Overview & status of methods for 3 particles

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Outline

- Motivations for studying 3-particles
- History and status of finite-volume formalism and applications
  [References at end of slides]
- Example of recent work: QC3 for 3 spin-½ particles
- Summary & Outlook

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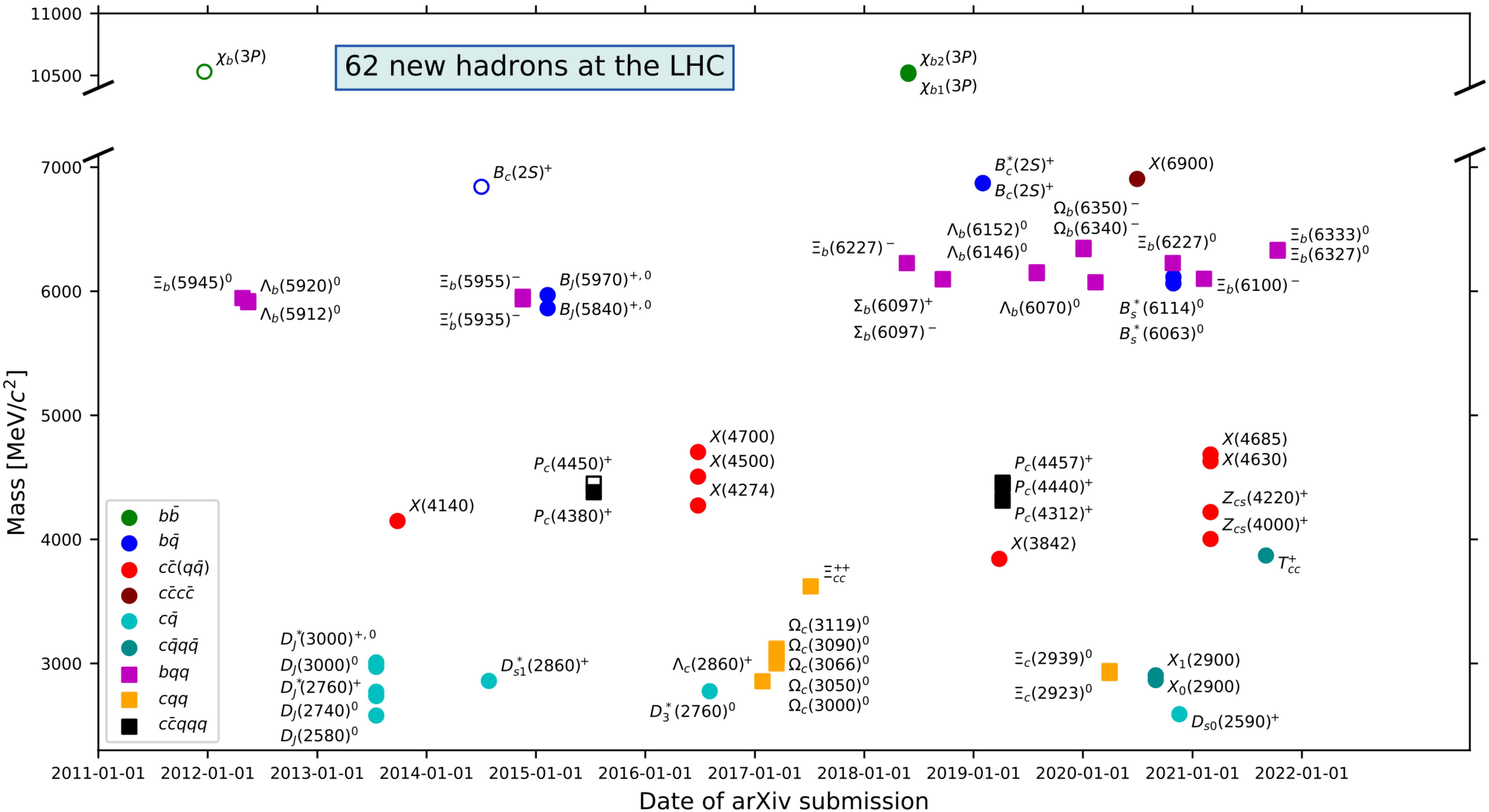
Will assume familiarity with QC2 (Lüscher quantization condition)

Motivations for studying three particles using LQCD

Cornucopia of exotics



[I. Danilkin, talk at INT workshop, March 23]

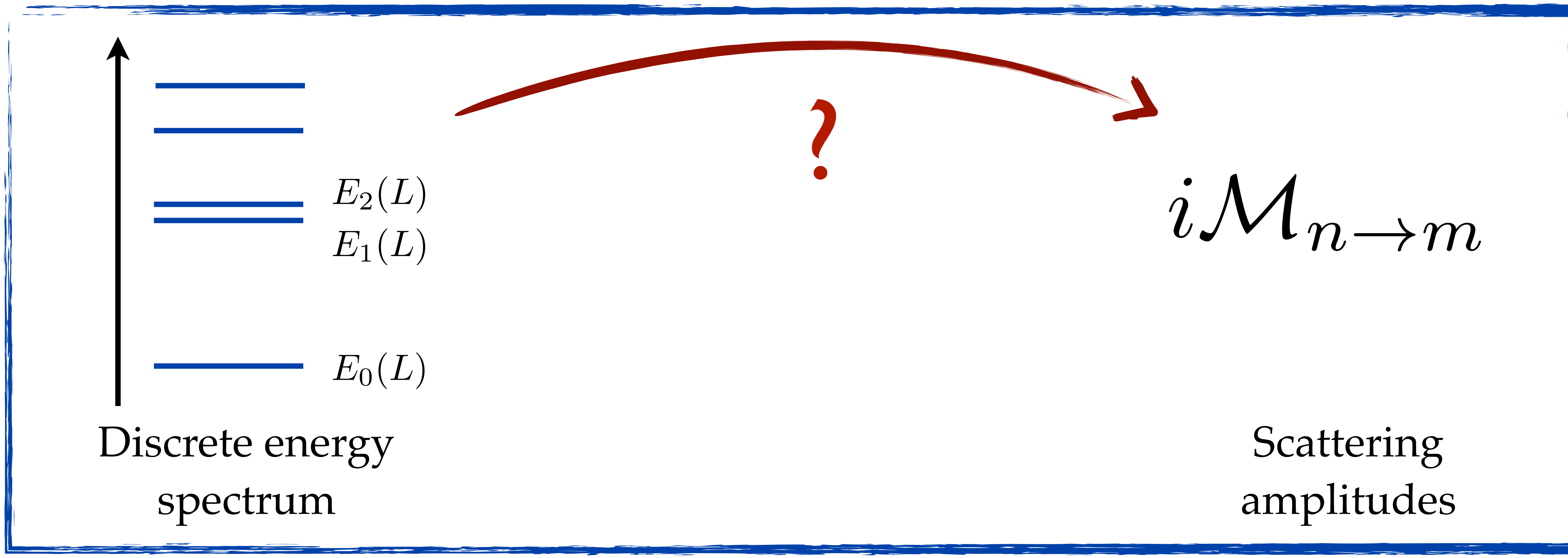
+ data from Babar, Belle, COMPASS, …

Motivations

- Studying resonances, most of which have decays involving 3 (or more) particles
  - $\omega(782,\ I^G J^{PC} = 0^- 1^{--}) \to 3\pi$
  - $N(1440,\ J^P = \frac{1}{2}^+) \to N\pi,\ N\pi\pi$
  - $T_{cc}(3875,\ I = 0,\ J^P = 1^+?) \to D^0 D^0 \pi^+$
- Determining 3-body "forces"
  - NNN interactions needed as input for EFT treatments of large nuclei, and for the neutron-star equation of state
  - $\pi\pi\pi, \pi K\bar{K}, \ldots$ interactions needed as input to study pion & kaon condensation
- Determining electroweak decay amplitudes involving 3 (or more) particles
  - $K \to 3\pi,\ D \to 2\pi, K\bar{K}, 4\pi, 2\eta, 6\pi,\ \ldots$
- Determine related 2-particle quantities that involve additional currents
  - $0 \xrightarrow{\mathcal{J}} 2,\quad 1 \xrightarrow{\mathcal{J}} 2,\quad 2 \xrightarrow{\mathcal{J}} 2$

History & status of formalism & applications for 3 particles

Problem in finite-volume QFT



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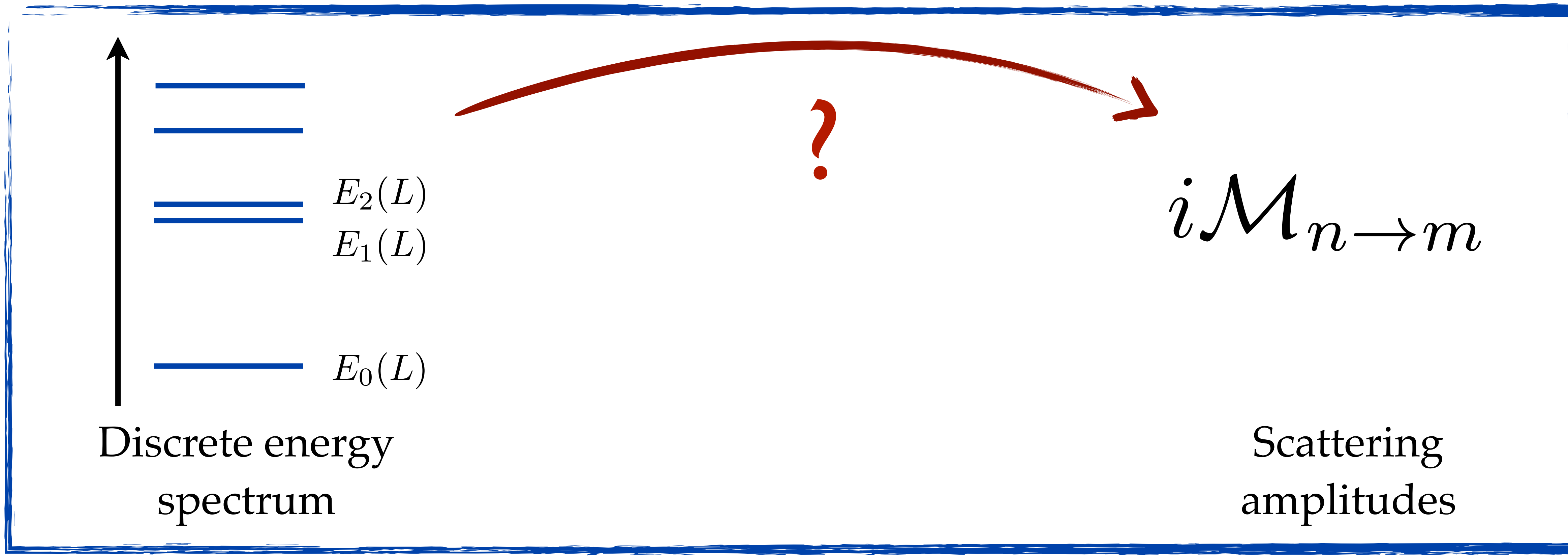

Assume discretization errors small, or have been extrapolated away

History of formalism for 3 particles

- [Beane, Detmold, Savage et al. 07-11] studied ground state energies of $N\pi^+, MK^+, N\pi^+ + MK^+$ systems, and determined 3-particle interactions for particles at rest
- [Polejaeva & Rusetsky 12] showed in NREFT that 3 body spectrum determined by $2 \to 2$ & $3 \to 3$ infinite-volume scattering amplitudes
- [Hansen & SRS 14, 15] derived quantization condition (QC3) for 3 identical scalars in generic, relativistic EFT, working to all orders in Feynman-diagram expansion, keeping all angular momenta—"RFT approach"
- [Hammer & Rusetsky 17] derived QC3 using NREFT—greatly simplified derivation
- [Mai & Döring 17] obtained QC3 using unitary, relativistic representation of $3 \to 3$ amplitude—"FVU approach"
- [Blanton & SRS 20] showed equivalence of RFT & FVU approaches
- [Müller, Pang, Rusetsky & Wu 20] relativized NREFT approach
- [Müller & Rusetsky 20; Hansen, Romero-López & SRS 21] derived formalism for determining $K \to 3\pi$ amplitude

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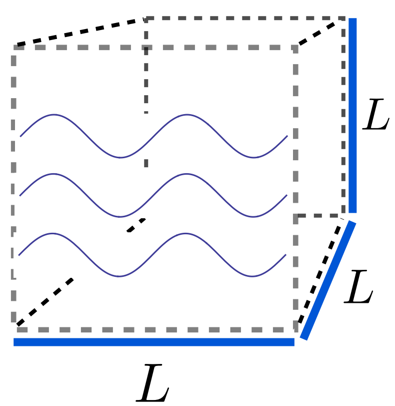
I will show examples using the RFT approach

Two-step method

2 & 3 particle
Spectra from LQCD



Quantization conditions

QC2: $\det\left[F^{-1} + \mathscr{K}_2\right] = 0$

QC3: $\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$

[These are the RFT forms, and assume $\mathbb{Z}_2$ symmetry]

Integral equations in infinite volume

Incorporates initial- and final-state interactions

Scattering amplitude $\mathscr{M}_3$

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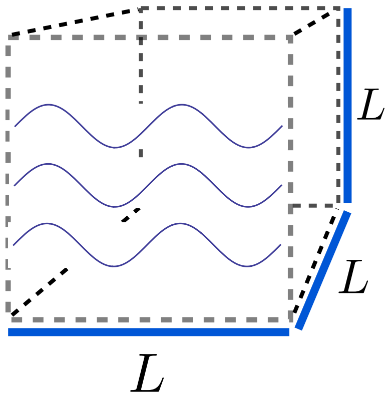

Infinite-volume K matrix:
Obtained from Feynman diagrams
using PV prescription for poles;
Real, free of unitary cuts

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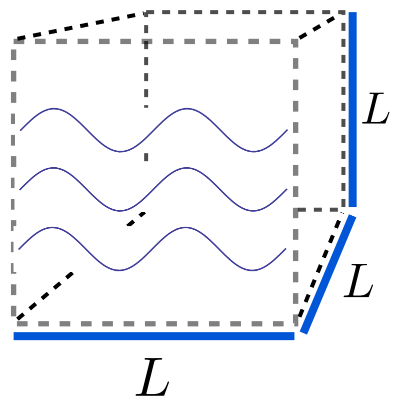

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[These are the RFT
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$\mathbb{Z}_2$ symmetry]

Intermediate infinite-volume K matrix:
A short-distance, real, three-particle
interaction free of unitary cuts, and
with physical divergences subtracted;
unphysical since depends on cutoff

Integral equations in
infinite volume

Incorporates initial- and
final-state interactions

Scattering amplitude $\mathcal{M}_3$

Matrix structure in QC3

- All quantities are infinite-dimensional matrices with indices $\mathbf{k}\ell m i$ describing 3 on-shell particles

[finite volume "spectator" momentum: $\boldsymbol{k} = (2\pi/L)\boldsymbol{n}$] x [2-particle CM angular momentum: ℓ, m] x [spectator flavor: i]

- For large $\boldsymbol{k}$ (at fixed E, L), the other two particles are below threshold
- Must include such configurations, by analytic continuation, up to a cut-off at $k \approx m$ [Polejaeva & Rusetsky, `12]

F_3 collects 2-particle interactions

$$F_3 = \frac{1}{2\omega L^3}\left[\frac{F}{3} - F\frac{1}{\mathcal{K}_2^{-1} + F + G}F\right]$$

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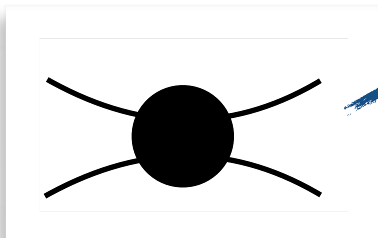

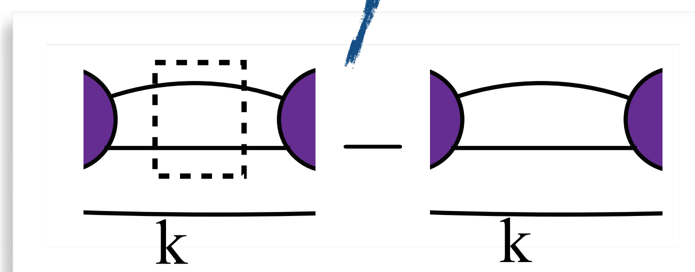

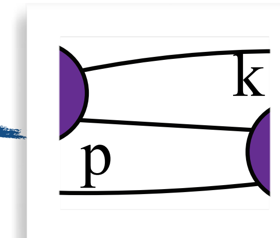

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$$F_3 = \frac{1}{2\omega L^3}\left[\frac{F}{3} - F\frac{1}{\mathscr{K}_2^{-1} + F + G}F\right]$$

- F & G are known geometrical functions, containing cutoff function $H(k)$

$$F_{p\ell'm';k\ell m} = \delta_{pk}\; H(\vec{k})\; F_{\mathrm{PV},\ell'm';\ell m}(E-\omega_k, \vec{P}-\vec{k}, L)$$

$$F_{\mathrm{PV};\ell'm';\ell m}(E,\vec{P},L) = \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \mathrm{PV}\int\frac{d^3k}{(2\pi)^3}\right)\frac{\mathscr{Y}_{\ell'm'}(\vec{k}^*)\mathscr{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k})}$$

$$\mathscr{Y}_{\ell m}(\vec{k}^*) = \sqrt{4\pi}\left(\frac{k^*}{q^*}\right)^{\ell} Y_{\ell m}(\hat{k}^*)$$

$$G_{p\ell'm';k\ell m} = \left(\frac{k^*}{q_p^*}\right)^{\ell'}\frac{4\pi Y_{\ell'm'}(\hat{k}^*)H(\vec{p})H(\vec{k})Y^*_{\ell m}(\hat{p}^*)}{(P-k-p)^2-m^2}\left(\frac{p^*}{q_k^*}\right)^{\ell}\frac{1}{2\omega_k L^3}$$

Relativistic form introduced in [BHS17]

Divergence-free K matrix

$$\det\left[F_3(E,\overrightarrow{P},L)^{-1}+\mathscr{K}_{\mathrm{df},3}(E^*)\right]=0$$

What is this? A quasi-local divergence-free 3-particle interaction

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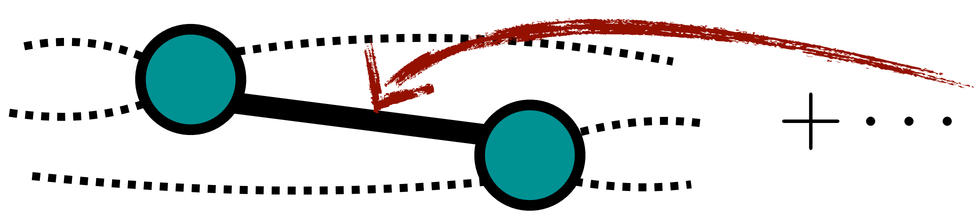
<u>Three-to-three amplitude has kinematic singularities!</u>

$i\mathcal{M}_{3\to3} \equiv$ fully connected correlator with six external legs amputated and projected on shell

$=$  $+\cdots$

Certain external momenta put this on-shell!

[Artwork from Hansen, HMI lectures]

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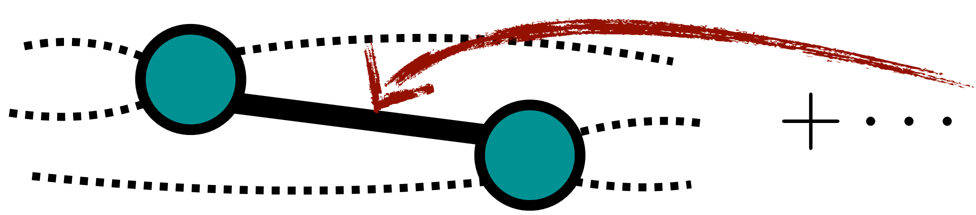


$+\cdots$

[Artwork from Hansen, HMI lectures]

- To have a nonsingular (divergence-free) quantity, need to subtract pole
- To obtain $\mathscr{K}_{\mathrm{df},3}$ also remove branch points associated with all cuts
- $\mathscr{K}_{\mathrm{df},3}$ has the same symmetries as $\mathscr{M}_{\mathrm{df},3}$, but depends on a cutoff function

Status: formalism

- 3 identical spinless particles [Hansen & SRS 14,15 (RFT); Hammer, Pang, Rusetsky 17 (NREFT); Mai, Döring 17 (FVU)]
  - Potential applications: $3\pi^+$, $3K^+$, $3D^+$, … as well as ϕ^4 theory
- Mixing of two- and three-particle channels for identical spinless particles [Briceño, Hansen, SRS 17]
  - Potential applications: Step on the way to $N(1440) \to N\pi,\ N\pi\pi$, etc.
- 3 degenerate but distinguishable spinless particles, e.g 3π with isospin 0, 1, 2, 3 [Hansen, Romero-López, SRS 20]; $I = 1$ case in FVU approach [Mai et al., 21]
  - Potential applications:

Resonance	$I_{\pi\pi\pi}$	J^P	Decays
$\omega(782)$	0	1^-	$\pi^+\pi^0\pi^-$
$h_1(1170)$	0	1^+	$\rho\pi \to 3\pi$
$\omega_3(1670)$	0	3^-	$3\pi,\ 5\pi$
$\pi(1300)$	1	0^-	$\rho\pi \to 3\pi$
$a_1(1260)$	1	1^+	$3\pi,\ K\bar{K}\pi$
$\pi_1(1400)$	1	1^-	$\eta\pi,\ 3\pi?$
$\pi_2(1670)$	1	2^-	$3\pi,\ K\bar{K}\pi$
$a_2(1320)$	1	2^+	$3\pi,\ K\bar{K},\ 5\pi,\ \eta\pi$
$a_4(1970)$	1	4^+	$3\pi,\ K\bar{K},\ 5\pi,\ \eta\pi$

Status: formalism (continued)

- 3 nondegenerate spinless particles [Blanton, SRS 20]
  - Potential applications: $D_s^+ D^0 \pi^-$
- 2 identical +1 different spinless particles [Blanton, SRS 21]
  - Potential applications: $\pi^+\pi^+K^+, K^+K^+\pi^+$
- 3 identical spin-½ particles [Draper, Hansen, Romero-López, SRS 23]
  - Potential applications: $3n,\ 3p,\ 3\Lambda$
- $DD\pi$ for all isospins (also $BB\pi,\ KK\pi$) [Draper, Hansen, Romero-López, SRS 23; talk by Fernando]
  - Potential applications: $T_{cc} \to D^*D$ incorporating LH cut

Status: applications

- $3\pi^+$: determined parameters in threshold expansion of $\mathcal{K}_{\mathrm{df},3}$, including pair interactions in s- and d-waves; integral equations solved for s-wave interactions only [Talk by Raúl]
- $3K^+$: determined s- and d-wave parameters in $\mathcal{K}_{\mathrm{df},3}$
- ϕ^4: extracted $\mathcal{K}_{\mathrm{df},3}$ in single-scalar theory; extracted 3-particle resonance parameters in two-scalar theory, using RFT and FVU approaches [Talk by Fernando]
- 3π with $I = 1$: first study of $a_1(1260)$ with formalism based on 2 levels; solved integral equations in FVU approach
- $\pi^+\pi^+K^+$ & $K^+K^+\pi^+$: determined s- and p-wave parameters in $\mathcal{K}_{\mathrm{df},3}$; found evidence for small discretization effects [Talk by Fernando]
- Integral equations solved for complex energies for simple system with near-unitary two-particle interactions and Efimov states (bound or resonant) [Talk by Raúl]
- ChPT: LO results for $3\pi^+, \pi^+\pi^+K^+, K^+K^+\pi^+, 3K^+$, including a^2 effects: agree in rough magnitude but not in detail with results from LQCD calculations [Talk by Fernando]
- ChPT: NLO result for $3\pi^+$; greatly improves agreement with LQCD results [Talk by Fernando]

QC3 for identical spin-½ particles

Three relativistic neutrons in a finite volume

Zachary T. Draper[a], Maxwell T. Hansen[b], Fernando Romero-López[c], and Stephen R. Sharpe[a]

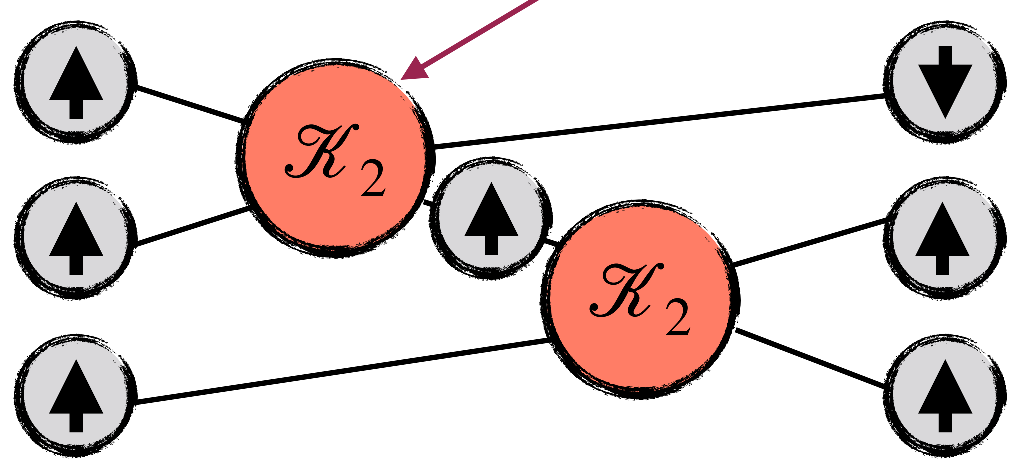
arXiv:2303.10219 (JHEP)

Motivation

- Determine 3 neutron interaction from first principles using LQCD
  - Important for neutron star EoS, heavy nuclei, …
- Incorporating spin into 3-particle formalism in a simple setting
  - Extensions to 3 nucleon interactions in isosymmetric QCD should be straightforward
  - Important step on the way to studying Roper: $N(1440) \to \pi N, \pi\pi N$
- Want relativistic approach since, for heavier than physical pions, the first inelastic threshold (where the formalism breaks down) can occur for relativistic nucleons
  - And for future applications such as the Roper, relativistic effects needed

New features for spin ½

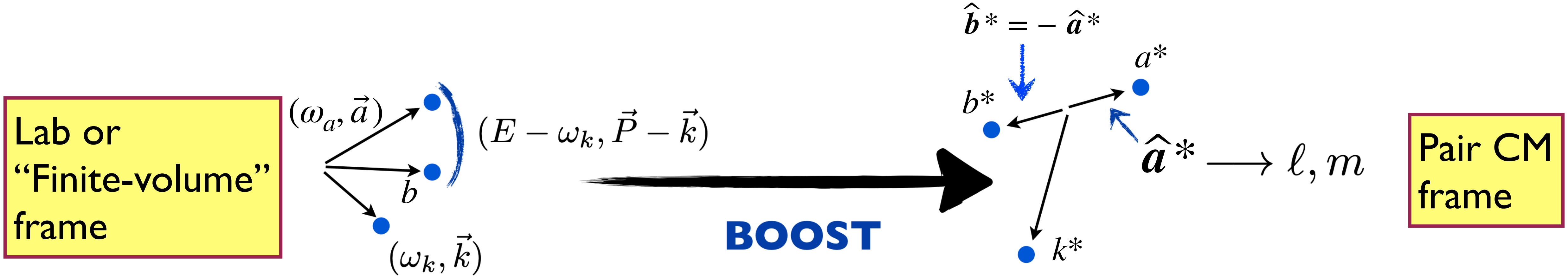
- Extra spin degree of freedom—gives extra matrix indices
- Total spin is conserved in NR limit; no longer true in relativistic system, due to Wigner rotations induced by boosts



- Antisymmetry of states due to Fermi statistics
- Inclusion of spin is much more complicated than for 2-particle QC [Briceño]

3-particle coordinates

- 3 scalars with total momentum $(E, \overrightarrow{P})$



[$\vec{k}$ of the spectator] x [ℓm of the “pair” or “dimer”]

- In finite volume, $\vec{k} = (2\pi/L)\mathbb{Z}^3 \Rightarrow$ matrix indices $\{k, \ell, m\}$
- What changes when include spin?

Describing spin ½ states

- Standard moving spin states: boost from CMF; corresponds to spinor $u(\boldsymbol{p}, s)$

$$|\boldsymbol{p}, sm_s\rangle = U(L(\boldsymbol{\beta}_p))\,|\boldsymbol{0}, sm_s\rangle \qquad L(\boldsymbol{\beta}_p) = R(\theta_p, \hat{\boldsymbol{n}}_p)\cdot L(\beta_p\hat{\boldsymbol{z}})\cdot R(\theta_p, \hat{\boldsymbol{n}}_p)^{-1}$$

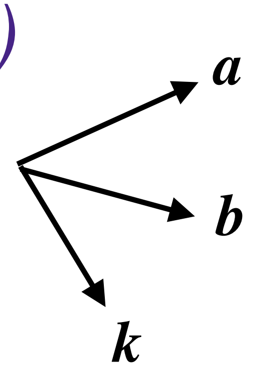
$$\equiv |\boldsymbol{p}, m_s(\boldsymbol{p})\rangle$$ for spin ½

  - Key property: rotates as nonrelativistic 2-component spinor

$$U(R)\,|\boldsymbol{p}, s\,m_s\rangle = |R\boldsymbol{p}, s\,m'_s\rangle\,\mathcal{D}^{(s)}_{m'_s,m_s}(R)$$

- Lab-frame description of 3 spin-½ particles (*lab-axis frame)*

$$|\boldsymbol{k}, m_s(\boldsymbol{k})\rangle \otimes |\boldsymbol{a}, m_s(\boldsymbol{a})\rangle \otimes |\boldsymbol{b}, m_s(\boldsymbol{b})\rangle$$



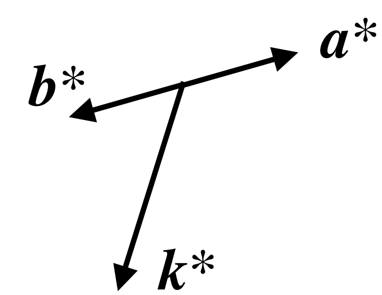
  - Natural choice for $\mathcal{K}_{\text{df},3}$
  - Collect spin indices into vector: $\boldsymbol{m}_s = (m_s(\boldsymbol{k}), m_s(\boldsymbol{a}), m_s(\boldsymbol{b}))$

Describing spin ½ states

- To combine spins of pair with orbital angular momentum ℓ, need a & b in pair CMF

$$|\boldsymbol{a}^*, m_s(\boldsymbol{a}^*)\rangle \equiv U(L(\boldsymbol{\beta}_{a^*}))|\boldsymbol{0}, m_s\rangle \quad \text{and} \quad |\boldsymbol{b}^*, m_s(\boldsymbol{b}^*)\rangle \equiv U(L(\boldsymbol{\beta}_{b^*}))|\boldsymbol{0}, m_s\rangle$$

  - Thus introduce *dimer-axis frame* spin indices



$$|\boldsymbol{k}, m_s(\boldsymbol{k})\rangle \otimes |\boldsymbol{a}^*, m_s(\boldsymbol{a}^*)\rangle \otimes |\boldsymbol{b}^*, m_s(\boldsymbol{b}^*)\rangle$$

  - Natural choice for $\mathcal{K}_2$, and for QC3
  - Collect spin indices into vector: $\boldsymbol{m}_s^* = (m_s(\boldsymbol{k}), m_s(\boldsymbol{a}^*), m_s(\boldsymbol{b}^*))$

- Relation between spin components involves Wigner rotations, e.g.

$$\begin{aligned}
|\boldsymbol{a}^*, m_s(\boldsymbol{a})\rangle &\equiv U(L(-\boldsymbol{\beta}_{P-k}))|\boldsymbol{a}, m_s(\boldsymbol{a})\rangle \\
&= U(L(-\boldsymbol{\beta}_{P-k}))U(L(\boldsymbol{\beta}_a))|\boldsymbol{0}, m_s\rangle \\
&= U(L(\boldsymbol{\beta}_{a^*}))U(R_a))|\boldsymbol{0}, m_s\rangle \\
&= |\boldsymbol{a}^*, m_s'(\boldsymbol{a}^*)\rangle \mathscr{D}(R_a)_{m_s' m_s}
\end{aligned}$$

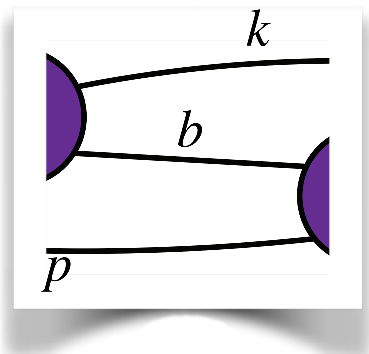
Wigner rotation

Spin ½ Wigner D-matrix representing Wigner rotation

Impact on G

$$\det\left[F_3^{-1}(E,\mathbf{P},L) + \mathscr{K}_{\mathrm{df},3}(E^*)\right] = 0$$

$$F_3 = \frac{F}{3} - F\frac{1}{\overline{\mathscr{K}}_{2,L}^{-1} + F + G}F$$



- Arises when spectator is switched
- Spin components conserved in lab frame

$$\Delta_{L,\alpha\beta}(b) = i\frac{(\not b + m)_{\alpha\beta}}{b^2 - m^2 + i\epsilon} + R_{L,\alpha\beta}(b)$$

$$(\not b + m)_{\alpha\beta}\Big|_{b^0=\omega_b} = \sum_{r=1}^{2} u_\alpha^r(\boldsymbol{b})\overline{u}_\beta^r(\boldsymbol{b})$$

Fully dressed propagator

Nonsingular residue

- Leads to Wigner D-matrices when express in *dimer-axis frame*

Spin indices match in lab frame

$$[\mathbf{G}^{\mathsf{lab}}]_{p\ell' m' \boldsymbol{m}_s';k\ell m \boldsymbol{m}_s}(E,\boldsymbol{P},L) \equiv -\delta_{m_s'(\boldsymbol{p}),m_s(\boldsymbol{p})}\delta_{m_s'(\boldsymbol{k}),m_s(\boldsymbol{k})}\delta_{m_s'(\boldsymbol{b}),m_s(\boldsymbol{b})} \times \frac{i}{4\omega_p\omega_k L^6}\frac{H(\boldsymbol{p})H(\boldsymbol{k})}{b^2-m^2}\frac{4\pi\mathcal{Y}_{\ell' m'}(\boldsymbol{k}_p^*)\mathcal{Y}_{\ell m}^*(\boldsymbol{p}_k^*)}{q_{2,p}^{*\ell'}\, q_{2,k}^{*\ell}}$$

Sign from Fermi Statistics

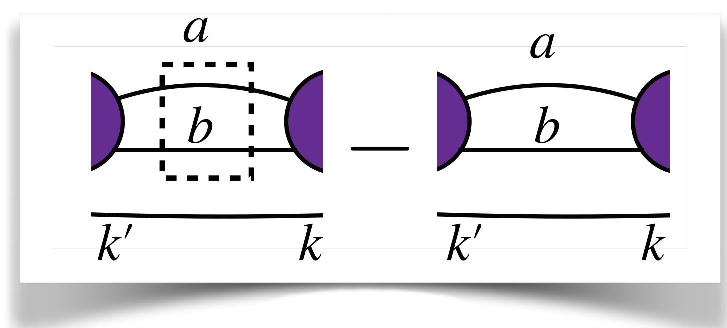
$$\mathbf{G}_{p\ell' m' \boldsymbol{m}_s'^*;k\ell m \boldsymbol{m}_s^*} = \mathcal{D}^{(p,k)\dagger}_{\boldsymbol{m}_s'^*\boldsymbol{m}_s''}\ \mathbf{G}^{\mathsf{lab}}_{p\ell' m' \boldsymbol{m}_s'';k\ell m \boldsymbol{m}_s'''}\ \mathcal{D}^{(k,p)}_{\boldsymbol{m}_s'''\boldsymbol{m}_s^*}$$

Product of two Wigner D-matrices (one for each Member of pair)

Impact on F

$$\det\left[F_3^{-1}(E,\mathbf{P},L) + \mathcal{K}_{\mathrm{df},3}(E^*)\right] = 0$$

$$F_3 = \frac{F}{3} - F\frac{1}{\overline{\mathcal{K}}_{2,L}^{-1} + F + G}F$$



Spin indices match
in lab frame

$$[\mathbf{F}^{\mathsf{lab}}]_{k'\ell'm'\boldsymbol{m}'_s;k\ell m\boldsymbol{m}_s}(E,\boldsymbol{P},L) \equiv \delta_{\boldsymbol{m}'_s\boldsymbol{m}_s}\delta_{k'k}\frac{iH(\boldsymbol{k})}{2\omega_k L^3}\frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{a}} - \mathrm{p.v.}\int_{\boldsymbol{a}}\right] \times \frac{4\pi\mathcal{Y}_{\ell'm'}(\boldsymbol{a}_k^*)\mathcal{Y}_{\ell m}^*(\boldsymbol{a}_k^*)}{2\omega_a(b^2-m^2)}\frac{1}{(q_{2,k}^*)^{\ell+\ell'}}$$

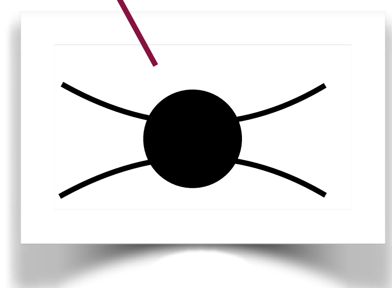
$$\mathbf{F} = \mathbf{F}^{\mathsf{lab}}$$

Wigner D-matrices cancel

Impact on $\mathscr{K}_2$

$$\det\left[F_3^{-1}(E,\mathbf{P},L)+\mathscr{K}_{\mathrm{df},3}(E^*)\right]=0$$

$$F_3=\frac{F}{3}-F\frac{1}{\overline{\mathscr{K}}_{2,L}^{-1}+F+G}F$$



- Naturally expressed in *dimer-axis frame*

$$[\mathbf{K}_2]_{k'\ell'm'\boldsymbol{m}_s'^*;k\ell m\boldsymbol{m}_s^*}(E,\boldsymbol{P})=i\delta_{k'k}2\omega_k L^3\mathcal{K}_2^{(\ell'm'\boldsymbol{m}_s'^*,\ell m\boldsymbol{m}_s^*)}(E_{2,k}^*)$$

$$\mathcal{K}_2^{(\ell'm'\boldsymbol{m}_s'^*,\ell m\boldsymbol{m}_s^*)}(E_{2,k}^*)=\delta_{m_s'(\boldsymbol{k})m_s(\boldsymbol{k})}\ \mathcal{K}_2^{[\ell'm'm_s'(\boldsymbol{a}'^*)m_s'(\boldsymbol{b}'^*)],\,[\ell m m_s(\boldsymbol{a}^*)m_s(\boldsymbol{b}^*)]}(E_{2,k}^*)$$

- Can convert $\mathscr{K}_2$ indices to total dimer spin: $\{\ell m s \mu_s\}$
  - Antisymmetry $\Rightarrow s=0$ and $s=1$ have opposite parities and do not mix
- And then to total dimer angular momentum: $\{j\mu_j\}$
  - $s=0 \ \Rightarrow\ $ even $\ell=j\ \Rightarrow$ single channel described by phase shift
  - $s=1 \ \Rightarrow\ $ odd $\ell\ \Rightarrow j=\ell-1,\ell,\ell+1\ \Rightarrow$ for even $j>0$ have two-channel mixing

Final results

- Quantization condition (**boldface** quantities absorb factors of i, 2ω, L^3)

$$\det_{\boldsymbol{k},\ell,m,\boldsymbol{m}_s^*}[\boldsymbol{F}_3^{-1} - \boldsymbol{K}_{\text{df},3}] = 0 \qquad \boldsymbol{F}_3 = \frac{\boldsymbol{F}}{3} + \boldsymbol{F}\frac{1}{\boldsymbol{K}_2^{-1} - \boldsymbol{F} - \boldsymbol{G}}\boldsymbol{F}$$

  - In practice, must truncate in ℓ so that matrices have finite dimension

- Integral equations relating $\boldsymbol{K}_{\text{df},3}$ and $\mathscr{M}_3$ take similar form to those for scalar particles, aside from extra spin indices and Wigner D-matrices

- Range of validity for (isosymmetric) QCD

$$\sqrt{4m_N^2 - m_\pi^2} + m_N < E_3^* < 3m_N + m_\pi$$

2-particle subchannel LH cut — Inelastic threshold

Threshold expansion for $\mathcal{K}_{\text{df},3}$

- Need parametrization of $\mathcal{K}_{\text{df},3}$ in order to apply QC3 in practice
- Expand about threshold; analogous to effective-range expansion for $\mathcal{K}_2$
  - Similar to NR expansion in pionless EFT, except using relativistic fields
- Method: use neutron field operators $\mathcal{N}$
  - Write down all operators of the form $(\overline{\mathcal{N}}\mathcal{N})^3$ with arbitrary gamma-matrix structure and 0, 2, 4,… derivatives, requiring Lorentz and parity invariance
  - Take matrix elements of these operators between lab-frame states, leading to completely antisymmetric expressions in terms of Dirac spinors (in lab frame)
  - Determine which are independent
  - Insert NR expression for Dirac spinors, and expand in 3-momenta

Operators without derivatives

- 12 operators (have not used Fierz identities)

$$
\begin{aligned}
\mathcal{O}_{\rm SSS}(x) &= [\overline{\mathcal{N}}(x)\mathcal{N}(x)]^3\,,\\
\mathcal{O}_{\rm SPP} &= [\overline{\mathcal{N}}\mathcal{N}][\overline{\mathcal{N}}\gamma_5\mathcal{N}][\overline{\mathcal{N}}\gamma_5\mathcal{N}]\,,\\
\mathcal{O}_{\rm SVV} &= [\overline{\mathcal{N}}\mathcal{N}][\overline{\mathcal{N}}\gamma_\mu\mathcal{N}][\overline{\mathcal{N}}\gamma^\mu\mathcal{N}]\,,\\
\mathcal{O}_{\rm SAA} &= [\overline{\mathcal{N}}\mathcal{N}][\overline{\mathcal{N}}\gamma_\mu\gamma_5\mathcal{N}][\overline{\mathcal{N}}\gamma^\mu\gamma_5\mathcal{N}]\,,\\
\mathcal{O}_{\rm STT} &= [\overline{\mathcal{N}}\mathcal{N}][\overline{\mathcal{N}}\sigma_{\mu\nu}\mathcal{N}][\overline{\mathcal{N}}\sigma^{\mu\nu}\mathcal{N}]\,,\\
\mathcal{O}_{\rm PVA} &= [\overline{\mathcal{N}}\gamma_5\mathcal{N}][\overline{\mathcal{N}}\gamma_\mu\mathcal{N}][\overline{\mathcal{N}}\gamma^\mu\gamma_5\mathcal{N}]\,,\\
\mathcal{O}_{\rm PTT'} &= [\overline{\mathcal{N}}\gamma_5\mathcal{N}][\overline{\mathcal{N}}\sigma_{\mu\nu}\mathcal{N}][\overline{\mathcal{N}}\sigma^{\mu\nu}\gamma_5\mathcal{N}]\,,\\
\mathcal{O}_{\rm TVV} &= [\overline{\mathcal{N}}\sigma_{\mu\nu}\mathcal{N}][\overline{\mathcal{N}}\gamma^\mu\mathcal{N}][\overline{\mathcal{N}}\gamma^\nu\mathcal{N}]\,,\\
\mathcal{O}_{\rm TAA} &= [\overline{\mathcal{N}}\sigma_{\mu\nu}\mathcal{N}][\overline{\mathcal{N}}\gamma^\mu\gamma_5\mathcal{N}][\overline{\mathcal{N}}\gamma^\nu\gamma_5\mathcal{N}]\,,\\
\mathcal{O}_{\rm T'VA} &= [\overline{\mathcal{N}}\sigma_{\mu\nu}\gamma_5\mathcal{N}][\overline{\mathcal{N}}\gamma^\mu\mathcal{N}][\overline{\mathcal{N}}\gamma^\nu\gamma_5\mathcal{N}]\,,\\
\mathcal{O}_{\rm TTT} &= [\overline{\mathcal{N}}\sigma_{\mu\nu}\mathcal{N}][\overline{\mathcal{N}}\sigma^{\nu\rho}\mathcal{N}][\overline{\mathcal{N}}\sigma_\rho{}^\mu\mathcal{N}]\,,\\
\mathcal{O}_{\rm TT'T'} &= [\overline{\mathcal{N}}\sigma_{\mu\nu}\mathcal{N}][\overline{\mathcal{N}}\sigma^{\nu\rho}\gamma_5\mathcal{N}][\overline{\mathcal{N}}\sigma_\rho{}^\mu\gamma_5\mathcal{N}]\,,
\end{aligned}
$$

- All lead to **identical** $3 \to 3$ amplitudes (*cf.* four independent forms for $2 \to 2$)
- Insert NR on-shell form, and find leading contribution to $\mathcal{K}_{\rm df,3}$ involves 2 derivatives:

$$
u_k = \sqrt{2\omega_k}\begin{pmatrix}\chi_k \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{\omega_k+m}\chi_k\end{pmatrix} \quad \Rightarrow \quad \mathcal{K}_A = \overline{\mathcal{A}}\left[(\chi_{k'}^\dagger\,\boldsymbol{\sigma}\cdot\boldsymbol{k'}\,\boldsymbol{\sigma}\cdot\boldsymbol{k}\,\chi_k)(\chi_{a'}^\dagger\chi_a)(\chi_{b'}^\dagger\chi_b)\right]
$$

antisymmetrization

Operators with 2 derivatives

- For consistency, need to consider operators with 2 derivatives
  - Using Fierz identities and EoM, find 22 independent operators

$$
\begin{aligned}
SSS &= (\partial^\mu \overline{\mathcal{N}} \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\mathcal{N})(\overline{\mathcal{N}}\mathcal{N}) ,\\
SPP &= (\partial^\mu \overline{\mathcal{N}} \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\gamma_5\mathcal{N})(\overline{\mathcal{N}}\gamma_5\mathcal{N}) ,\\
PSP &= (\partial^\mu \overline{\mathcal{N}}\gamma_5 \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\mathcal{N})(\overline{\mathcal{N}}\gamma_5\mathcal{N}) ,\\
SVV &= (\partial^\mu \overline{\mathcal{N}} \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\gamma_\nu\mathcal{N})(\overline{\mathcal{N}}\gamma^\nu\mathcal{N}) ,\\
VSV &= (\partial^\mu \overline{\mathcal{N}}\gamma_\nu \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\mathcal{N})(\overline{\mathcal{N}}\gamma^\nu\mathcal{N}) ,\\
ASA &= (\partial^\mu \overline{\mathcal{N}}\gamma_\nu\gamma_5 \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\mathcal{N})(\overline{\mathcal{N}}\gamma^\nu\gamma_5\mathcal{N}) ,\\
TST &= (\partial^\mu \overline{\mathcal{N}}\sigma_{\nu\rho} \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\mathcal{N})(\overline{\mathcal{N}}\sigma^{\nu\rho}\mathcal{N}) ,\\
PVA &= (\partial^\mu \overline{\mathcal{N}}\gamma_5 \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\gamma_\nu\mathcal{N})(\overline{\mathcal{N}}\gamma^\nu\gamma_5\mathcal{N}) ,\\
VAP &= (\partial^\mu \overline{\mathcal{N}}\gamma^\nu \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\gamma_\nu\gamma_5\mathcal{N})(\overline{\mathcal{N}}\gamma_5\mathcal{N}) ,\\
APV &= (\partial^\mu \overline{\mathcal{N}}\gamma^\nu\gamma_5 \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\gamma_5\mathcal{N})(\overline{\mathcal{N}}\gamma_\nu\mathcal{N}) ,
\end{aligned}
$$

$$
\begin{aligned}
SVV' &= (\partial^\mu \overline{\mathcal{N}} \partial_\nu \mathcal{N})(\overline{\mathcal{N}}\gamma_\mu\mathcal{N})(\overline{\mathcal{N}}\gamma^\nu\mathcal{N}) ,\\
TST' &= (\partial_\nu \overline{\mathcal{N}}\sigma_{\mu\rho} \partial^\mu \mathcal{N})(\overline{\mathcal{N}}\mathcal{N})(\overline{\mathcal{N}}\sigma^{\nu\rho}\mathcal{N}) ,\\
PVA' &= (\partial^\mu \overline{\mathcal{N}}\gamma_5 \partial_\nu \mathcal{N})(\overline{\mathcal{N}}\gamma_\mu\mathcal{N})(\overline{\mathcal{N}}\gamma^\nu\gamma_5\mathcal{N}) ,\\
VTV' &= (\partial^\mu \overline{\mathcal{N}}\gamma_\rho \partial_\nu \mathcal{N})(\overline{\mathcal{N}}\sigma^{\nu\rho}\mathcal{N})(\overline{\mathcal{N}}\gamma_\mu\mathcal{N}) ,\\
ATA' &= (\partial^\mu \overline{\mathcal{N}}\gamma_\rho\gamma_5 \partial_\nu \mathcal{N})(\overline{\mathcal{N}}\sigma^{\nu\rho}\mathcal{N})(\overline{\mathcal{N}}\gamma_\mu\gamma_5\mathcal{N}) ,\\
TTT' &= (\partial^\mu \overline{\mathcal{N}}\sigma_{\mu\eta} \partial^\eta \mathcal{N})(\overline{\mathcal{N}}\sigma^{\nu\rho}\mathcal{N})(\overline{\mathcal{N}}\sigma_{\nu\rho}\mathcal{N}) ,\\
TTT'' &= (\partial_\mu \overline{\mathcal{N}}\sigma^{\mu\nu} \partial_\eta \mathcal{N})(\overline{\mathcal{N}}\sigma^{\eta\rho}\mathcal{N})(\overline{\mathcal{N}}\sigma_{\rho\nu}\mathcal{N}) ,\\
TTT'' &= (\partial_\eta \overline{\mathcal{N}}\sigma^{\mu\nu} \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\sigma^{\eta\rho}\mathcal{N})(\overline{\mathcal{N}}\sigma_{\rho\nu}\mathcal{N}) ,\\
TT_5P' &= (\partial^\mu \overline{\mathcal{N}}\sigma_{\mu\rho} \partial_\nu \mathcal{N})(\overline{\mathcal{N}}\sigma^{\nu\rho}\gamma_5\mathcal{N})(\overline{\mathcal{N}}\gamma_5\mathcal{N}) ,\\
T_5VA' &= (\partial^\mu \overline{\mathcal{N}}\sigma_{\mu\nu}\gamma_5 \partial^\nu \mathcal{N})(\overline{\mathcal{N}}\gamma^\rho\mathcal{N})(\overline{\mathcal{N}}\gamma_\rho\gamma_5\mathcal{N}) ,\\
T_5VA'' &= (\partial^\mu \overline{\mathcal{N}}\sigma_{\rho\nu}\gamma_5 \partial^\nu \mathcal{N})(\overline{\mathcal{N}}\gamma^\mu\mathcal{N})(\overline{\mathcal{N}}\gamma^\rho\gamma_5\mathcal{N}) ,\\
T_5TT_5' &= (\partial^\mu \overline{\mathcal{N}}\sigma_{\mu\nu}\gamma_5 \partial^\nu \mathcal{N})(\overline{\mathcal{N}}\sigma^{\eta\rho}\mathcal{N})(\overline{\mathcal{N}}\sigma_{\eta\rho}\gamma_5\mathcal{N})
\end{aligned}
$$

- Inserting NR on-shell form, find two independent 2-derivatives forms:

$$
\mathcal{K}_A = \overline{\mathcal{A}}\left[(\chi^\dagger_{k'}\, \boldsymbol{\sigma}\cdot\boldsymbol{k'}\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_k)(\chi^\dagger_{a'}\chi_a)(\chi^\dagger_{b'}\chi_b)\right] \qquad \mathcal{K}_B = \overline{\mathcal{A}}\left[\boldsymbol{k'}\cdot\boldsymbol{k}(\chi^\dagger_{k'}\chi_k)(\chi^\dagger_{a'}\chi_a)(\chi^\dagger_{b'}\chi_b)\right]
$$

Same as from 0-derivative operators

Summary for $\mathscr{K}_{\mathrm{df},3}$

- 0-derivative operators contribute

Dimensionless combination

Unknown, dimensionless constant

$$m_N^2 \mathscr{K}_{\mathrm{df},3}^{\mathsf{lab}} \supset \frac{c_0}{m_N^2} \mathscr{K}_A + \mathscr{O}(\boldsymbol{k}^4/m_N^4)$$

- 2-derivative operators imply

$$m_N^2 \mathscr{K}_{\mathrm{df},3}^{\mathsf{lab}} \supset \frac{c_1}{\Lambda_{\mathrm{EFT}}^2} \mathscr{K}_A + \frac{c_2}{\Lambda_{\mathrm{EFT}}^2} \mathscr{K}_B + \mathscr{O}(\boldsymbol{k}^4/m_N^2 \Lambda_{\mathrm{EFT}}^2)$$

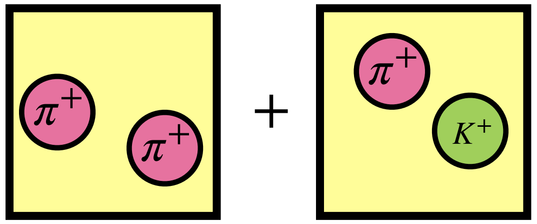
- Since expect $\Lambda_{\mathrm{EFT}} \sim m_\pi$, the 2-derivative operators dominate
- The form of the allowed operators could more easily have been determined directly using a NR expansion, but this would lose the implications of relativity at higher order

Summary & Outlook for 3N

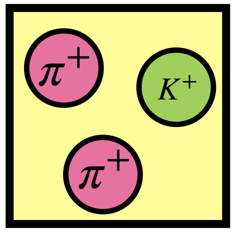
- Including spin in the formalism involves additional subtleties not present for 2 particles
  - Wigner rotations and fermion signs
- Implementing the QC3 is underway for toy interactions [including Wilder Schaaf]
- Various generalizations should be straightforward
  - 3 nucleons of arbitrary isospin [underway]
  - $N\pi\pi$ at maximal isospin (no 3-particle resonance, but includes $\Delta\pi$)
  - $N\pi\pi + N\pi$ (for the Roper)
  - Higher spins (e.g. ρ if stable)—though hard to think of applications
- Need to extend methods for solving integral equations
- Need to relate parameters in $\mathscr{K}_{\rm df,3}$ to those in chiral EFTs used to study light nuclei

Summary & Outlook

Summary



- Two-particle sector is entering precision phase
  - Frontier is two nucleons, which are more challenging for LQCD
- Major steps have been taken in the three-particle sector



  - Formalism well established & cross checked, and almost complete
  - Several applications to three-particle spectra from LQCD
  - Initial discrepancy with LO ChPT explained by large NLO contributions
  - Integral equations solved in several cases
  - Path to a calculation of $K \to 3\pi$ decay amplitudes is now open

Outlook

- Generalize formalism to broaden applications
  - 3 nucleons with $I = \frac{1}{2}$ (nnp & ppn)
  - Accessing the WZW term: $K\bar{K} \leftrightarrow \pi^+\pi^0\pi^-(I=0)$
  - $N(1440, J^P = \frac{1}{2}^+) \rightarrow N\pi,\ N\pi\pi$
  - $J^{PC}, I^G = 1^{-+}, 1^-:\ \ \pi_1(1600) \rightarrow \eta\pi,\ 3\pi,\ KK\pi\pi,\ \eta\pi\pi\pi,\ 5\pi$
- Extend implementations using LQCD simulations
  - $3\pi^+,\ 3K^+,\ \pi^+\pi^+K^+,\ K^+K^+\pi^+$ at physical quark masses
  - I=0,1 three-particle resonances ($\omega,\ a_1, \ldots$)
- Extend applications of integral equations in the presence of three-particle resonances, e.g. T_{cc}
- Move on to 4 particles!

ExoHad collaboration

ExoHad Collaboration exohad.org People Events Talks Publications

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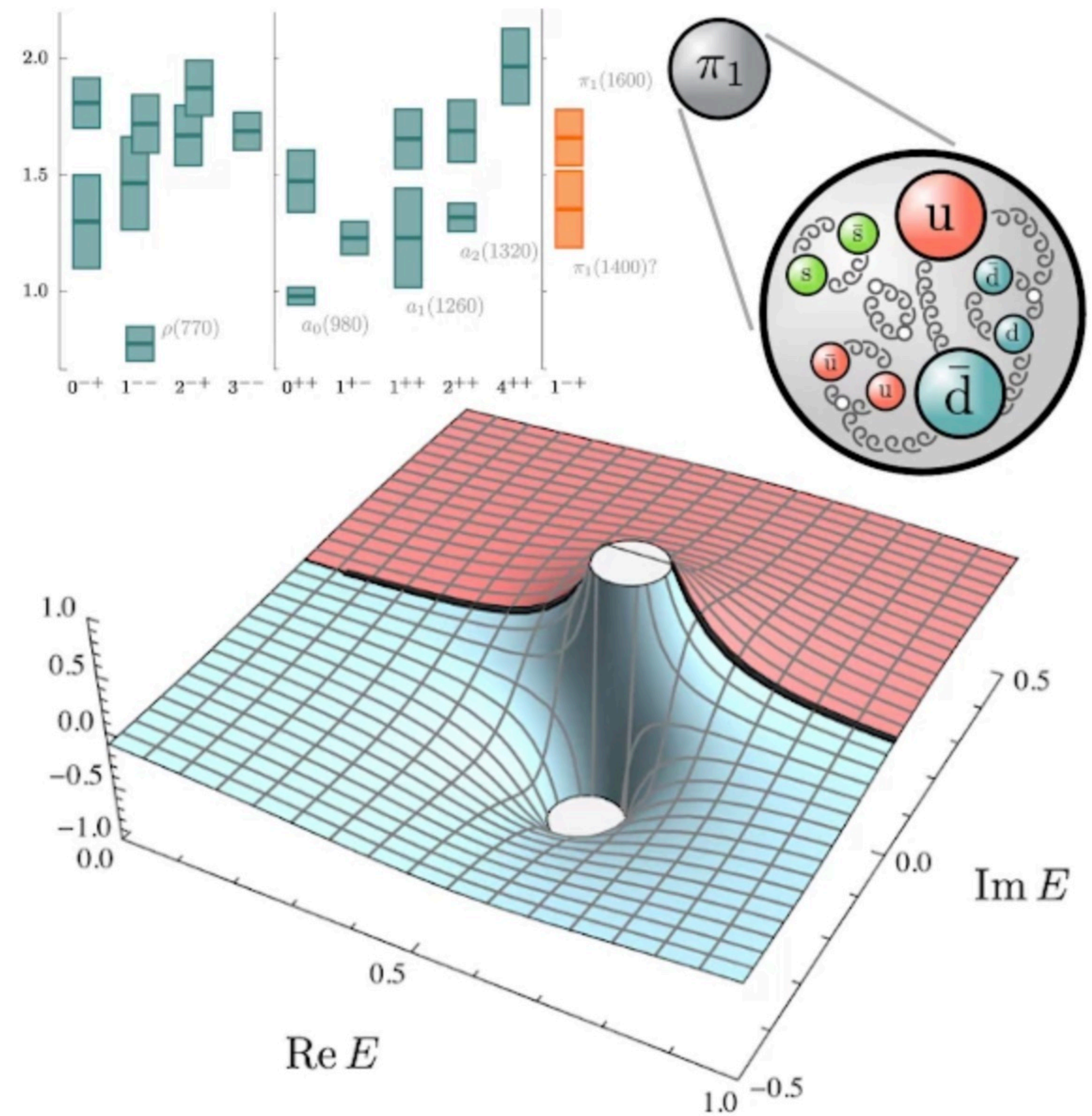
EXOTIC HADRONS TOPICAL COLLABORATION



The Exo(tic) Had(ron) Collaboration started in 2023 to explore all aspects of exotic hadron physics, from predictions within lattice QCD, through reliable extraction of their existence and properties from experimental data, to descriptions of their structure within phenomenological models.

Thank you!
Questions?

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"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

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SRS

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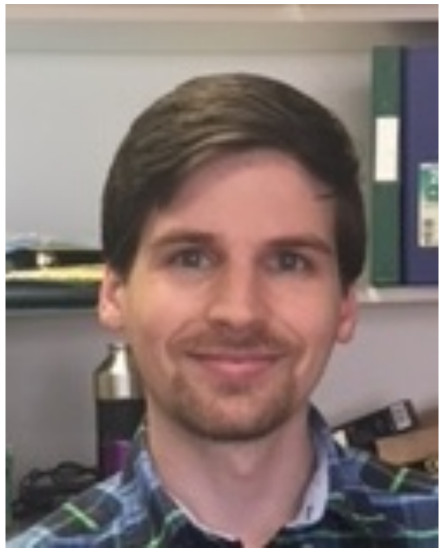

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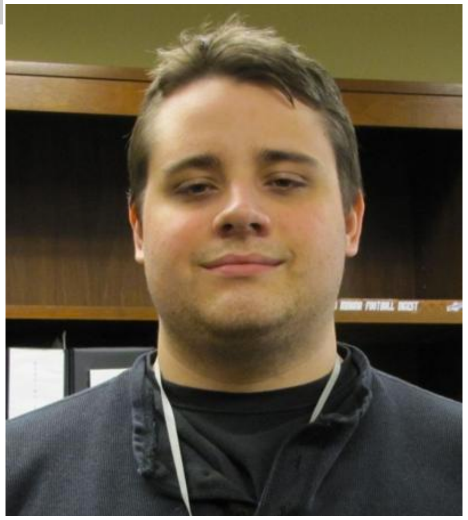

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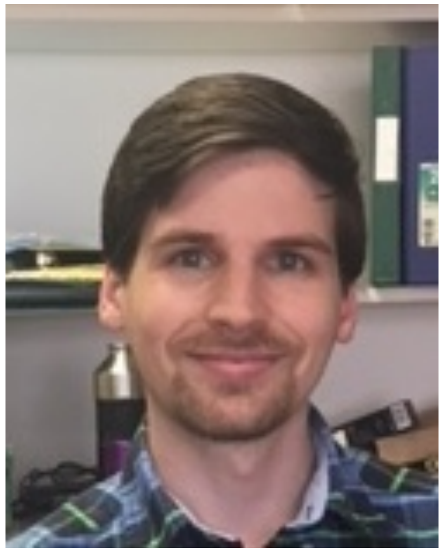

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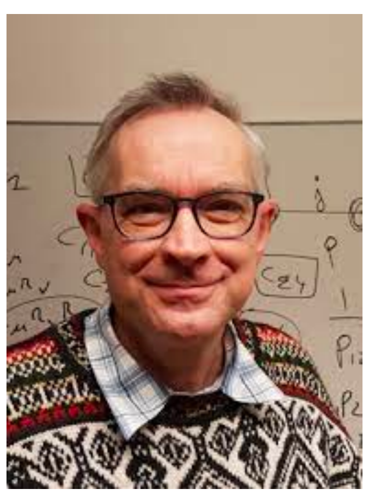

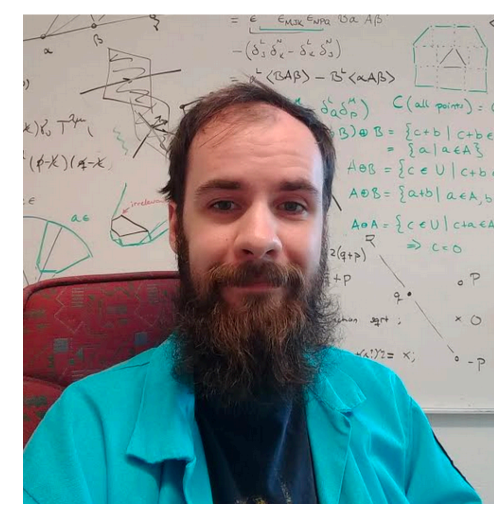

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Alternate 3-particle approaches

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- M. Mai et al., 1706.06118, EPJA [unitary parametrization of M_3 involving R matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of R matrix parametrization]
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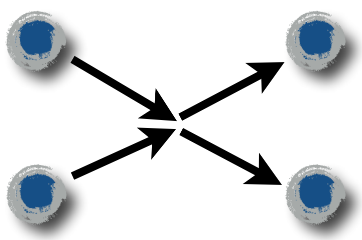
Backup slides

Divergence-free K matrix

- $K_{df,3}$ has the same symmetries as M_3: relativistic invariance, particle interchange, T-reversal

M_2, K_2



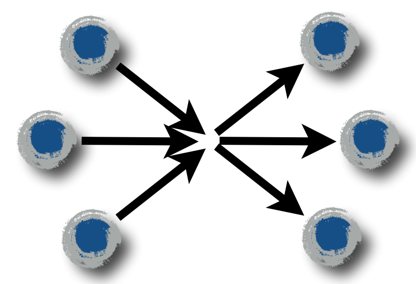
12 momentum components

-10 Poincaré generators

2 degrees of freedom

$s=E^{*2}$ + θ

$M_3, K_{df,3}$



18 momentum components

-10 Poincaré generators

8 degrees of freedom

$s=E^{*2}$ + 7 “angles”

- Need more parameters to describe $\mathscr{K}_{\rm df,3}$ than $\mathscr{K}_2$ (will be discussed in lecture 3)
- Why $\mathscr{K}_2$ and $\mathscr{K}_{\rm df,3}$ appear in QC3, rather than $\mathscr{M}_2$ and $\mathscr{M}_{\rm df,3}$, will be explained shortly