heller & Neuberger ‘82]

- Qualitatively: Small $L \Leftrightarrow$ High $T \Rightarrow$ deconfinement $\Rightarrow$ $\text{tr}(U_\mu) \neq 0$

- Can understand in weak coupling limit as due to clustering of eigenvalues of $U_\mu$ [BHN ‘82, Kazakov & Migdal ‘82]

$$U_\mu = V_\mu^\dagger \Lambda_\mu V_\mu$$

$$\Lambda_\mu = \text{diag}[e^{i\theta_1^\mu}, \ldots, e^{i\theta_N^\mu}]$$

$Z_N$ symmetry: $\theta_\mu^a \rightarrow \theta_\mu^a + \frac{2\pi}{N}$

$$F_{EK} \xrightarrow{b \rightarrow \infty} (d-2) \sum \log \left[ \sum_{a<b} \sin^2 \left( \frac{\theta_\mu^a - \theta_\mu^b}{2} \right) \right]$$

$\Rightarrow$ Eigenvalues attract for $d>2 \Rightarrow \theta_\mu^a = \theta_\mu^b$ and so $\text{tr} \ U_\mu \neq 0$

- For reduction to hold need uniform distribution of eigenvalues, uncorrelated in different directions

- Role of momenta played by $\theta_\mu^a - \theta_\mu^b$
Can reduction be rescued?

QCD (fund) $N=3$

$N_f$ Dirac

$N \rightarrow \infty$

$N=\infty$ Yang-Mills quarks “quenched”

$N=\infty$ Y-M

SINGLE SITE

Eguchi-Kawai

“Orbifold”
Can reduction be rescued?

\[ \text{QCD (fund) } N = 3 \]
\[ N_f \text{ Dirac} \]

\[ N \rightarrow \infty \]

\[ N = \infty \text{ Yang-Mills quarks "quenched"} \]

\[ \text{N} = \infty \text{ Y-M SINGLE SITE} \]
\[ \text{Eguchi-Kawai} \]

\[ \text{FAILS "Orbifold"} \]

\[ 't \text{ Hooft} \]
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SINGLE SITE
Eguchi-Kawai

Quenched EK
SINGLE SITE
Bhanot, Heller, Neuberger

Twisted EK
SINGLE SITE
Gonzales-Arroyo, Okawa
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Friday, June 28, 13

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Can reduction be rescued?

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FAILS"

"Orbifold"

REBORN?

FAILS

FAILS

'\text{t} \text{ Hooft}
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SINGLE SITE

Gonzales-Arroyo, Okawa

Deformed EK

SINGLE-SITE

Unsal & Yaffe

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Can reduction be rescued?

QCD (fund) $N=3$

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$N = \infty$ Yang-Mills quarks “quenched”

$N \rightarrow \infty$

Fails

Fails

Fails

Fails

Fails?

Reborn

$N = \infty$ Y-M $aL > 1$ fm

Narayanan & Neuberger

Can reduction be rescued?

Eguchi-Kawai

SINGLE SITE

Deformed EK

SINGLE-SITE

Unsal & Yaffe

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An alternative approach: AEK

QCD (N=3)
2N_f Dirac fermions in AS irrep (q^{ab})
infinite volume
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QCD (N=3) 2N_f Dirac fermions in AS irrep (q^{ab})
infinite volume

\[ \frac{N}{\xi} \rightarrow \infty \]

QCD (N=\infty) 2N_f Dirac fermions in AS irrep (q^{ab})
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[Corrigan-Ramond Armoni-Shifman-Veneziano]
An alternative approach: AEK

QCD ($N=3$)
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"Orientifold" equivalence (in even C sectors)
[ASV]

QCD ($N=\infty$)
$N_f$ Dirac fermions in Adjoint irrep
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[Corrigan-Ramond Armoni-Shifman-Veneziano]

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"Orientifold" equivalence
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[Corrigan-Ramond Armoni-Shifman-Veneziano]

QCD ($N=\infty$)
$N_f$ Dirac fermions
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"Orbifold" equivalence
[Kovtun, Unsal, Yaffe]

QCD ($N=\infty$)
$N_f$ Dirac fermions
in Adjoint irrep infinite volume
An alternative approach: AEK

QCD ($N=3$) $2N_f$ Dirac fermions in AS irrep ($q^{ab}$) infinite volume

Agree within $1/N!$

QCD ($N=\infty$) $N_f$ Dirac fermions in Adjoint irrep SINGLE SITE

$N \rightarrow \infty$

QCD ($N=\infty$) $2N_f$ Dirac fermions in AS irrep ($q^{ab}$) infinite volume

“Orientifold” equivalence (in even C sectors) [Corrigan-Ramond Armoni-Shifman-Veneziano]

QCD ($N=\infty$) $N_f$ Dirac fermions in Adjoint irrep infinite volume

“Orbifold” equivalence [Kovtun,Unsal,Yaffe]
Why do adjoint fermions help?

- Adjoint fermions survive in large N limit (unlike fundamentals)

- At one-loop order, fermions lead to repulsion between link eigenvalues, as long as use periodic (non-thermal) BC \([K, U & Y]\)

- Repulsion wins for \(N_f > 1/2\) massless Dirac fermions

  - Usually leads to uniform distribution of \(\theta_\mu\), but depends on details of fermion action [Lohmayer & Narayan, 2013]

- Any non-zero mass \([|m_{phys}| > 1/(aN)]\) leads to attraction at small \(\theta^a_\mu - \theta^b_\mu\)
  and thus to center-symmetry breaking

  - Need massless fermions?

- However, perturbation theory not reliable for small \(\mid \theta^a_\mu - \theta^b_\mu \mid\), nor in stronger coupling region of interest

  - Need non-perturbative simulation
What would we learn?

What would we learn?

QCD (AS) \(N = 3\)
2\(N_f\) Dirac

\(N \rightarrow \infty\)

QCD (AS) \(N = \infty\)

“Orientifold”

QCD (Adj) \(N = \infty\)

“Orbifold”

N\(_f\) Dirac

SINGLE SITE

QCD (Adj) \(N = \infty\)

N\(_f\) Dirac

Use single-site QCD(Adj) for \(N\) large to learn about 3 theories of great interest

- \(N_f = 1\): learn about QCD with 2 flavors in Corrigan-Ramond large-\(N\) limit
- \(N_f = 2\): alternative window on “minimal” walking technicolor theory
- \([N_f = 1/2\): equivalent to SYM, for which exact results are known]

Even though “matrix model” lives on a single site, one can calculate many physical quantities (string tension, pion mass, ...)

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Conditions for equivalences to hold

1. Large-N factorization holds

2. Orientifold: C not broken in QCD(AS,Adj)

3. Orbifold: Translation invariance unbroken in QCD(Adj.) in infinite volume

4. Orbifold: \((Z_N)^4\) center symmetry unbroken in QCD(Adj.) on a single site
1. Large-N factorization holds

2. Orientifold: $C$ not broken in QCD(AS,Adj)

3. Orbifold: Translation invariance unbroken in QCD(Adj.) in infinite volume

4. Orbifold: $(\mathbb{Z}_N)^4$ center symmetry unbroken in QCD(Adj.) on a single site

IN THIS TALK:
We assume the first three hold and study the last
Results for $N_f=1$ & $2$
adjoint Eguchi-Kawai (AEK) model


A. Gonzalez-Arroyo & M. Okawa, arXiv: 1305.6253
Action of AEK model

Wilson gauge and fermion action

\[ S_{\text{gauge}} = 2N b \sum_{\mu < \nu} \text{Re} \text{Tr} U_{\mu} U_{\nu} U^{\dagger}_{\mu} U^{\dagger}_{\nu}, \quad b = 1/(g^2 N) \]

\[ S_F = \sum_{j=1,N_f} \bar{\psi}_j D_W \psi_j \]

\[ D_W = 1 - \kappa \sum_{\mu=1}^4 \left[ (1 - \gamma_\mu) U^{\text{adj}}_{\mu} + (1 + \gamma_\mu) U^{\dagger \text{adj}}_{\mu} \right] \]

Parameters

\[ k \sim 1/m \]

Symmetries:

gauge: \[ U_{\mu} \rightarrow \Omega U_{\mu} \Omega^\dagger \quad (\text{all } \mu) \quad \Omega \in SU(N) \]

center \((Z_N)^4\): \[ U_{\mu} \rightarrow U_{\mu} e^{2\pi i n_\mu/N} \quad n_\mu \in \mathbb{Z}_N \]
Scaling of CPU with $N$

- Original studies used Metropolis algorithm
  \[ P(U) = e^{S_{EK}(U)} (\det D_W)^{N_f} \]
  - Determinant real & positive; evaluate explicitly
  - Scaling is $\sim (N^2)^3 \times N^2 \sim N^8 \Rightarrow$ can reach $N \approx 15$ on PC

- Present studies use rHMC (HMC) for $N_f=1$ (2)
  - Using $U^{adj} \sim U \cdot U^{\dagger}$, scaling is $\sim (N^3) \times N^{1-1.5} \sim N^{4-4.5}$
  - Can reach $N=53$ on PC, $N=289$ on supercomputer
Order params for symm breaking

- traces of “open” loops

\[ \text{tr} \left( U_\mu \right), \text{tr} \left( U_\mu U_\nu \right), \text{tr} \left( U_\mu U_\nu^\dagger \right), \text{tr} \left( U_1^{n_1} U_2^{n_2} U_3^{n_3} U_4^{n_4} \right), \ldots \]

- histograms of eigenvalues of links: \( \theta_\mu^a \)

- also calculate plaquette and larger Wilson loops
Expected phase diagram (infinite volume)

\[
N_f = 1, \; N = \infty
\]

Continuum physics

\[
(1/g^2N =) \; b
\]

heavy quarks

\[
\kappa_c(b)
\]

heavy quarks

\[
b(\kappa)_{\text{bulk}}
\]

\[
\sim 0.36
\]

QCD simulations done at \( b \sim 1/3 \)

2 critical lines (Aoki phase) or first-order line: light quarks

pure-gauge theory
Conclusion for $N_f=1$ AEK model [B&S]

First-order transition: quarks are light here

Based on $N \leq 53$; shows weak $N$ dependence
Conclusion for $N_f=1$ AEK model [B&S]

First-order transition: quarks are light here

Identical to infinite volume theory (at large $N$) within “funnel”

Based on $N \leq 53$; shows weak $N$ dependence
Very surprising feature:

- Inconsistent with pert. thy (requires $m_{\text{phys}} = 0$ in general)
- Violates naive decoupling of heavy quarks

Heavy Quarks ($m_{\text{phys}} \sim 1/a$) can “save” large-N reduction!
Very surprising feature:

- Checked using rHMC [Azeyanagi, Hanada, Unsal & Yacoby; Koren & SS]
- Supported by analytic arguments going beyond PT [AHUY, Unsal & Yaffe]

\[ |am_{\text{phys}}| < \frac{1}{b^{1/4}} \]

Heavy Quarks (\(m_{\text{phys}} \sim 1/a\)) can “save” large-N reduction!
Infinite volume expectation for $N_f=2$?

- $N=2$ gauge theory ("minimal walking technicolor") subject of many recent studies

- Dependence on $N$ not known

[Heitanen et al]
Phase diagram of $N_f=2$ AEK model [B,K&S]

$16 \leq N \leq 53$
Phase diagram of $N_f=2$ AEK model \[ [B,K&S] \]

$16 \leq N \leq 53$

First-order transition for all $b$
Phase diagram of $N_f=2$ AEK model \([B,K&S]\)

$16 \leq N \leq 53$

Funnel in which volume indep. holds

First-order transition for all $b$

crossover or bulk transition?
Hysteresis scans at $b=1$ ($N=10, 16, 23, 30$)

First order transition at $\kappa_c$
Hysteresis scans at $b=1$ (N=10,16,23,30)

First order transition at $\kappa_c$

Funnel

plaquette

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Hysteresis scans at $b=1$ ($N=10, 16, 23, 30$)

$|\text{tr}(U_\mu U_\nu)|$

Scaling to zero $\sim 1/N$

Funnel

$K_c$
Funnel width finite as $N \to \infty$

Good fit gives intercept $\kappa_f = 0.0655 \ll \kappa_c$

Very poor fit if force $\kappa_f$ to equal $\kappa_c$ at $N=\infty$
Outside the “funnel”

Complicated pattern of clumping of link eigenvalues

Qualitatively consistent with analytic arguments

Several competing “vacua”

Examples on next slide
Distribution of link eigenvalues

\[ U_\mu = V_\mu^\dagger \Lambda_\mu V_\mu \]
\[ \Lambda_\mu = \text{diag} \left[ e^{i\theta^1_\mu}, \ldots, e^{i\theta^N_\mu} \right] \]

\( N = 24, \ b = 0.35, \ \kappa = 0.19 \)

\( N = 16, \ b = 0.35, \ \kappa = 0.22 \)

\((Z_{24})^4\) invariant inside funnel

5 clumps (e.g. 4,3,3,3 & 3)
all 4 links have “locked” clumping
Extreme weak coupling

Funnel narrows in accord with [AHUY] prediction

\[ a_{mf} \sim b^{-1/4} \]

\[ \kappa_c(g=0) \]

In fact, funnel closes before \( b = \infty \) due to non-universal UV effect:
\[ \text{tr}(U_1U_2U_3U_4) \neq 0 \quad [\text{Lohmayer & Narayanan}] \]

Can fix by small change to fermion action.
Conclusions for $N_f=2$ AEK model

* In range of interesting values of $b$ (and beyond) volume independence works for $|m| < O(1/a)$

  ➡ Crucial first test of reduction has been passed

  ➡ Also seen on $2^4$ lattice by [Catterall, Galvez & Unsal, JHEP 1008 (2010) 010]

  ➡ By tuning quark mass can use reduction to study both pure gauge theory and (nearly) conformal theory

  ➡ Semi-analytic understanding of phase diagram

* Phase diagram similar to that for $N_f=1$

  ➡ No sign of 2nd-order transition seen for $N=2$
Problems at very large $N$?

- Extrapolate average plaquette to $N=\infty$ using $N\leq 53$
- Extrapolation requires $1/N$ term
- Result should lie close to pure gauge value

$N_f = 2$, $b=0.35$, $\kappa=0.12$ (heavy adjoint)

[Bringoltz, Koren & SS]
Problems at very large $N$?

- New results with $N$ up to 289 [Gonzalez-Arroyo & Okawa]
- Non uniform behavior in $N$!? ($k=0$ points in plot)

$k=1,3,5$ points are with Twisted AEK model---have better behavior

$N_f = 2$, $b=0.35$, $\kappa=0.12$ (heavy adjoint)
Problems at very large $N$?

- New results with $N$ up to 289 [Gonzalez-Arroyo & Okawa]
- Form of $N$ dependence varies with parameters

$k=1,3,5$ points are with Twisted AEK model—have better behavior

$N_f = 2$
$b = 0.35$
$\kappa = 0.14$
(lighter adjoints)
Twisted Adjoint Eguchi-Kawai (TAEK) model: recent results

A. Gonzalez-Arroyo & M. Okawa, arXiv: 1304.0306, 1305.6253
**Action of single-site TAEK model**

Only change from AEK is twist in gauge action:

\[
S_{\text{gauge}} = 2Nb \sum_{\mu < \nu} \text{Re} \text{Tr} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger Z_{\mu\nu}
\]

\[
S_F = \sum_{j=1,N_f} \bar{\psi}_j D_W \psi_j
\]

\[
D_W = 1 - \kappa \sum_{\mu=1}^4 \left[ (1 - \gamma_\mu) U_\mu^{\text{adj}} + (1 + \gamma_\mu) U_\mu^{\dagger \text{adj}} \right]
\]

- **Weak coupling:** $Z_{N^4}$ broken to $Z_{L^4}$; perturbation theory as $L \rightarrow \infty$ reproduces that on $L^4$ lattice
- **Spectrum of $D_W$** in weak coupling identical to that on an $L^4$ lattice
- **Pure gauge:** $k=1$ theory fails at large $N$; revived by using $k/L > 0.1$
- **Adjoints not necessary for reduction---used because of physics interest**
Reduction works for TEK model

- Pure gauge: $16^4$ with $N<16$ vs $1^4$ with $N=289, 529, 841$ ($L=17, 23, 29$) $k=5, 7, 9$
Wilson loops and string tension

\[ W(4,4) = \]

\[ W(M, M') \xrightarrow{M' \to \infty} e^{-V(M)M'} \]

\[ V(M) \xrightarrow{M \to \infty} \sigma M \]

- Can reach loops of size L/2 x L/2 (since “volume” is L^4)
- Use smearing to get good signal for large loops (standard method)
Reduction works for TEK model

- Pure gauge: $32^4$ with 3, 5, 6, 8 vs $1^4$ with $N=841$, $k=9$ ($L=29$)
Improved $N$ dependence for TAEK model

$N_f = 2$, $b=0.35$, $\kappa=0.14$

(Same plot as shown above)
Search for conformality

TAEK model
$N_f=2$
$N=289, k=5$

Preliminary

Quark mass & string tension vanish!

S. Sharpe, “Large N reduction” 6/29/13 @ Cracow School of Theoretical Physics, Zakopane, Poland
Cross-check

[Гонсалес-Арройо & Окава, arXiv:1304.0306]

TAEK model
$N_f=1$
$N=289, k=5$

Preliminary

Quark mass vanishes but string tension does not!
Conclusions & Outlook
EK reduction appears practical

- Need large values of N (e.g. $289 = 17^2$, $841 = 29^2$)
  - Not surprising once accept that $L = \sqrt{N}$ (no free lunch!)

- Twisted model appears to be the “model of choice”
  - Only downside is that it is difficult to include fundamental fermions
  - Without twisting, can use heavy adjoints to stabilize center symmetry

- Successful calculation of string tension
  - First application: indications of conformal fixed-point for 2 adjoints
**Future directions & issues**

- **Calculation of hadron properties in $N_f=1$ TAEK**
  - In principle, can calculate hadron masses, glueball-qq-bar mixing, ... on a single site, although it may be easier to extend in one direction
  - Window into hadron resonances where decays widths are small

- **Efficient implementation on supercomputers?**
  - Use $2^4$ (or larger) to allow parallelism

- **Scaling vs standard large $N$ extrapolation?**
  - We find $CPU \sim N^{4.5} L^5 N^2$ vs. standard $CPU \sim L^5 N^3$

- **Extend calculation to $N_f=1/2$ using overlap fermions**
  - [Heitanen & Narayanan] have taken first steps
Thank you!

Any questions?