Scattering amplitudes from Euclidean Lattice QCD



Steve Sharpe
University of Washington



Outline

☑ Lecture 1

- Motivation/Background/Overview
- Deriving the two-particle quantization condition (QC2)

☑ Lecture 2

- Applying the QC2, in brief
- Deriving the three-particle quantization condition (QC3)
- Status of three-particle formalism & applications

☐ Lecture 3: Research Talk

- Recent applications of the QC3
- Outlook

Outline for Lecture 3

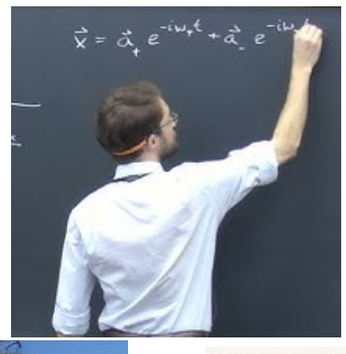
- Recent applications of three-particle formalism
  - Three-particle amplitudes involving pions & kaons at physical quark masses
  - $DD\pi$ scattering, relevant for the T_{cc}^{+}
  - Dreaming of three neutrons
- Summary & Outlook

Recent applications of three-particle formalism

QCD predictions for physical multimeson scattering amplitudes

Sebastian M. Dawid,[1] Zachary T. Draper,[1] Andrew D. Hanlon,[2] Ben Hörz,[3] Colin Morningstar,[4] Fernando Romero-López,[5,6] Stephen R. Sharpe,[1] and Sarah Skinner[4]

[2502.14348 (PRL) & 2502.17976 (PRD)]

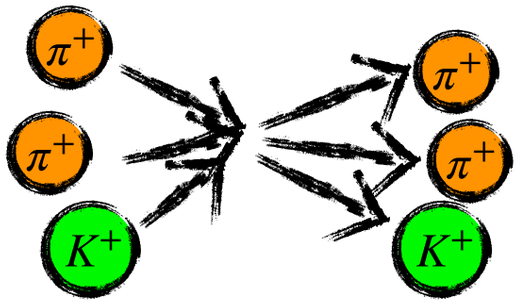








Three PGBs at maximal isospin



$$3\pi^+, \quad 3K^+, \quad \pi^+\pi^+K^+, \quad K^+K^+\pi^+$$

- Benchmark system, with simple repulsive dynamics
  - First calculation at physical quark masses
  - Study $J^P = 0^-, 1^+, 2^-$, including subchannels with $\ell = 0, 1, 2$
  - Compare to expectations from ChPT
- Use CLS ensembles ($\mathcal{O}(a)$ improved Wilson) + GEVP

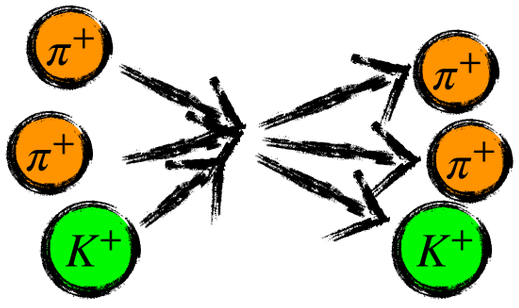
	$(L/a)^3 \times (T/a)$	M_π[MeV]	M_K[MeV]	$N_{\rm cfg}$	$M_\pi L$
N203	$48^3 \times 128$	340	440	771	5.41
N200	$48^3 \times 128$	280	460	1712	4.42
D200	$64^3 \times 128$	200	480	2000	4.20
E250	$96^3 \times 192$	130	500	505	4.05

[Blanton, SRS, et al., PRL 2020 & JHEP 2021]
[Draper, SRS, et al., JHEP 2023]

New work

$$a \simeq 0.063 \text{ fm}$$

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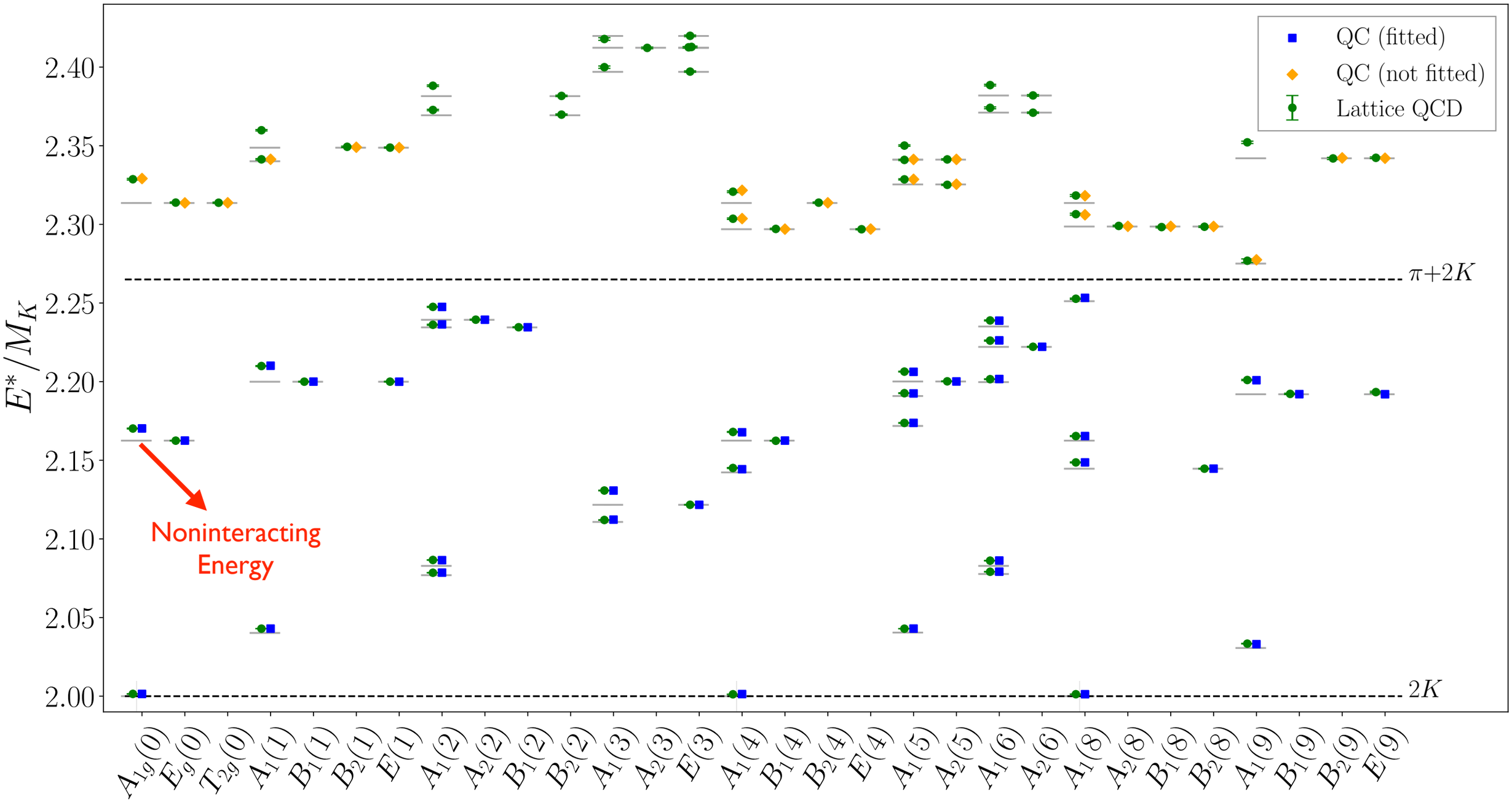
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New work

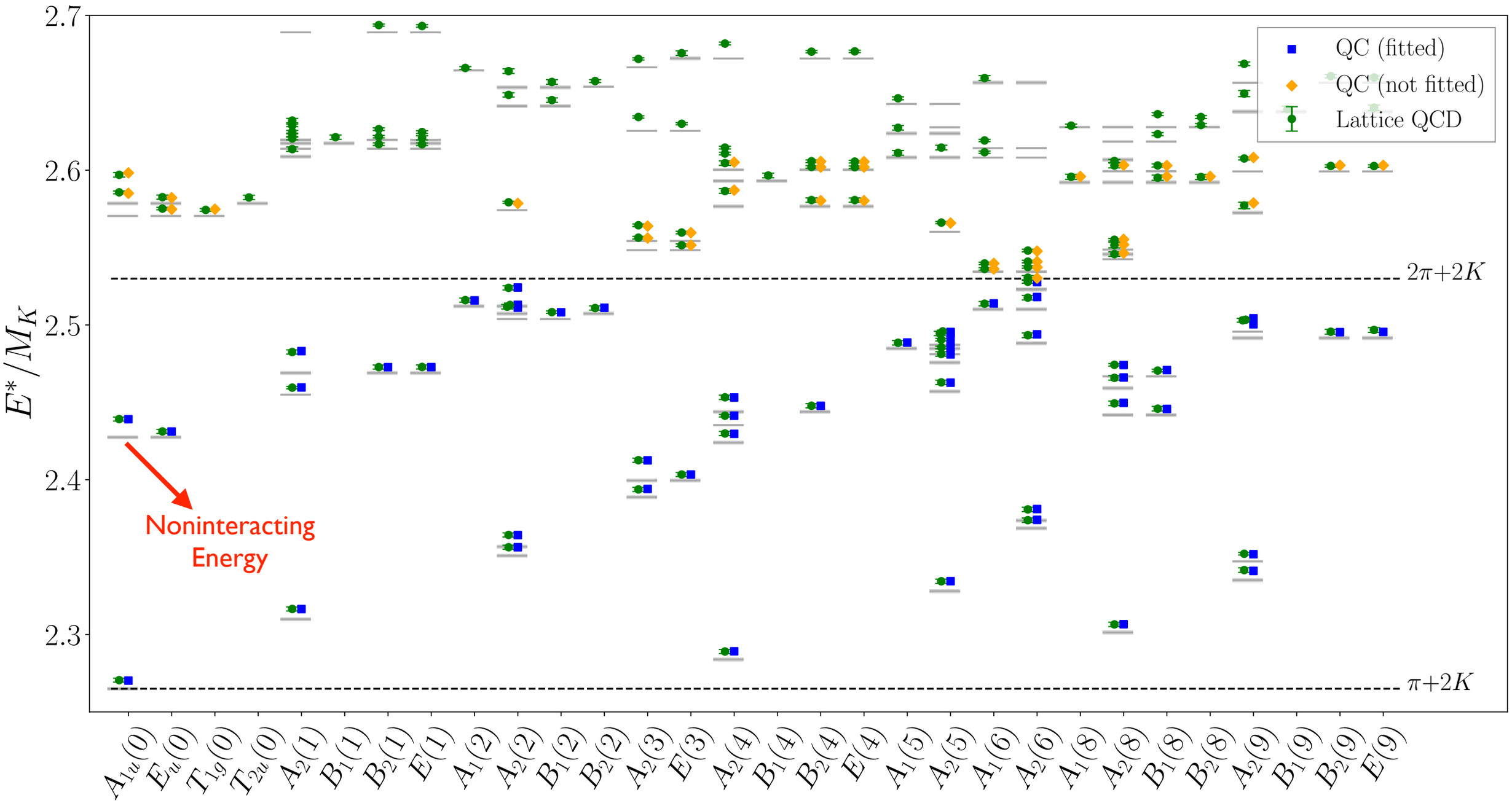
$$a \simeq 0.063 \text{ fm}$$
$$\text{tr } m_q = 2m_{ud} + m_s \simeq \text{const}$$

Example of spectrum



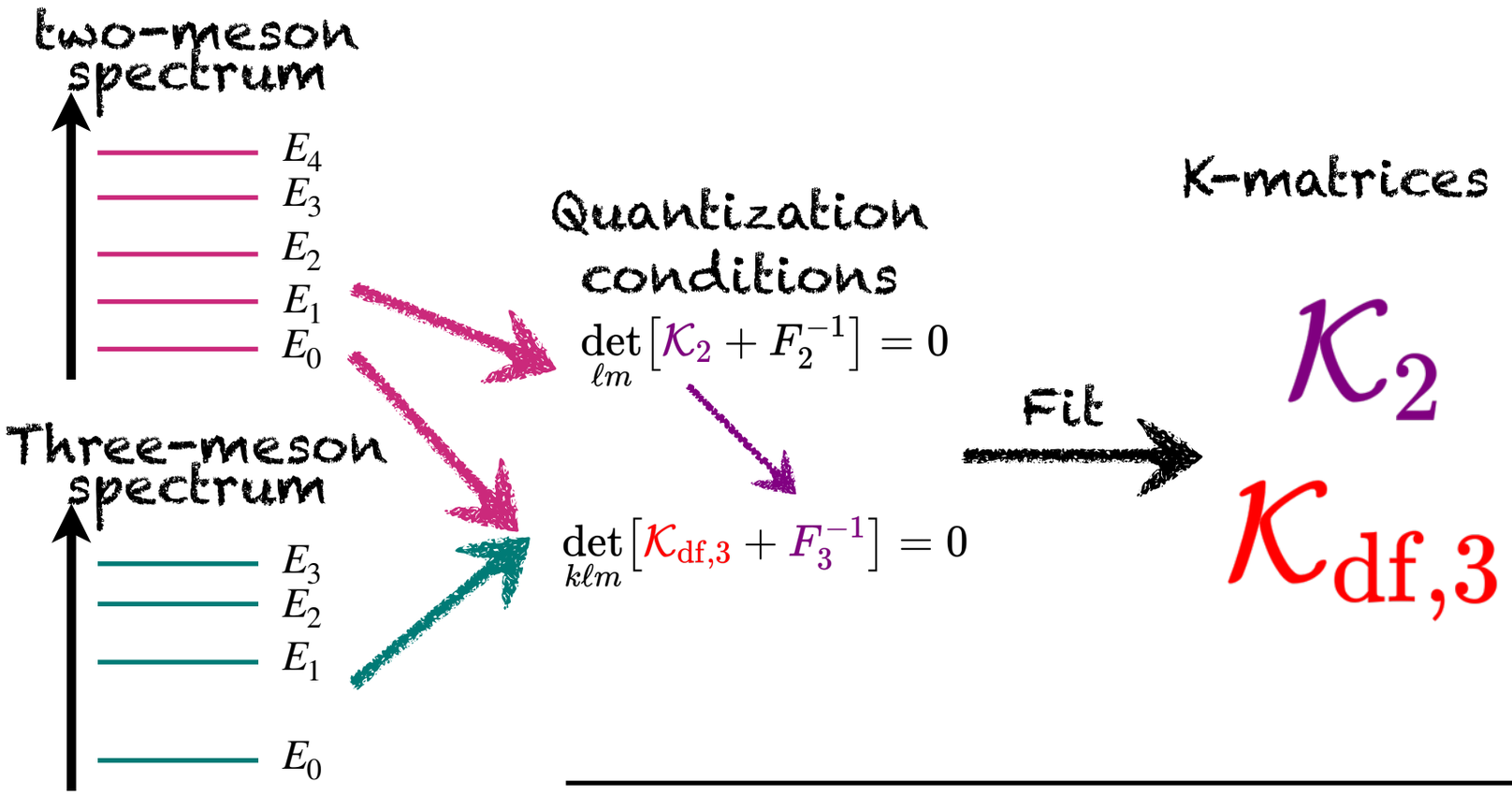
E250: K^+K^+

Example of spectrum



E250: $K^+K^+\pi^+$

Fits to spectra

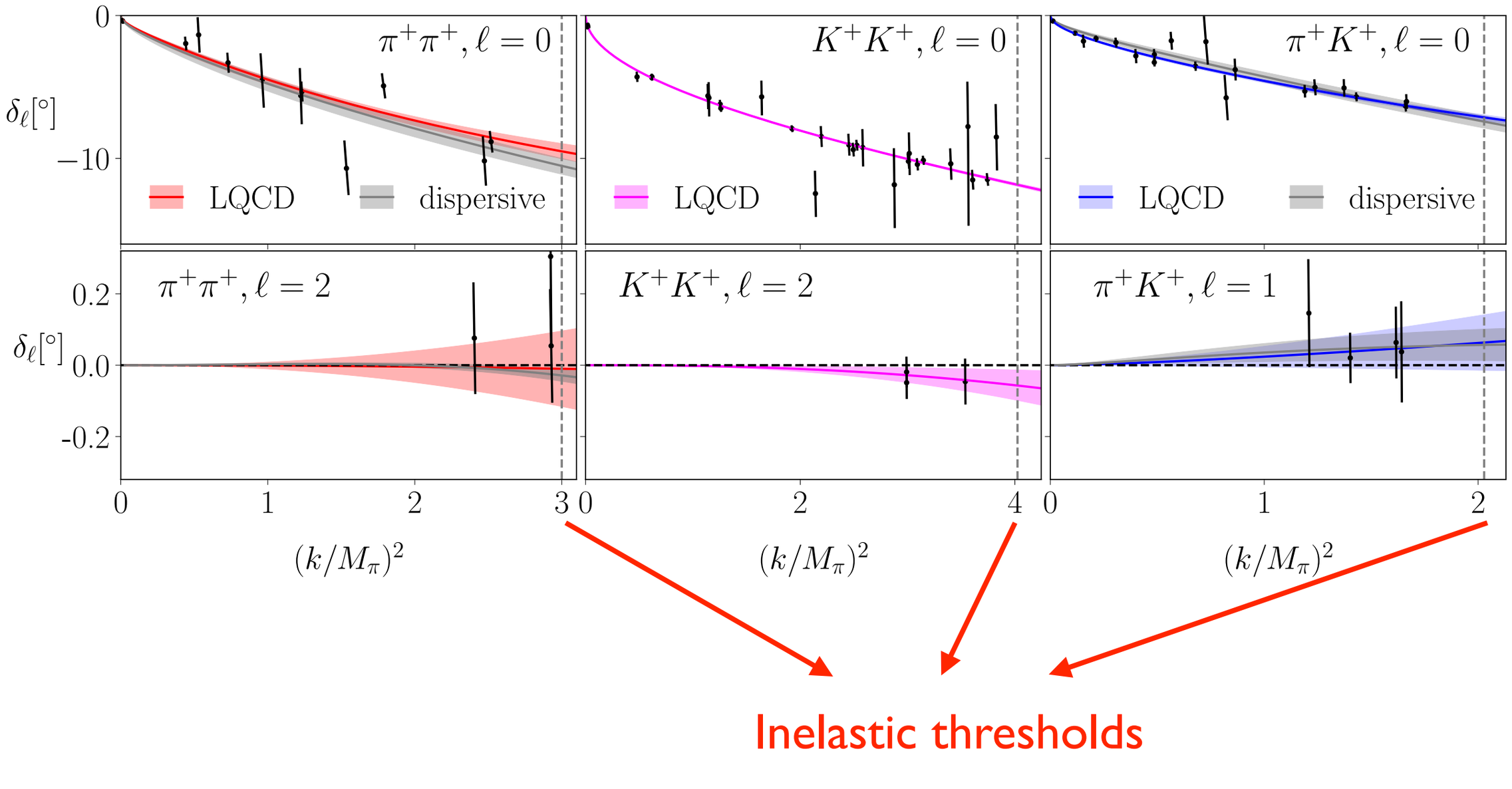


channels	# levels	# parameters	χ^2/DOF
$2\pi/3\pi$	$34+32$	6	1.14
$2\pi/\pi K/2\pi K$	$24+25+23$	9	1.23
$2K/\pi K/2K\pi$	$25+40+50$	10	1.95
$2K/3K$	$40+53$	6	1.49

Parameters in $\mathscr{K}_2$ and $\mathscr{K}_3$: use threshold expansions

2-particle interactions

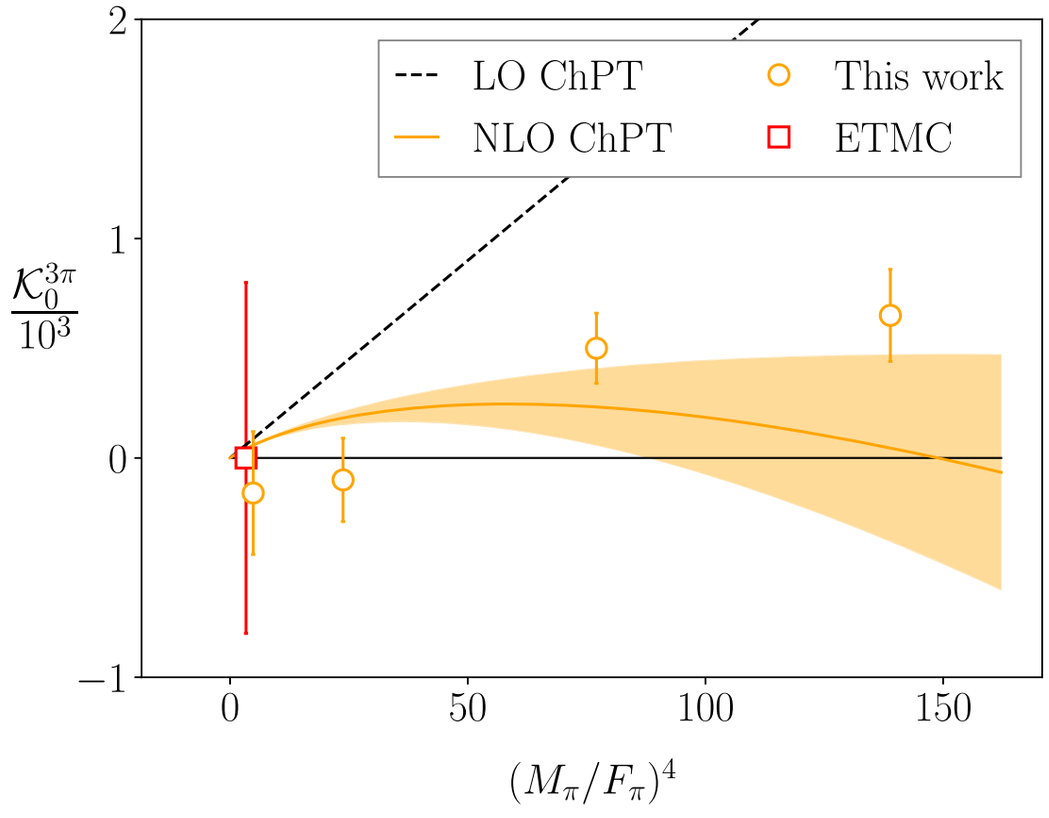
E250: ~physical quark masses

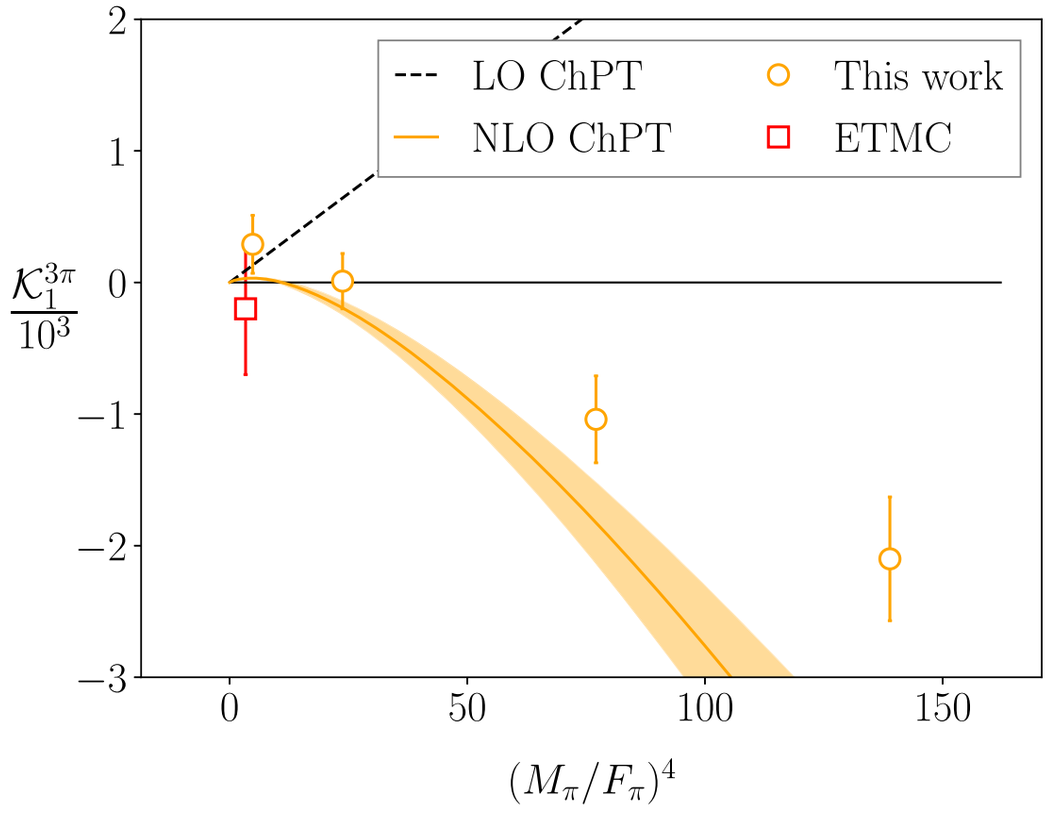


Comparing $\mathcal{K}_{\mathrm{df},3}$ to ChPT

- Parametrize $\mathcal{K}_{\mathrm{df},3}$ in a threshold expansion, keeping first two terms
- From fits to spectra, we find non vanishing results for these terms at heavier quark masses
- Compare to NLO Chiral Perturbation theory calculation of $\mathcal{K}_{\mathrm{df},3}$ for $3\pi \to 3\pi$

[Baeza-Ballesteros, Bijnens, Husek, Romero-López, SRS, Sjö, 2303.13206 (JHEP) & 2401.14293 (JHEP)]

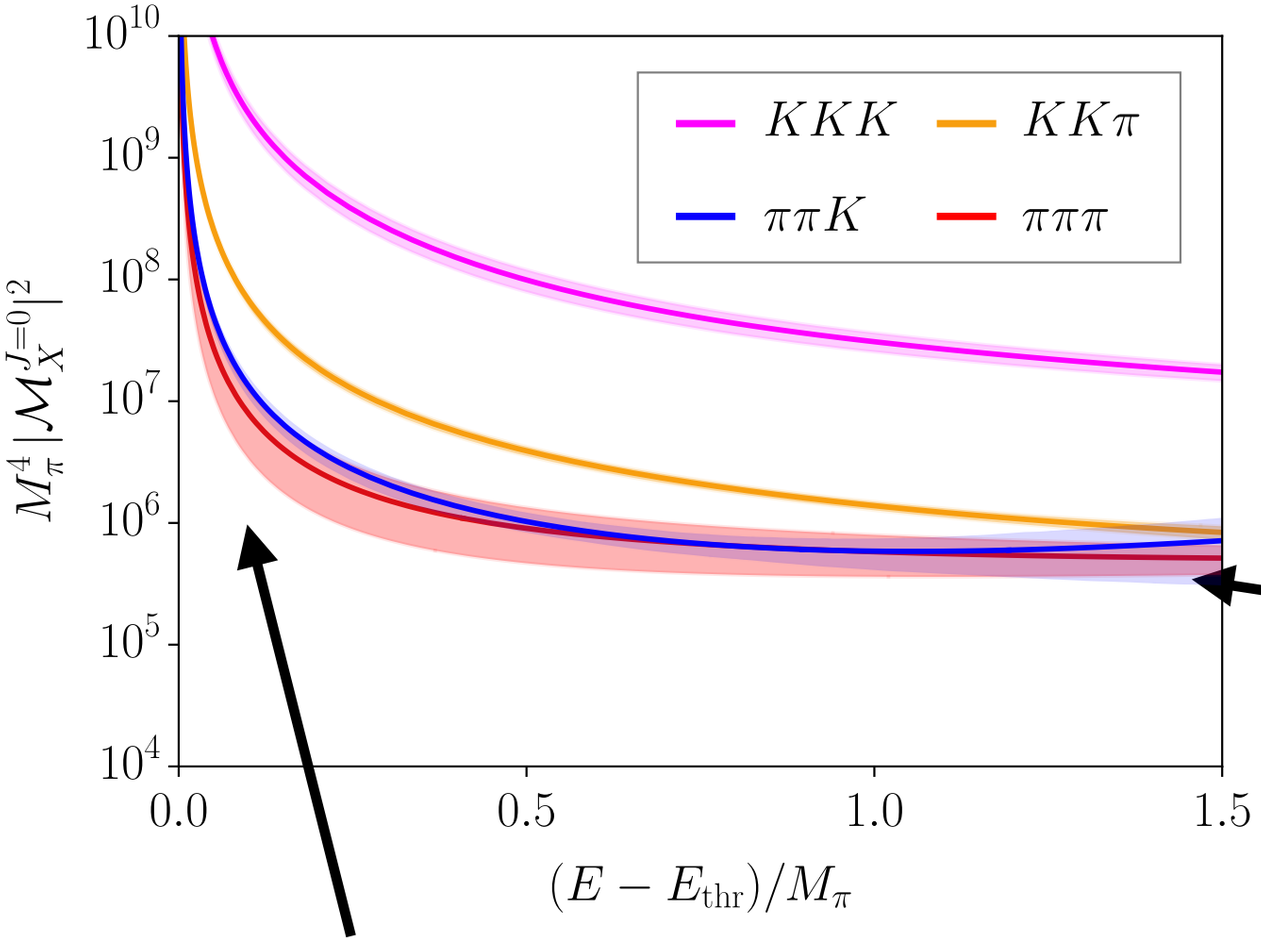




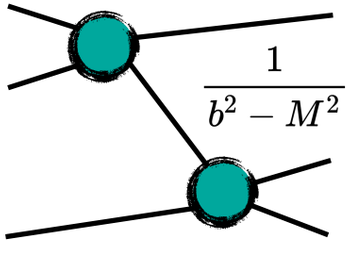
Large NLO corrections in ChPT resolve LO inconsistency

Physical 3-particle amplitudes

$M_\pi^4 |\text{Amplitude}|^2$



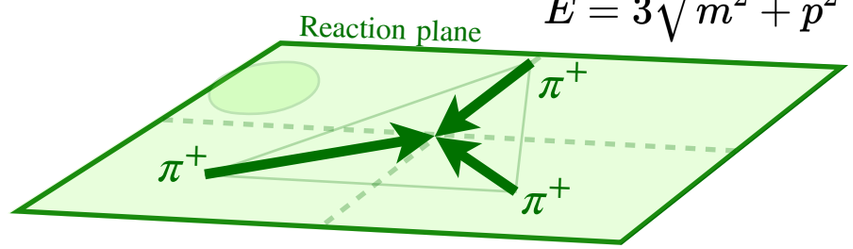
Divergent at threshold



$J^P = 0^-$

Equilateral kinematic configuration

$E = 3\sqrt{m^2 + p^2}$



Pion interactions are chirally suppressed

$$\mathcal{M}_3 = \mathcal{D} + \mathcal{M}_{\text{df,3}}$$

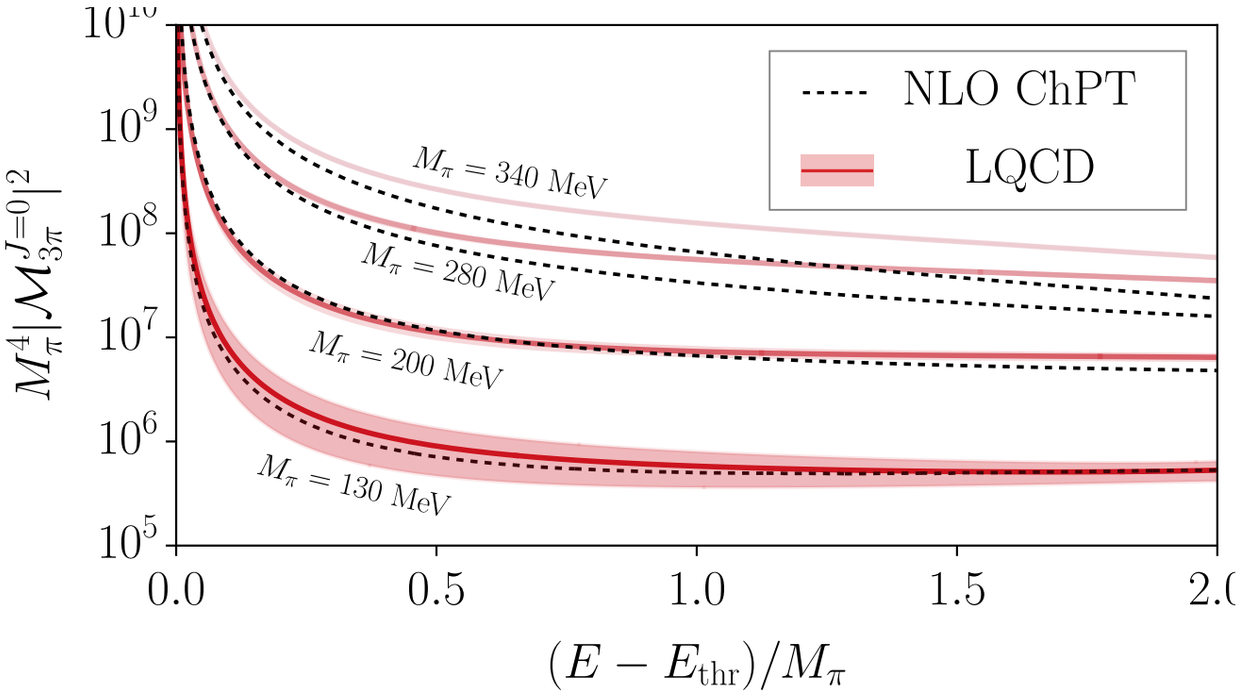
$$\mathcal{D} = \mathcal{S}\left\{ \mathcal{M}_2\, G\, \mathcal{M}_2 + \mathcal{M}_2\, G\, G\, \mathcal{M}_2 + \ldots \right\}$$

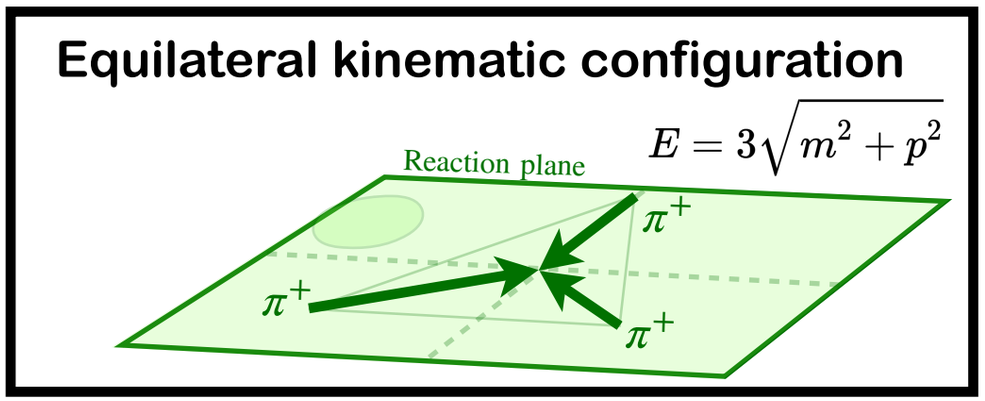
$$\mathcal{M}_{\text{df,3}} = \mathcal{K}_{\text{df,3}} + \mathcal{S}\left\{ \mathcal{M}_2\, \rho\, \mathcal{K}_{\text{df,3}} + \mathcal{K}_{\text{df,3}}\, \rho\, \mathcal{M}_2 + \ldots \right\}$$

Comparison with ChPT

$M_\pi^4 |\text{Amplitude}|^2$

$J^P = 0^-$





Observe expected breakdown of ChPT as M_π or p_π increase

Using three-particle methods to study the $T_{cc}^+(3875)$



Incorporating DDπ effects and left-hand cuts in lattice QCD studies of the $T_{cc}(3875)^+$

Maxwell T. Hansen,[a] Fernando Romero-López[b] and Stephen R. Sharpe[c]

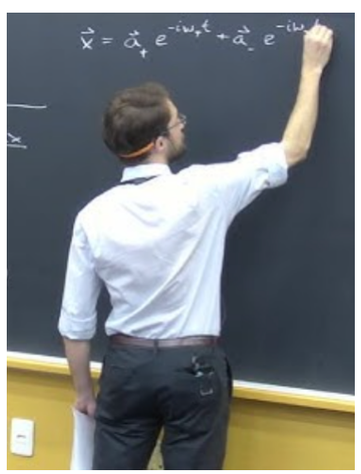
[2401.06609, JHEP]



Finite- and infinite-volume study of $DD\pi$ scattering

Sebastian M. Dawid,[a] Fernando Romero-López[b,c] and Stephen R. Sharpe[a]

[2409.17059, JHEP]



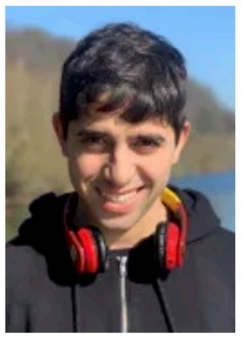
Comparison of integral equations used to study T_{cc}^+ for a stable D^*

Sebastian M. Dawid,[a] Fernando Romero-López[b] and Stephen R. Sharpe[a]

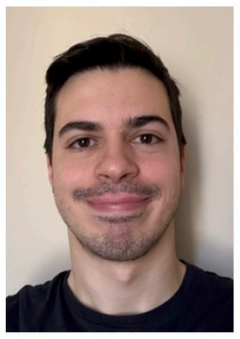
[2505.05466, JHEP]

Using three-particle methods to study the $T_{cc}^{+}(3875)$

+ work in progress



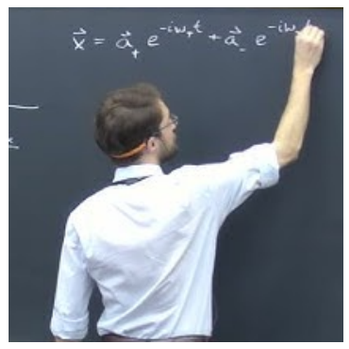
Herzallah Alharazin (Bochum)



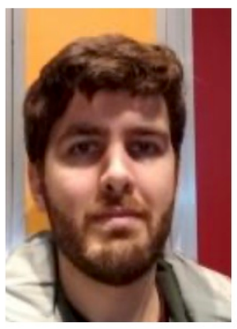
André Baião-Raposo (Bochum)



John Bulava (Bochum)



Sebastian Dawid (Indiana)



Jeremy Green (DESY)



Colin Morningstar (CMU)



Davide Laudicina (Bochum)



Fernando Romero-López (U. Bern)



Miguel Salg (U. Bern)



Andres Stump (Humboldt U.)

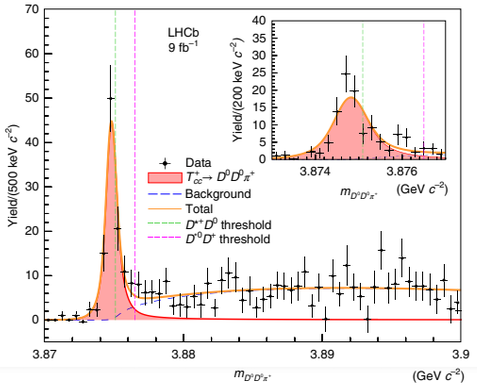
Doubly-charmed tetraquark

NATURE PHYSICS | VOL 18 | JULY 2022 | 751–754 | www.nature.com/naturephysics

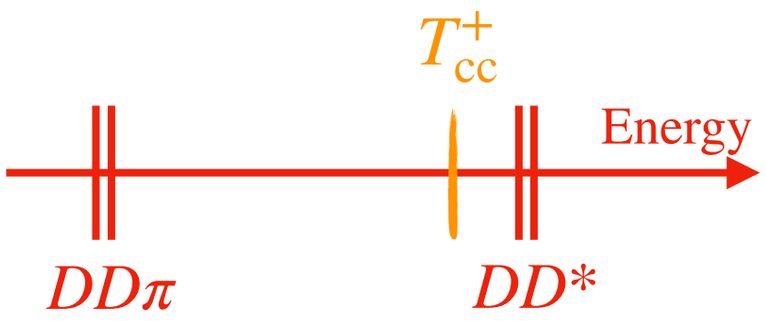
Observation of an exotic narrow doubly charmed tetraquark

LHCb Collaboration*

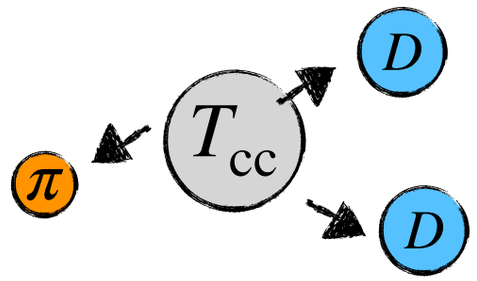
$$T_{cc}^{+}(3875)$$
$$I = 0,\ J^P = 1^+$$
$$cc\bar{u}\bar{d}$$



Experiment



- **For physical quark masses is a three-body resonance**



need three-body formalism!

Adapted from: Fernando Romero-López

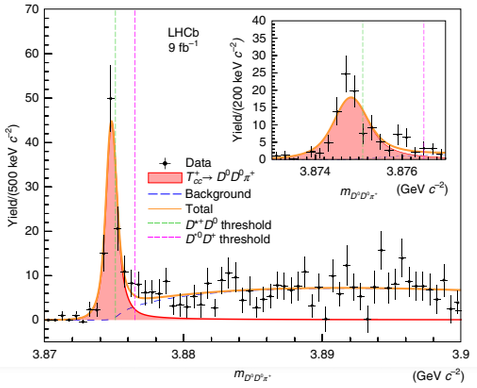
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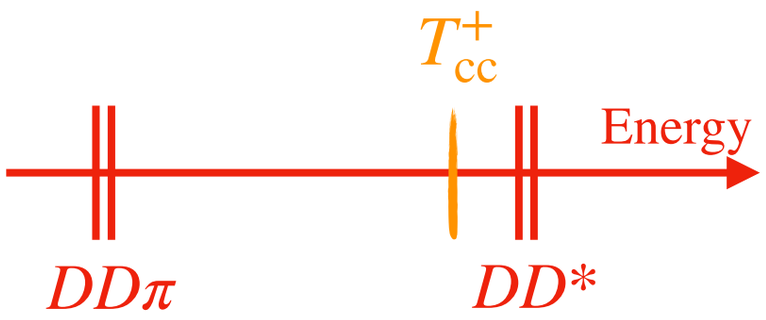
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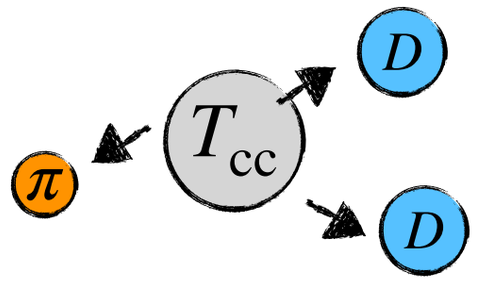
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$$I = 0,\ J^P = 1^+$$
$$cc\bar{u}\bar{d}$$



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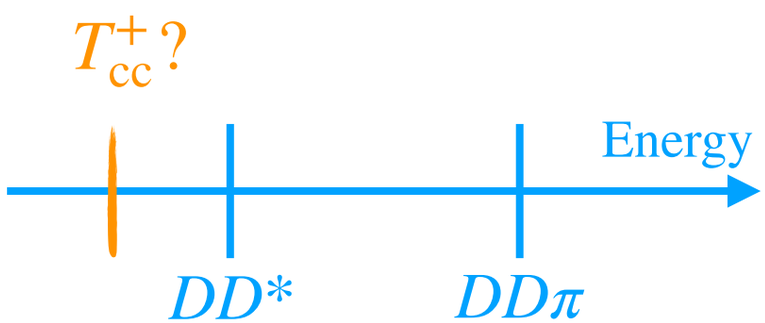


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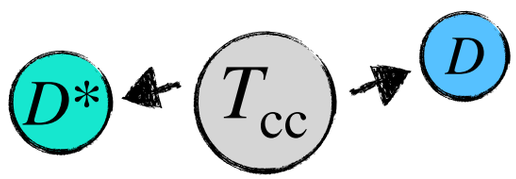


need three-body formalism!

LQCD calculations with heavier-than-physical quarks



- D* is stable for slightly heavier-than-physical quarks



suitable for the two-body finite-volume formalism?

Adapted from: Fernando Romero-López

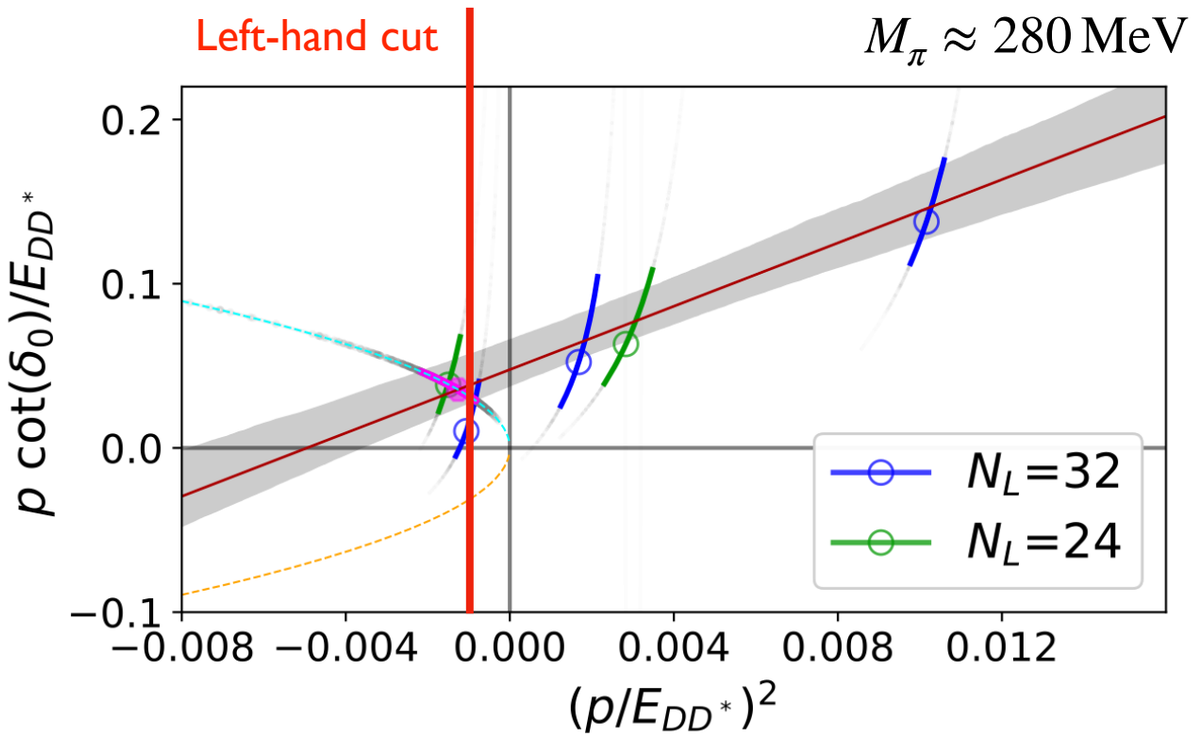
DD^* scattering

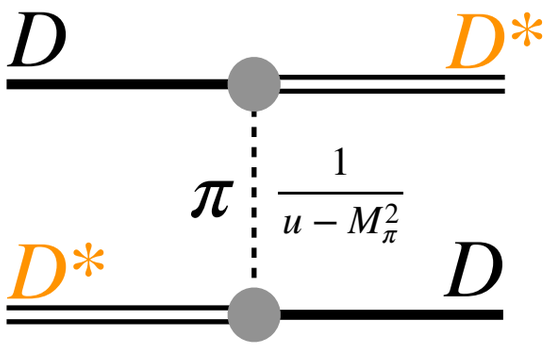
- Several LQCD studies using QC2 in T_{cc} channel

  [Chen et al., 2206.06185]
  [Lyu et al. (HALQCD), 2302.04505]
  [Padmanath & Prelovsek, 2202.10110]
  [Whyte, Thomas, Wilson, 2405.15741]
  [Several talks at Lattice 2025]

  - Use heavier-than-physical quarks
  - Find signature of virtual bound state
  - But QC2 fails at left-hand cut, below which phase shift becomes complex

[Padmanath, Prelovsek, arXiv:2202.10110]





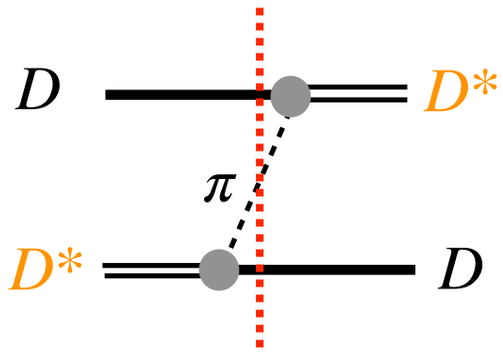
- Several solutions to failure have been proposed, by generalizing the QC2

  [Du et al (2408.09375); Abolnikov et al. (2407.04649), Bubna et al. (2402.12985);
  Meng et al. (2312.01930); Raposo, Hansen (2311.18793); Raposo, Hansen, Briceño, Jackura (2502.19375)]

3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

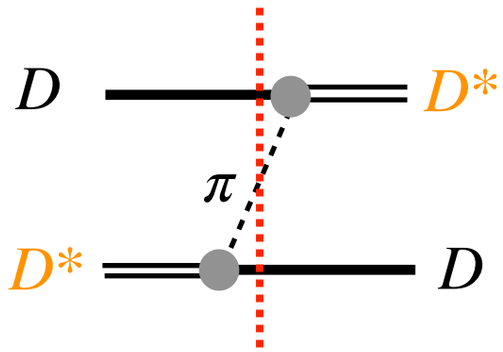
- Left-hand cut is a three-body effect



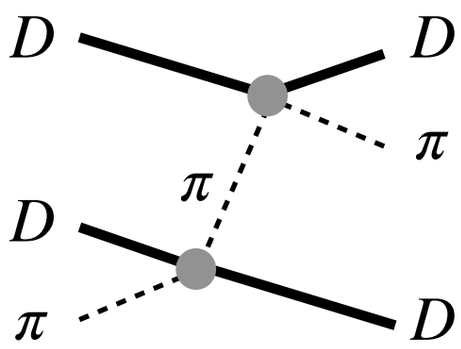
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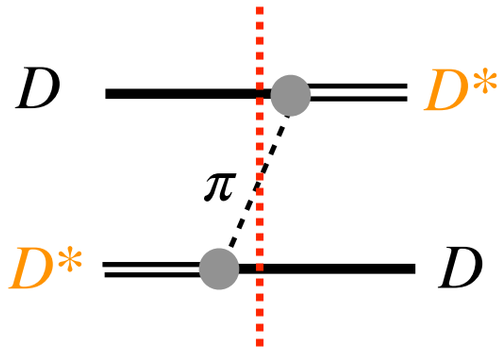
- Use $DD\pi$ QC3, but include D^* as a bound state in p-wave $D\pi$ channel



3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

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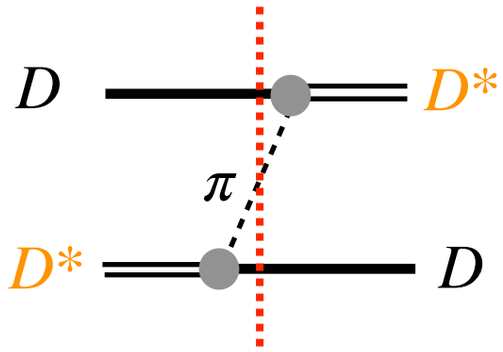


- Finite-volume effects from u-channel pion exchange naturally incorporated

3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

- Left-hand cut is a three-body effect



- Use $DD\pi$ QC3, but include D^* as a bound state in p-wave $D\pi$ channel



  - Finite-volume effects from u-channel pion exchange naturally incorporated

- Another advantage of QC3 approach:
  - Works for all choices of quark masses, including physical case of unbound D^*
- Successfully implemented in [Dawid, Romero-López, SRS, 2409.17059, JHEP & 2505.05466, JHEP; plus work in progress]

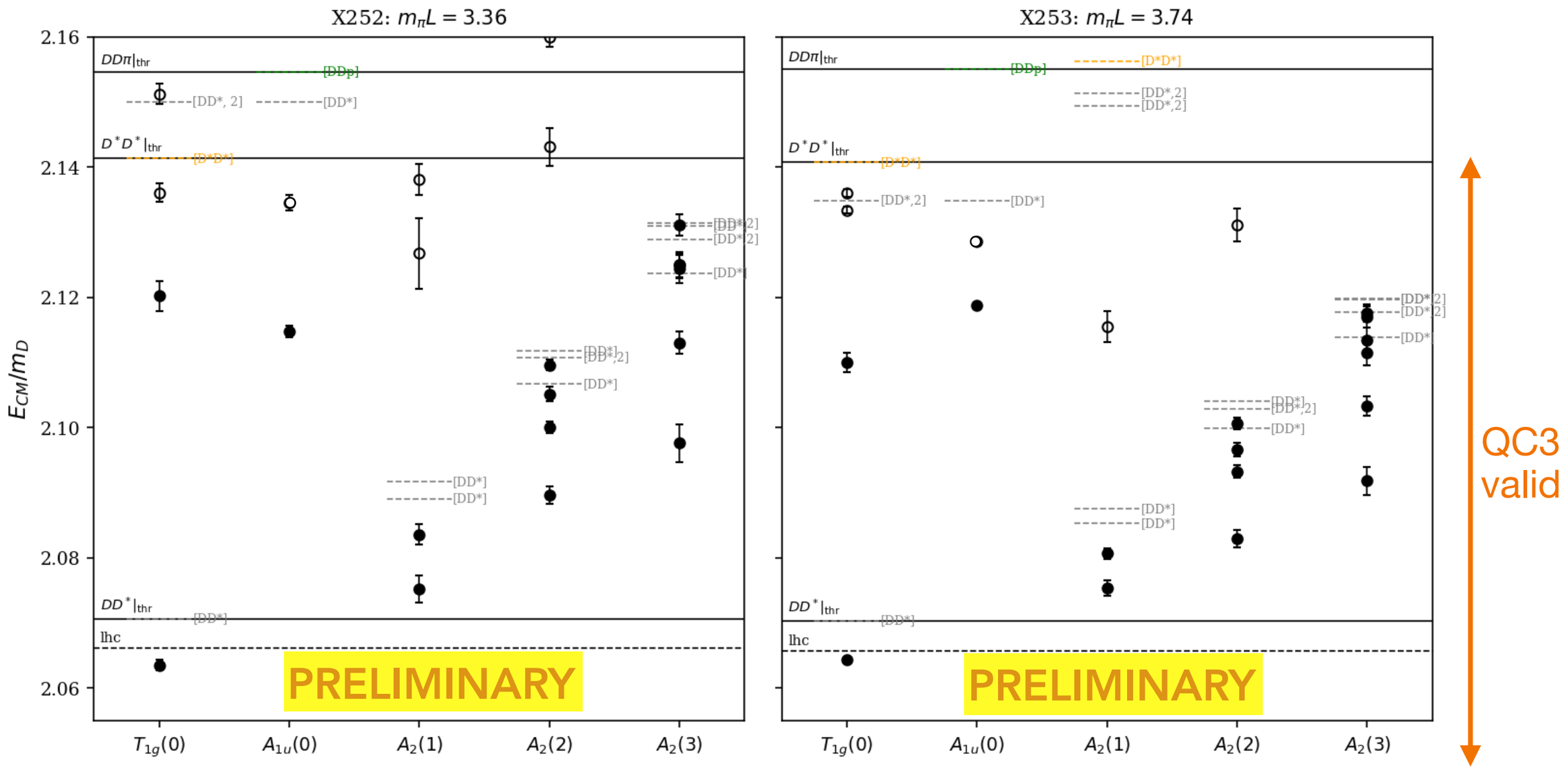
Simulation details

- Use two CLS ensembles with same pion mass but different volume

	L/a	T/a	$m_\pi L$	m_π[MeV]	a[fm]	M_D[MeV]	$(M_{D^*} - M_D)$[MeV]	Configs
X252	36	128	3.4	280	0.064	1880	130	175
X253	40	128	3.7	280	0.064	1880	130	301
Physical values:				135		1870	140-145	

- Use frames (0,0,0), (0,0,1), (1,1,0), (1,1,1) and irreps coupling dominantly to $J^P = 1^+, 0^-$
- Use distillation, with multiple operators, and GEVP
- Simplest fit uses two-parameter effective range expansion in s- and p-wave $D\pi$, and two-parameter threshold expansion for $\mathcal{K}_{\mathrm{df},3}$
- All results are PRELIMINARY

$DD\pi$ spectra

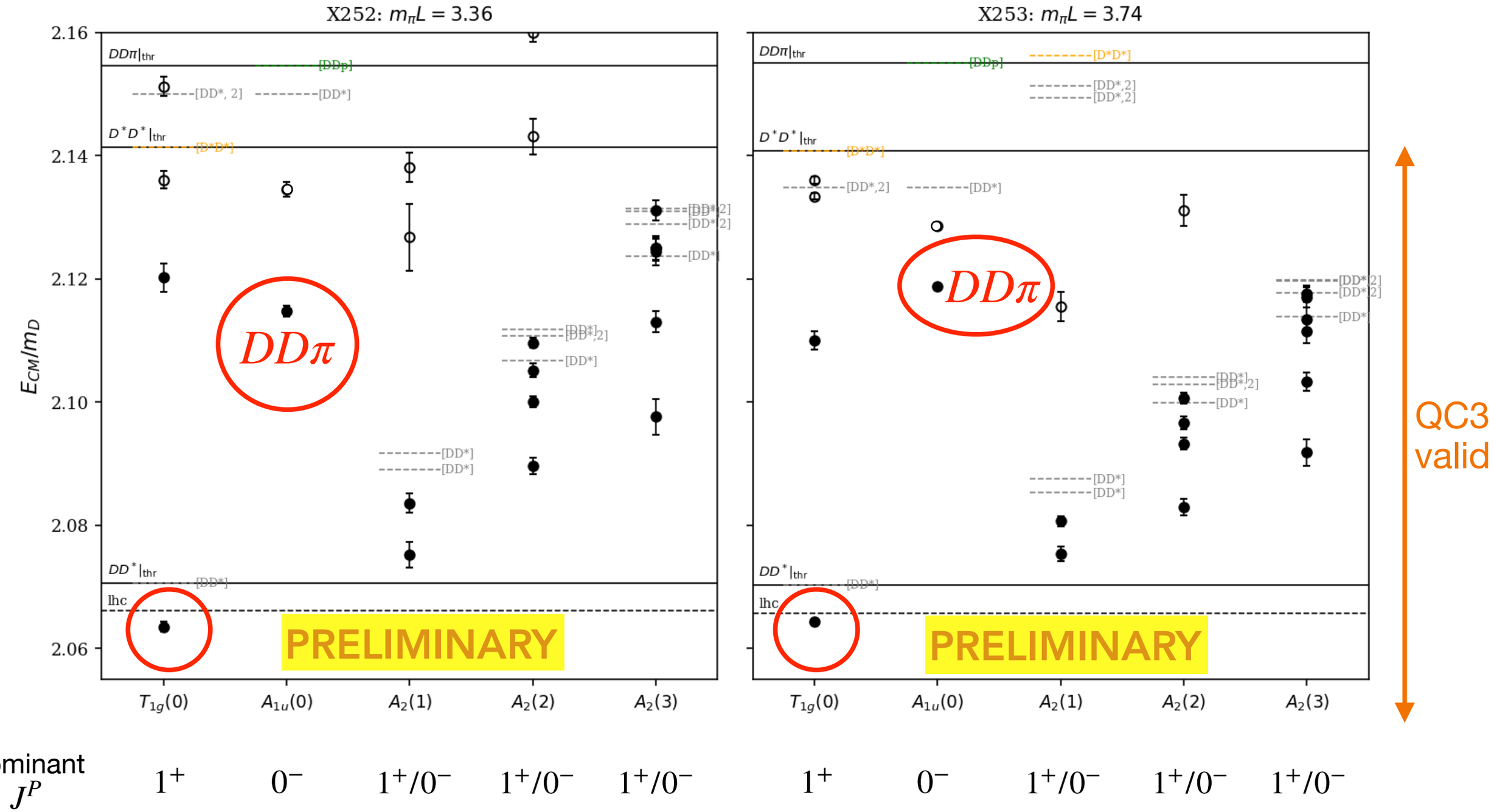


Dominant J^P	1^+	0^-	$1^+/0^-$	$1^+/0^-$	$1^+/0^-$	1^+	0^-	$1^+/0^-$	$1^+/0^-$	$1^+/0^-$

Keep $15 + 15$ $DD\pi$ levels in fit, together with $16 + 16$ $D\pi$ levels

$DD\pi$ spectra

3-particle effects crucial for some levels

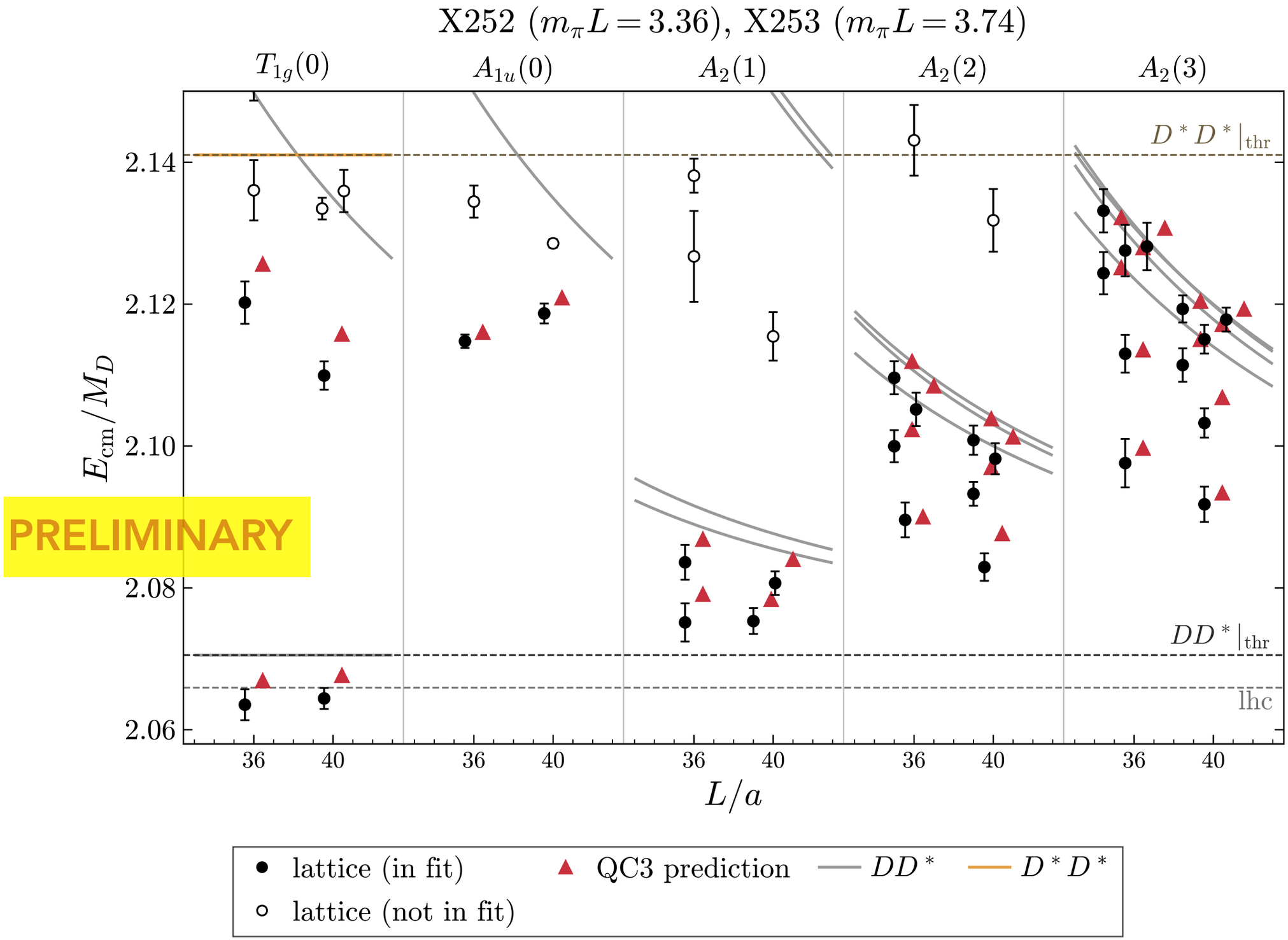


Dominant J^P	1^+	0^-	$1^+/0^-$	$1^+/0^-$	$1^+/0^-$	1^+	0^-	$1^+/0^-$	$1^+/0^-$	$1^+/0^-$

Keep $15 + 15$ $DD\pi$ levels in fit, together with $16 + 16$ $D\pi$ levels

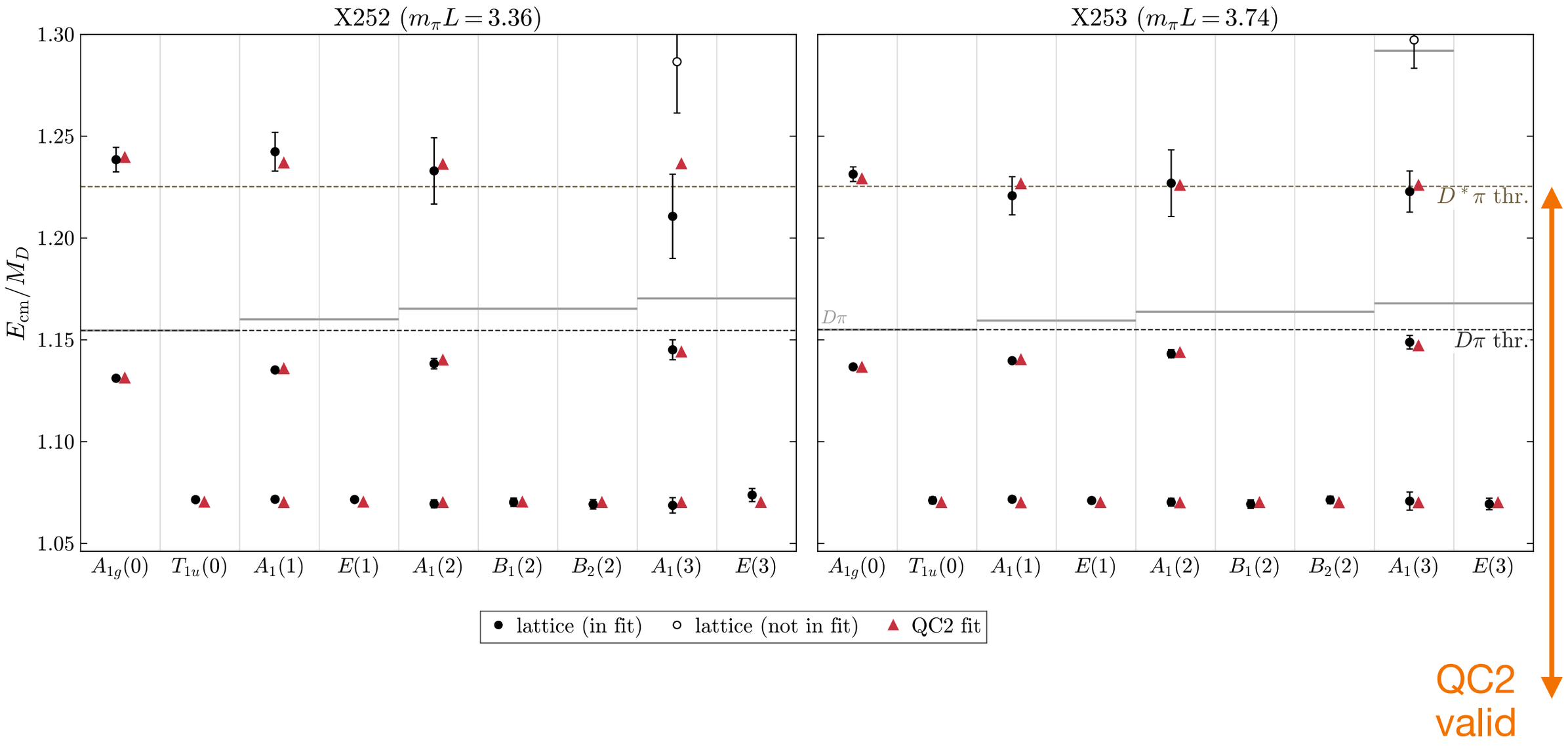
Global Fit ($DD\pi$ levels)

Combined fit to 62 levels with 6 params: $\chi^2/\mathrm{dof} = 88.5/56 = 1.58$



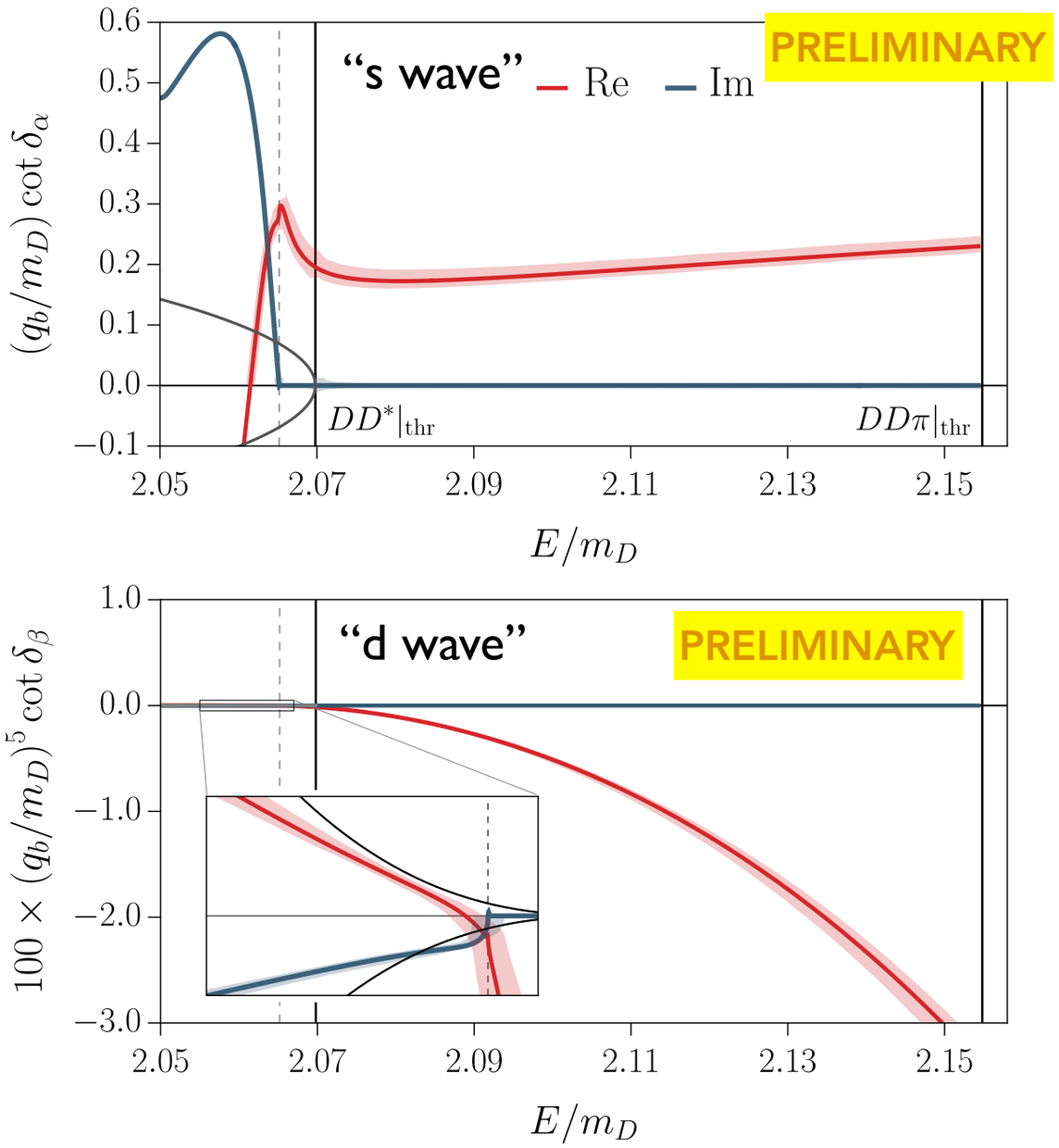
Global Fit ($D\pi$ levels)

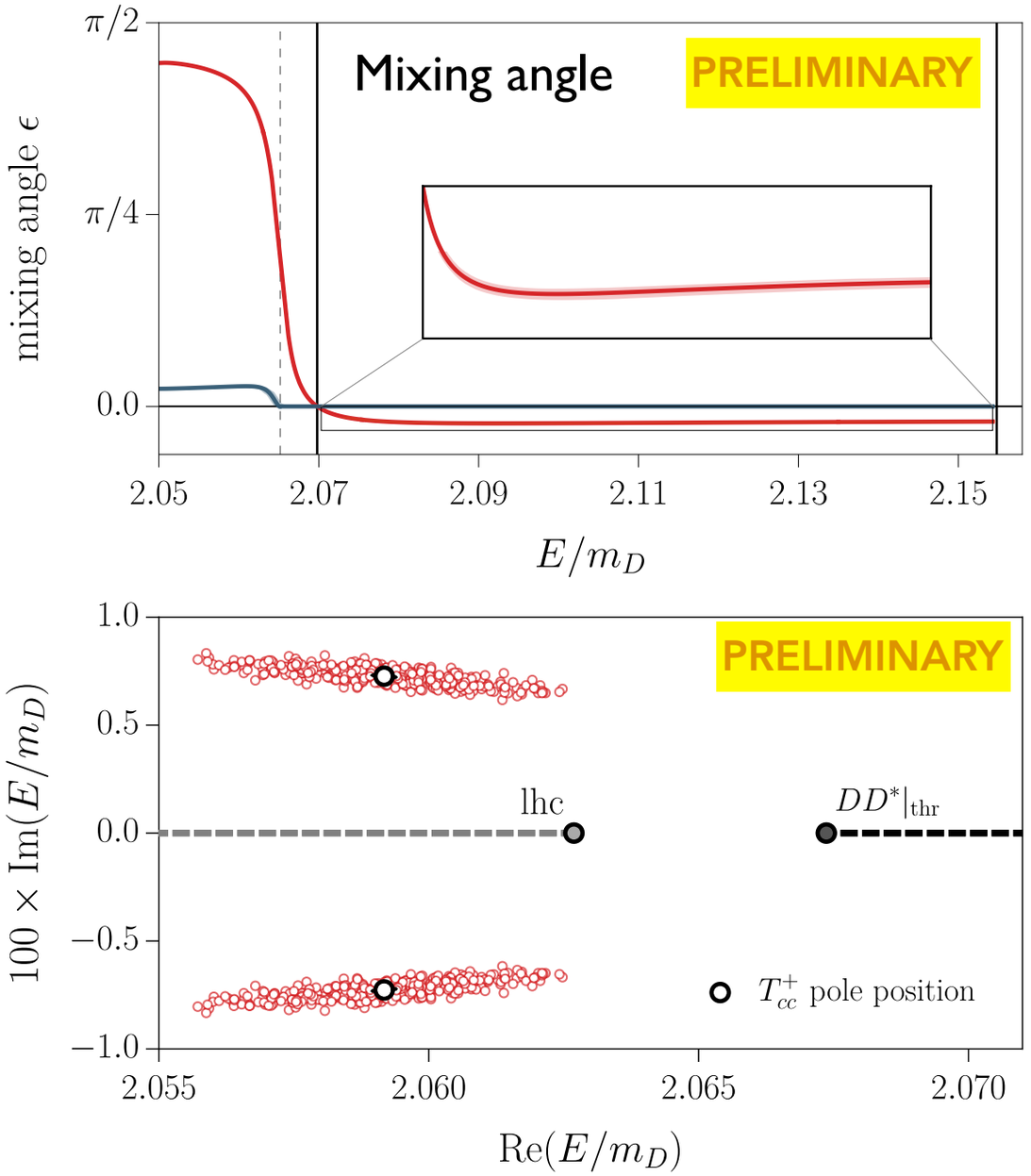
Combined fit to 62 levels with 6 params: $\chi^2/\mathrm{dof} = 88.5/56 = 1.58$



DD^* amplitude

- Given $\mathcal{K}_2$ and $\mathcal{K}_{\mathrm{df},3}$, solve integral equations to obtain DD^* amplitude using LSZ





- Both s & d waves combine with spin of D^* to give $J^P = 1^+$
- Use Blatt-Biedenharn parametrization of K matrix

$$(8\pi E)\left[\mathcal{K}_{DD^*}^{-1}\right] = \begin{pmatrix}\cos(\epsilon) & -\sin(\epsilon)\\ \sin(\epsilon) & \cos(\epsilon)\end{pmatrix}\begin{pmatrix}q_b\cot(\delta_\alpha) & 0\\ 0 & q_b\cot(\delta_\beta)\end{pmatrix}\begin{pmatrix}\cos(\epsilon) & \sin(\epsilon)\\ -\sin(\epsilon) & \cos(\epsilon)\end{pmatrix}$$

Dreaming of 3 neutrons



Three relativistic neutrons in a finite volume

Zachary T. Draper,[a] Maxwell T. Hansen,[b] Fernando Romero-López[c]
and Stephen R. Sharpe[a]





[2303.10219, JHEP]

Implementation of the three-neutron quantization condition

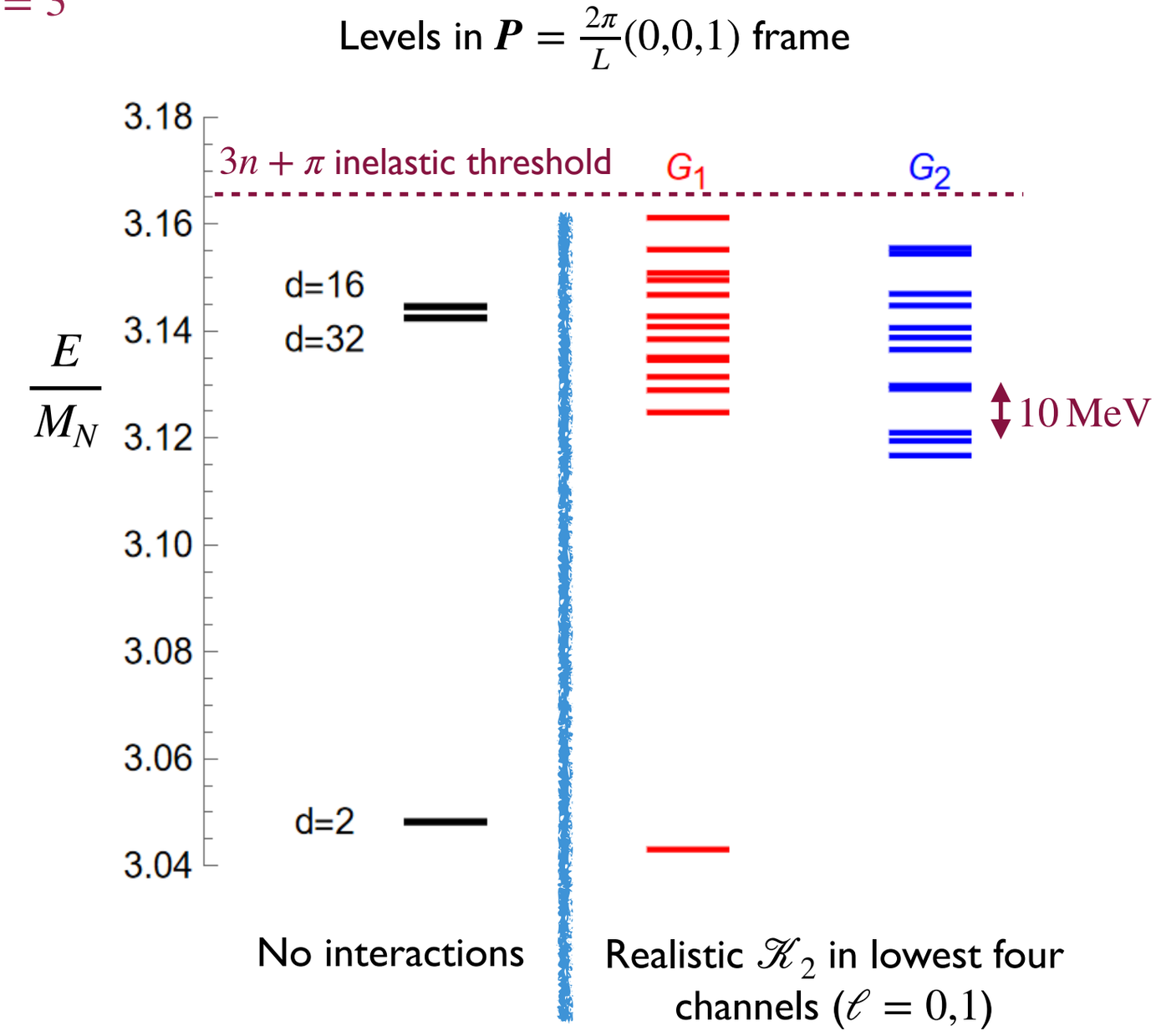
Wilder Schaaf[a] and Stephen R. Sharpe[a]



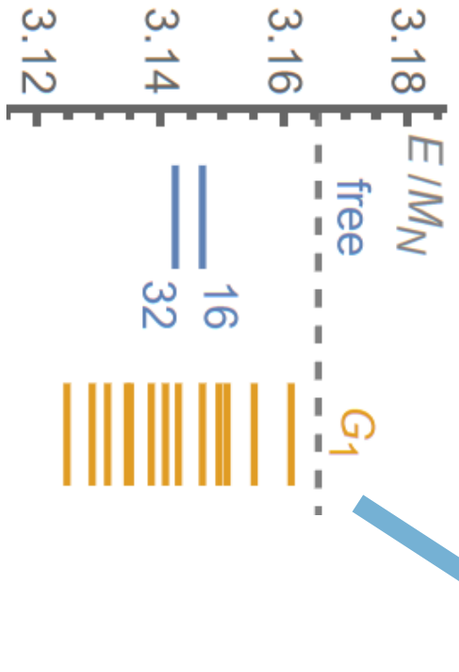
[2512.24508, JHEP]

Spectrum in realistic setup

- Physical masses, $L \approx 4.3$ fm:
  - $M_N L = 20,\ M_\pi = 0.15 M_N \Rightarrow M_\pi L = 3$

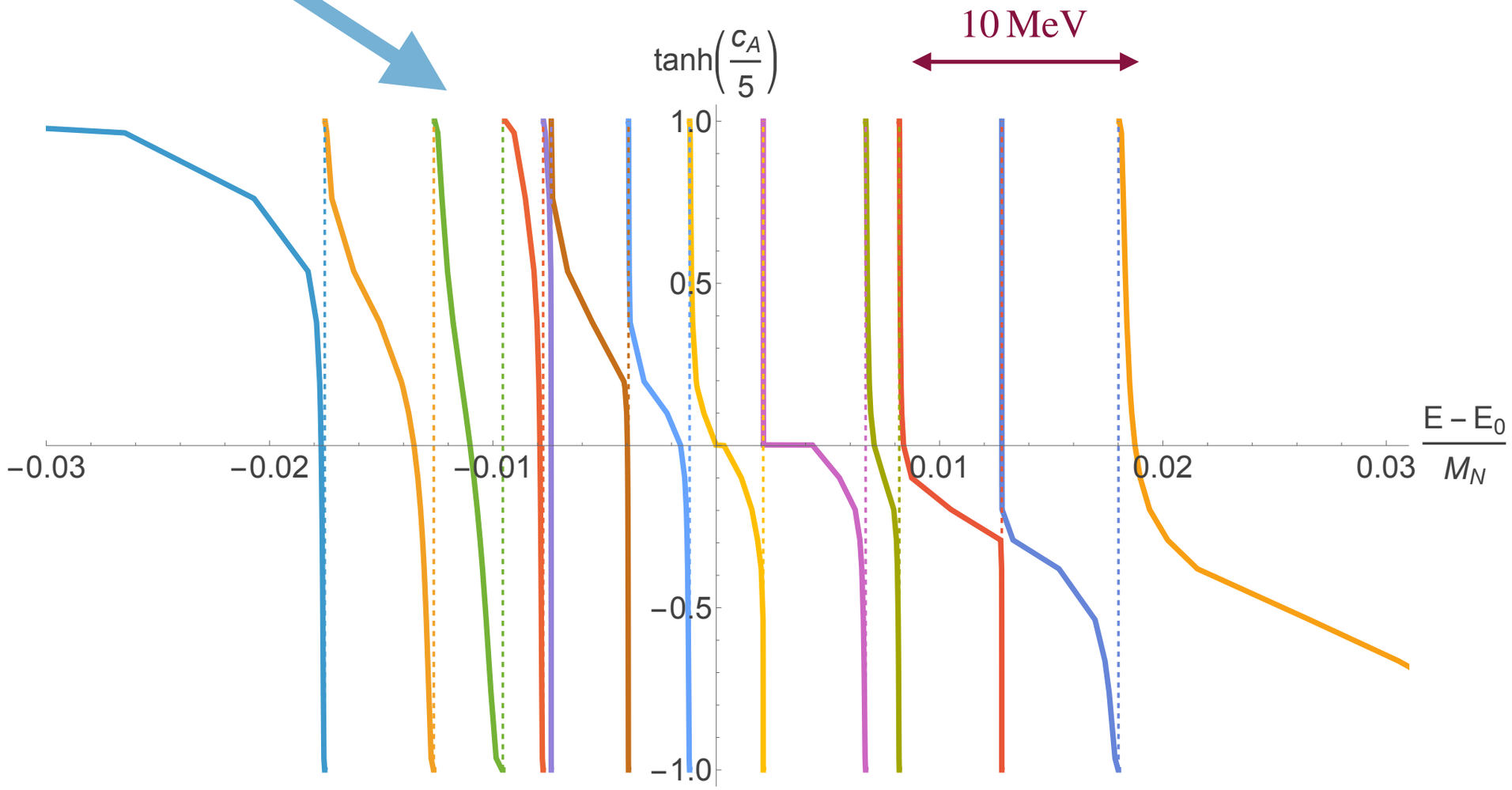


Spectrum in realistic setup



Add leading-order $\mathcal{K}_{\mathrm{df},3}$ in threshold expansion

Keep one term (out of two) with coefficient c_A



Summary & Outlook

Summary & Outlook

- Major steps have been taken in the three-particle sector
  - Formalism well established & cross checked, and almost complete
  - Several pioneering end-to-end applications to three-particle spectra from LQCD
  - Path to a calculation of $K \to 3\pi$ decay amplitudes is now open
- Near-term steps in implementation
  - $T_{cc}^+ \to D^*D \to DD\pi$ for lighter quark masses
  - Extended calculations for $3\pi(I = 0,1,2)$
  - $N\pi\pi \leftrightarrow \Delta\pi(I = 5/2)$
  - $K\pi \leftrightarrow K\pi\pi(I = 3/2), 3\pi(I = 0) \leftrightarrow K\overline{K}$ (WZW term)
- Final steps in three-particle formalism
  - $N\pi\pi + N\pi$ [for Roper], $NN\pi + NN$, and $NNN(I = \frac{1}{2})$
- Very recent review—talk by SRS at Lattice 2025 [2601.04147]

Challenges & Future

- Reducing model-dependence in fitting & analytic continuation
  - [Salg, Romero-López, Jay, 2506.16161] propose using Bayesian reconstruction + Nevanlinna interpolation
- Controlling all systematic errors (when will results be FLAG-ready?)
  - For which processes is precision important?
  - For three-nucleon interaction even a semiquantitative result would be useful
- Comparing formalisms in detail (e.g. RFT vs FVU)
  - Equivalent up to technical details (e.g. smooth vs hard cutoff functions)
  - Share data for cross-checking analysis? Bootcamp?
- Moving to more particles (needed, e.g., for $D \to \pi\pi, K\overline{K}$ & physical $\pi(1300)$)
  - QC4 (which is not yet known except in threshold expansion) may be practical
  - QCN seems impractical—combine QC2-4 with inclusive methods in some yet-to-developed manner?
    - "Scattering amplitudes from finite-volume spectral functions" [Bulava & Hansen, 1903.11735]
    - Squeeze as much information as we can out of the lattice simulations!
- Combine LQCD results with EFT and amplitude-analysis methods (Exohad collab.)

Thank you!

Questions?