Multiparticle systems in lattice QCD

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Outline

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Present Status
(1 or 2 particles)

Why 3+ particles?

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Overview of methods
& theoretical results

Summary

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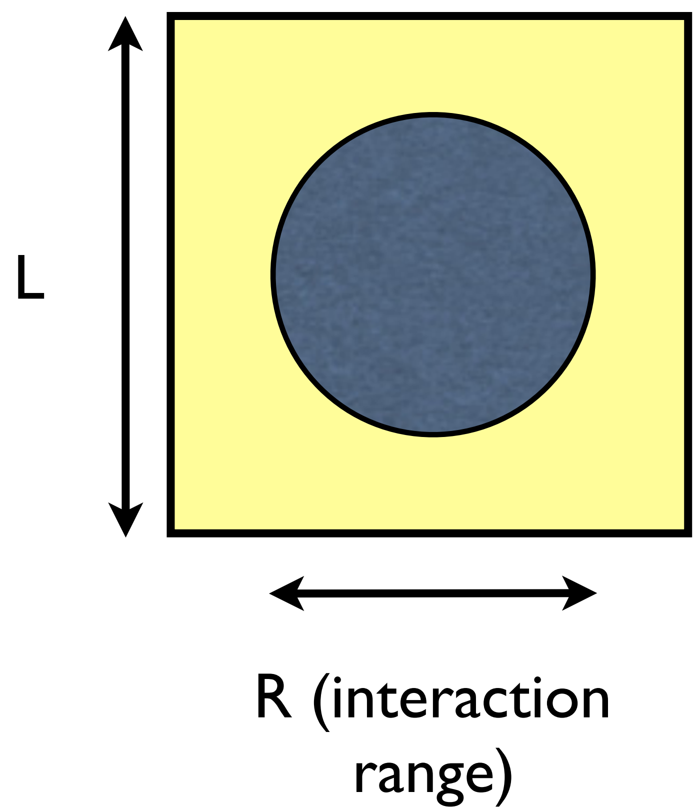
(Some details
of derivations)

90

Overview of present status

Well-controlled LQCD calculations

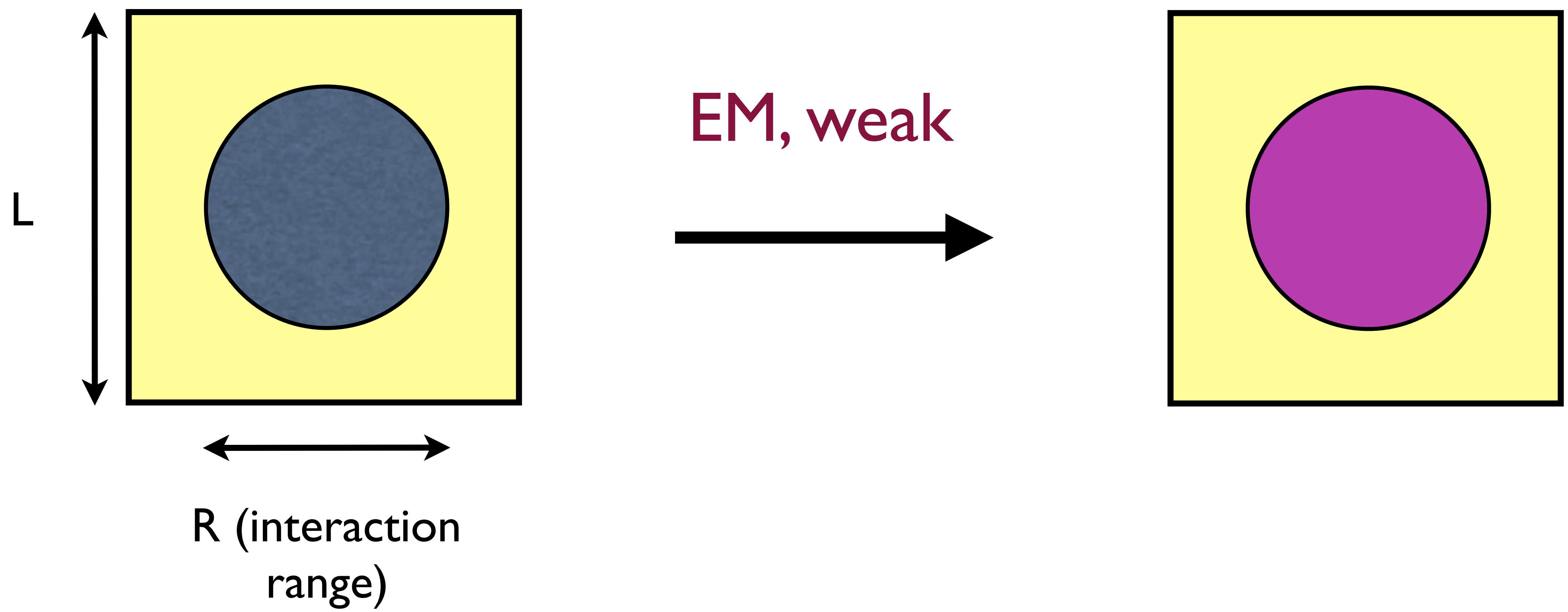
- Single particle masses



For large enough boxes (L>2R) dominant finite-volume effects for single-particle states fall as $\exp(-M_\pi L)$ [Lüscher 86,91] and can be made small

Well-controlled LQCD calculations

- Single particle masses and matrix elements



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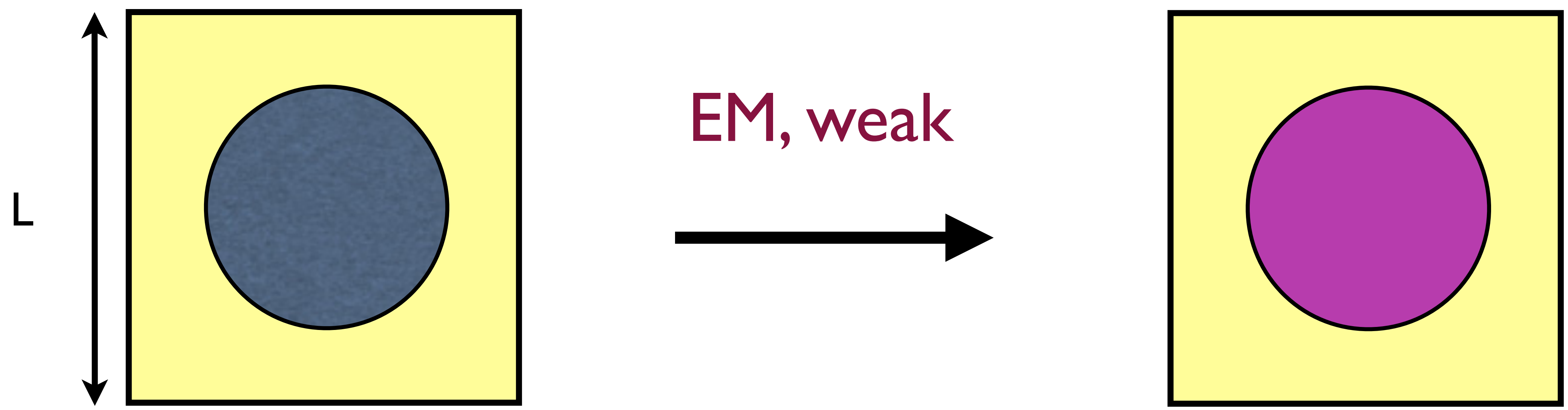
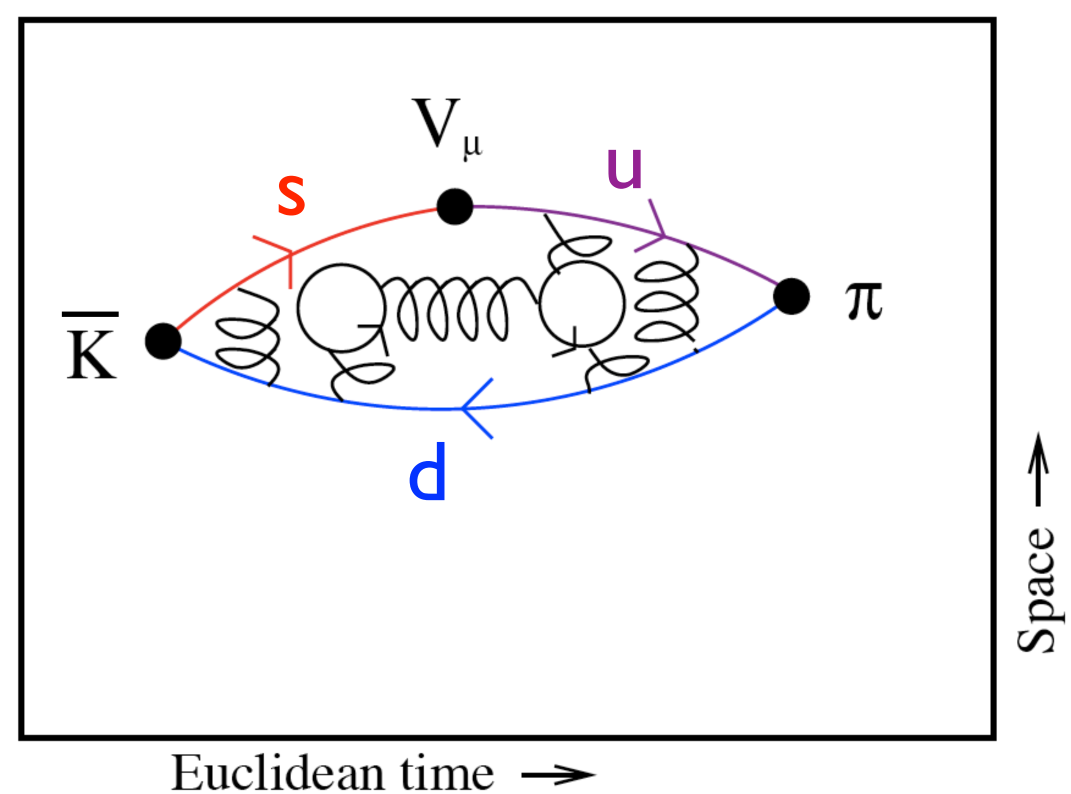


Example:
K→π form factor

$$\langle \pi(\vec{p}_2) | V_\mu(0) | \overline{K}(\vec{p}_1) \rangle$$



Flavo(u)r Lattice Averaging Group

Eur. Phys. J. C (2017) 77:112
DOI 10.1140/epjc/s10052-016-4509-7

THE EUROPEAN PHYSICAL JOURNAL C

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Review

Review of lattice results concerning low-energy particle physics

Flavour Lattice Averaging Group (FLAG)

S. Aoki[1], Y. Aoki[2,3,17], D. Bečirević[4], C. Bernard[5], T. Blum[3,6], G. Colangelo[7], M. Della Morte[8,9], P. Dimopoulos[10,11], S. Dürr[12,13], H. Fukaya[14], M. Golterman[15], Steven Gottlieb[16], S. Hashimoto[17,18], U. M. Heller[19], R. Horsley[20], A. Jüttner[21,a], T. Kaneko[17,18], L. Lellouch[22], H. Leutwyler[7], C.-J. D. Lin[22,23], V. Lubicz[24,25], E. Lunghi[16], R. Mawhinney[26], T. Onogi[14], C. Pena[27], C. T. Sachrajda[21], S. R. Sharpe[28], S. Simula[25], R. Sommer[29], A. Vladikas[30], U. Wenger[7], H. Wittig[31]

- Reviews every 2$^+$ years: provide "vetted" averages

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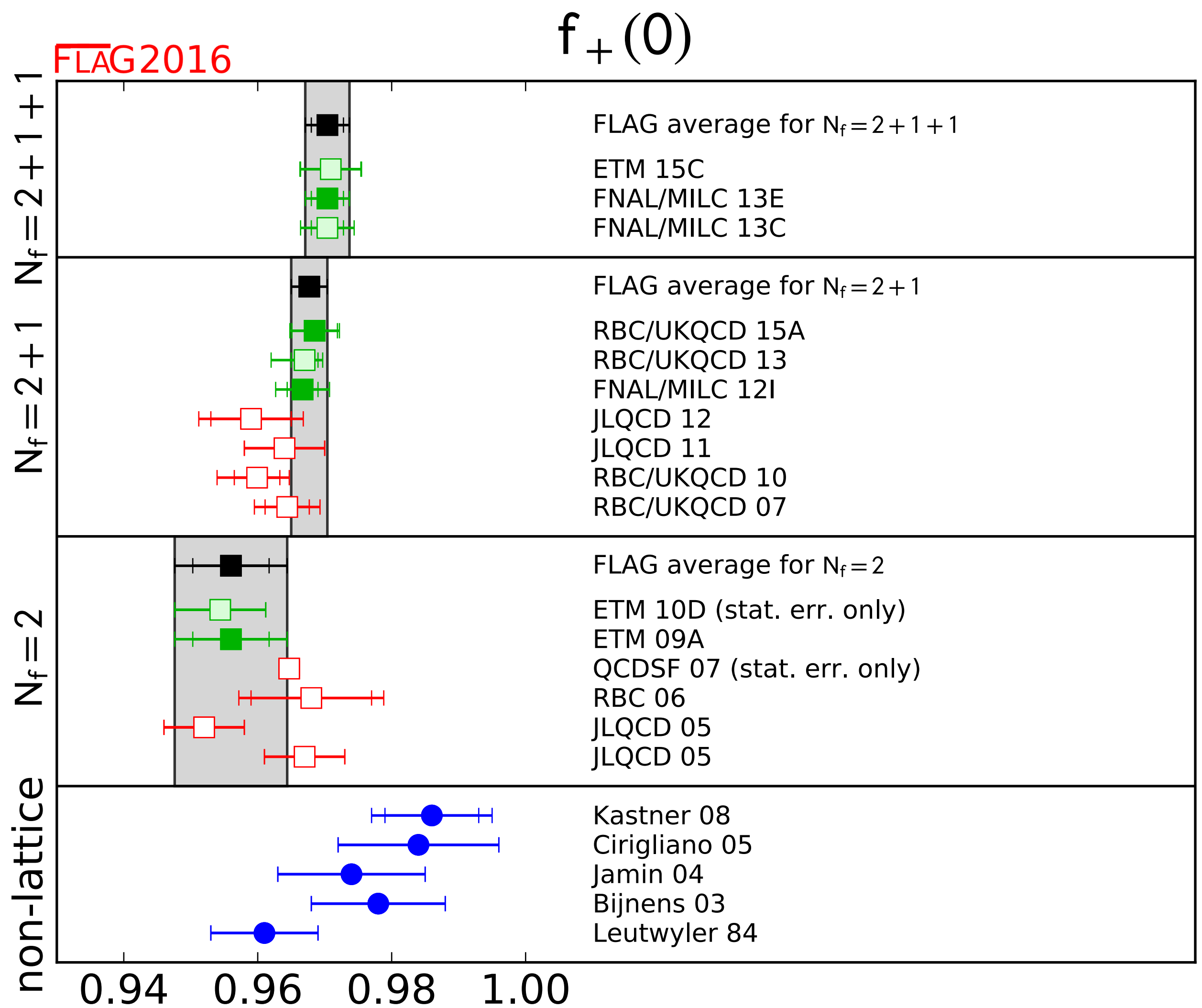
- Reviews every 2^+ years: provide "vetted" averages

Well-controlled LQCD calculations

- Example from FLAG16: K→π form factor



Present Frontier

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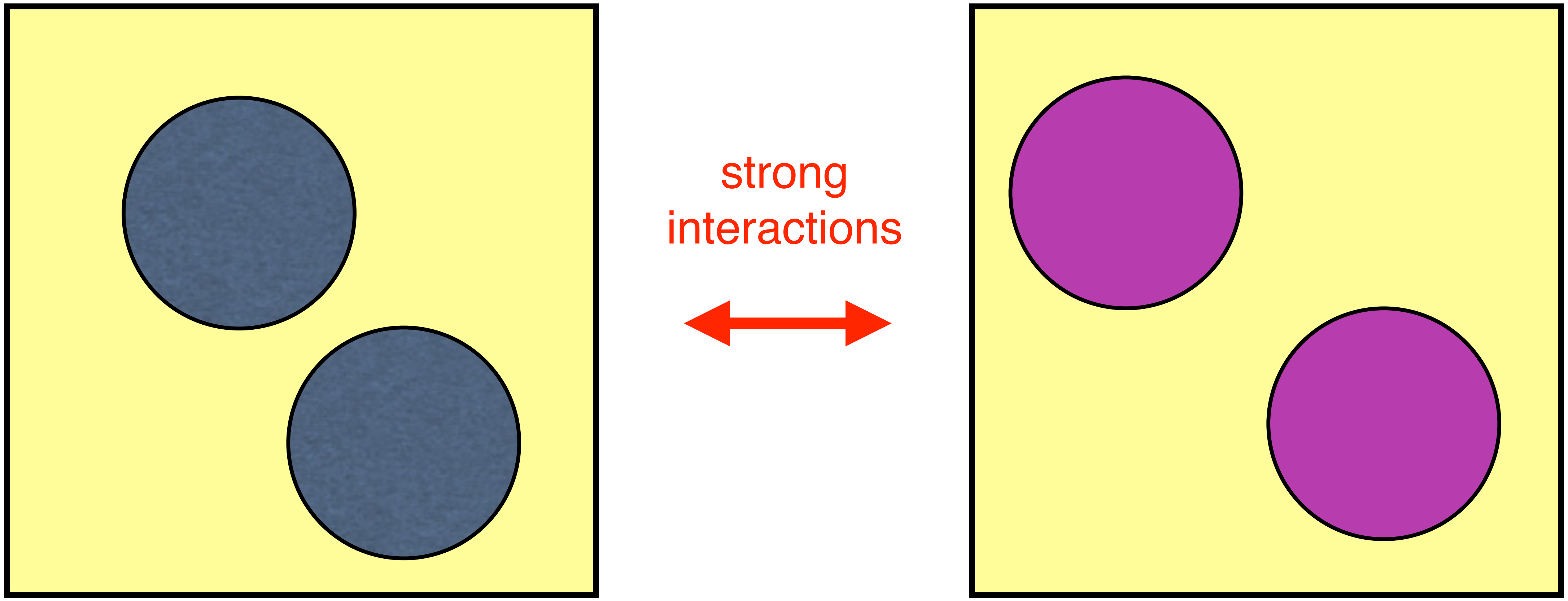
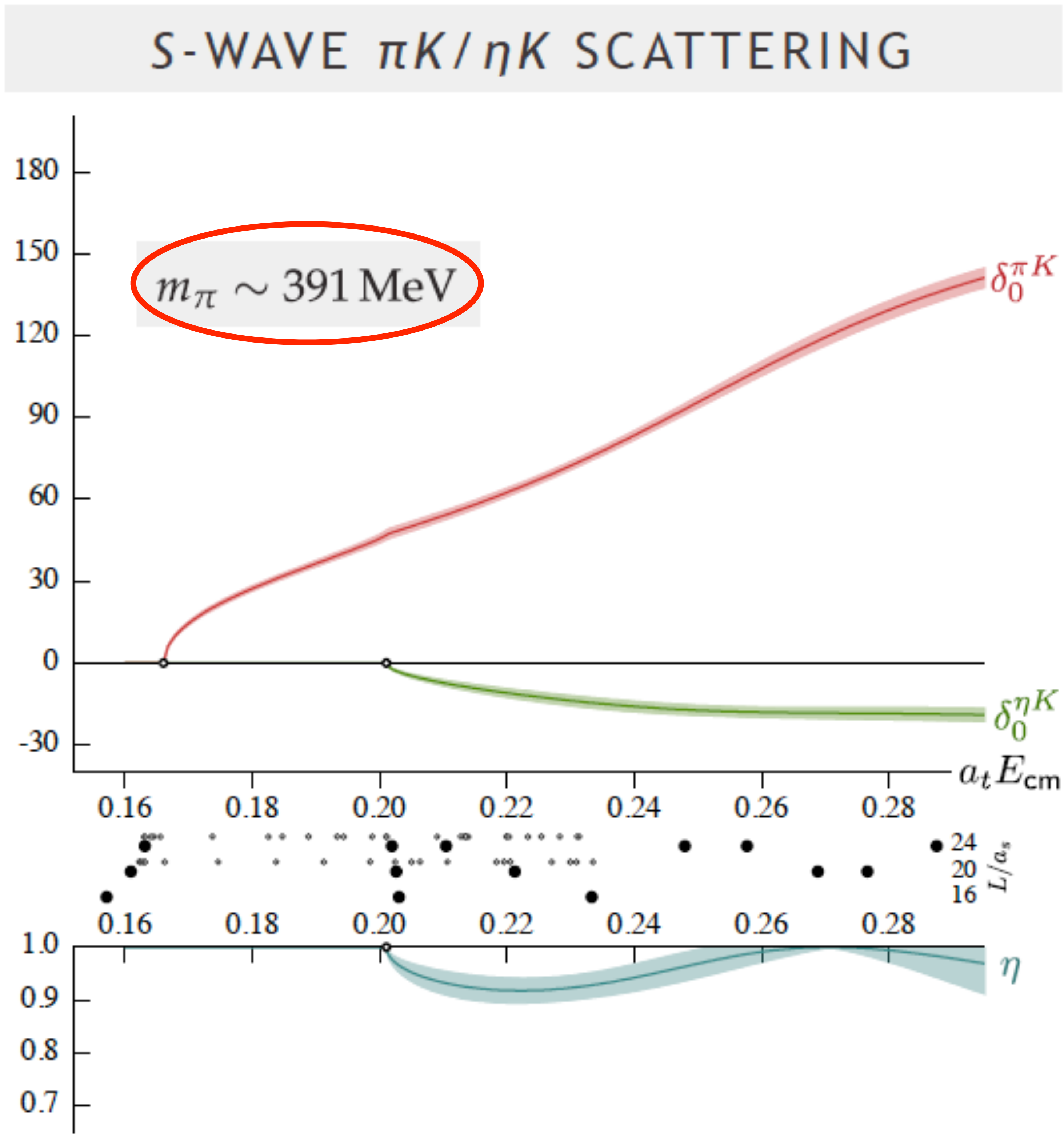


e.g. $\pi K \leftrightarrow \eta K$, $\pi\pi \leftrightarrow \bar{K}K$

- Issues associated with 2 particles ($1/L^n$ finite-volume effects,…) are theoretically understood [Lüscher, …]
- Can extract scattering amplitudes—infinite-volume quantities
- Numerical implementations expanding rapidly despite computational challenges

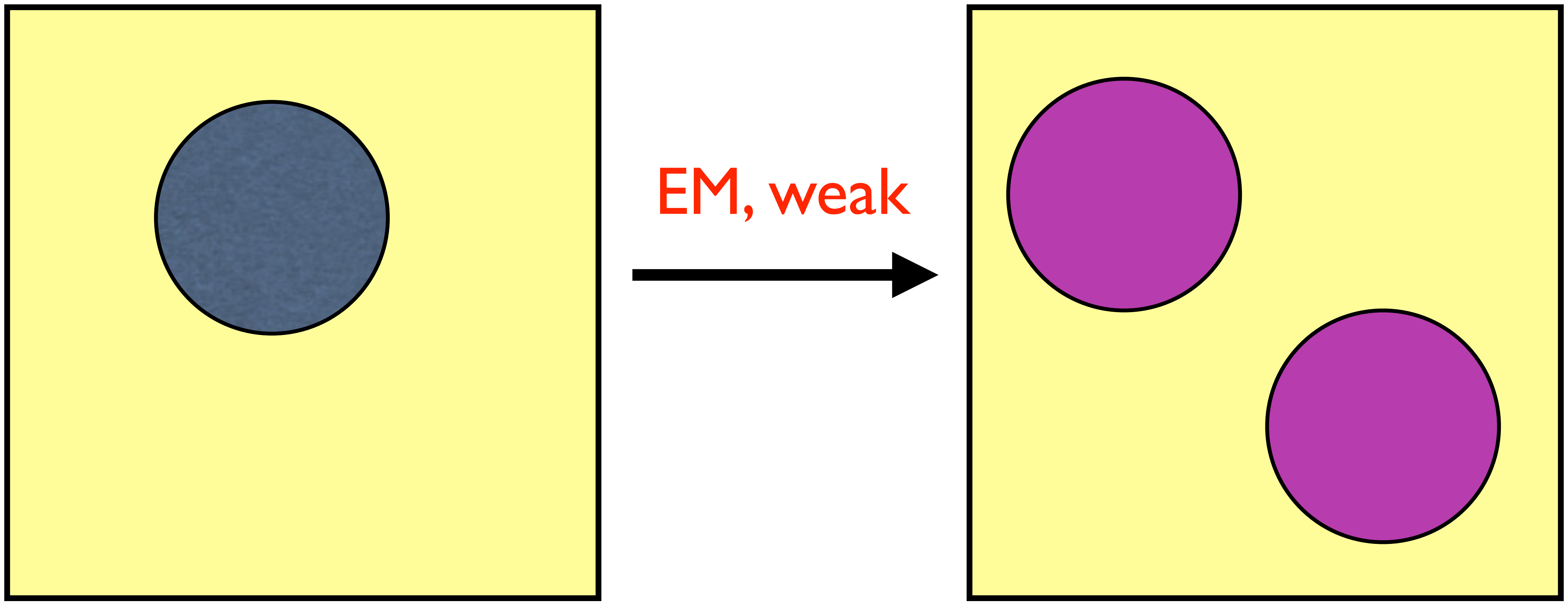
Present frontier



[Dudek, Edwards, Thomas & Wilson arXiv:1406.4158]

- Theory for multiple two-particle channels [He, Feng, Liu 05; Briceño & Davoudi 12; Hansen & SRS 12]

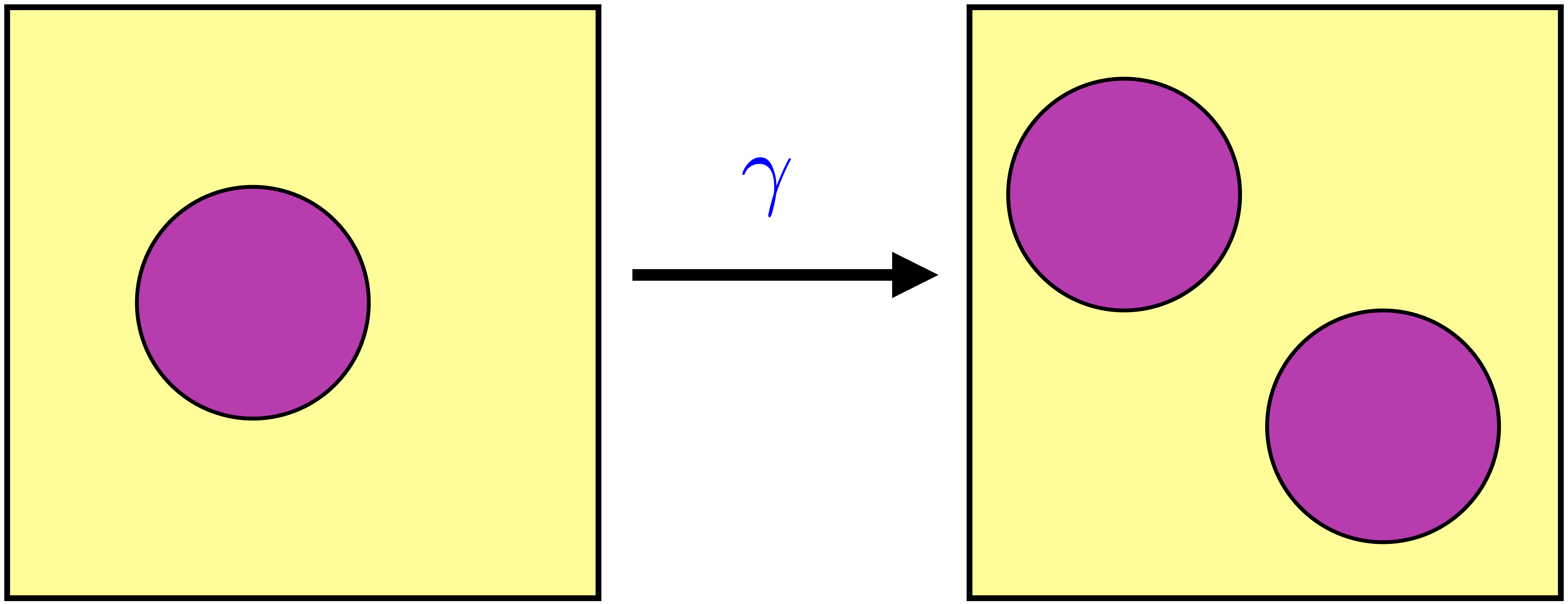
Present frontier



e.g. $K \rightarrow \pi\pi$ decay amplitudes

- Issues associated with 2 particles ($1/L^n$ finite-volume effects,…) are theoretically understood [Lellouch & Lüscher, …]
- First lattice results obtained for decay rates (consistent with $\Delta I = \frac{1}{2}$ rule) and for ε'/ε [RBC/UKQCD]

Present frontier



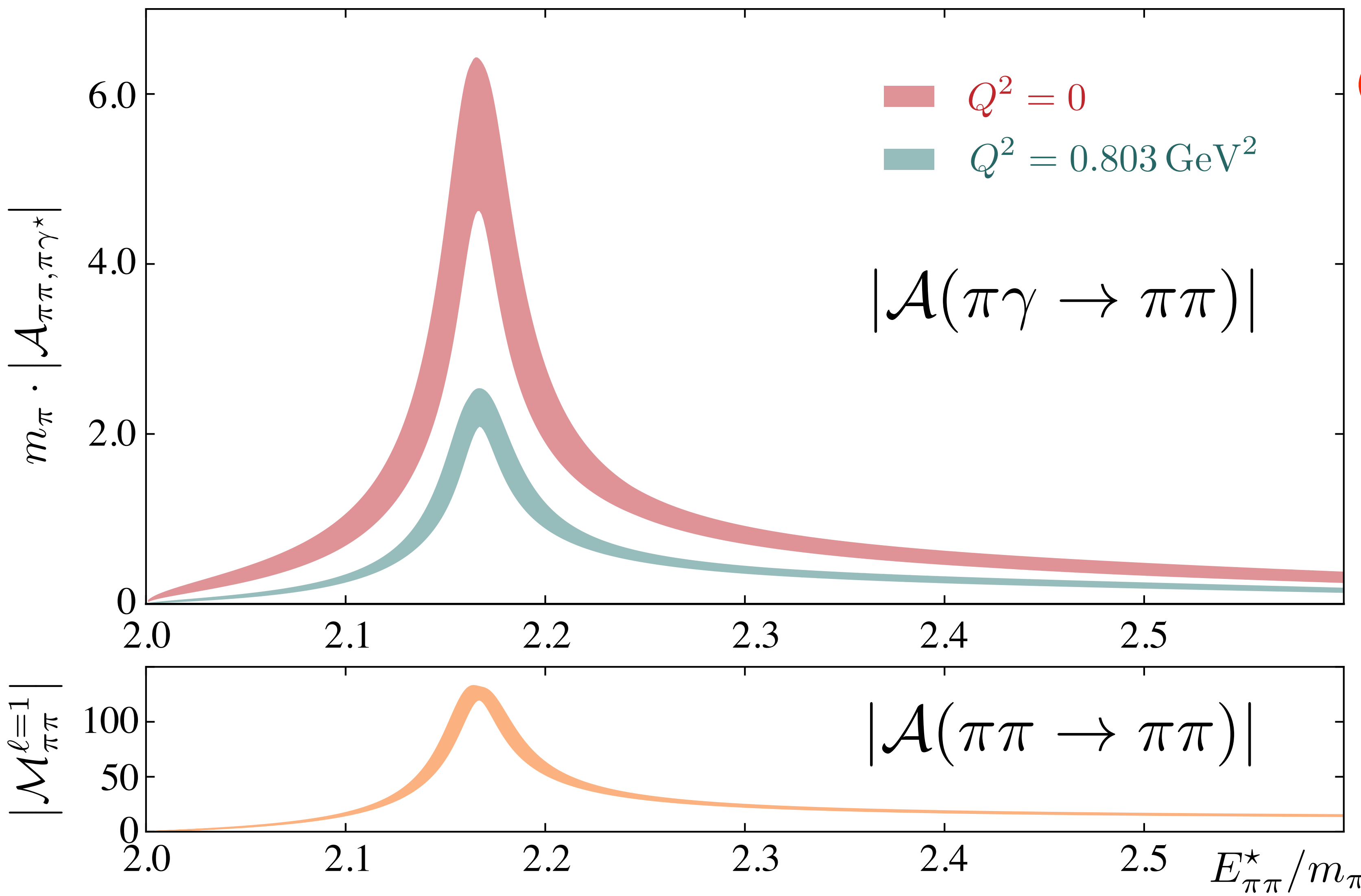
e.g. $\pi\gamma \rightarrow$ ``ρ'' amplitude

- Issues associated with 2 particles ($1/L^n$ finite-volume effects,…) are theoretically understood [Briceño, Hansen & Walker-Loud, …]

Present frontier

$\pi\gamma \to \rho$

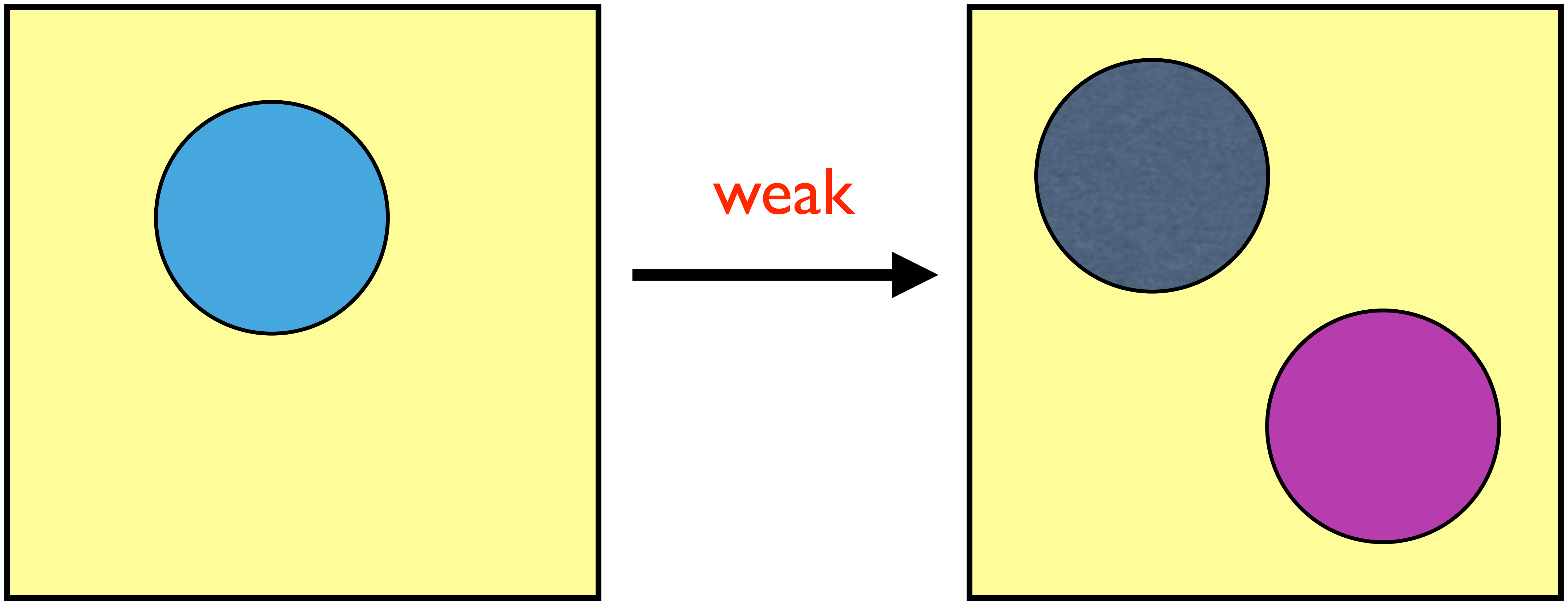
$m_\pi \approx 390$ MeV



Briceño, Dudek, Edwards, Shultz, Thomas, Wilson [HadSpec collab.]
arXiv:1604.03530

- Results also from [Leskovic, …, Meinel, …., arXiv:1611:00282]

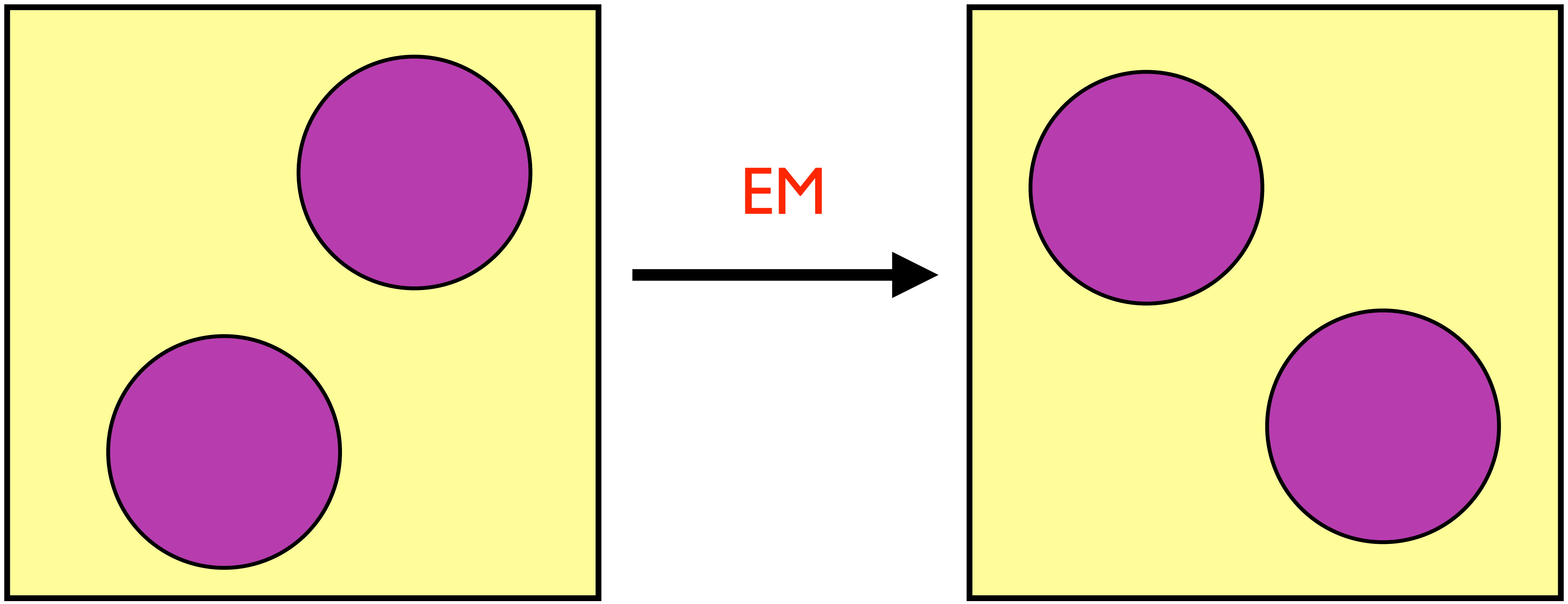
Present frontier



e.g. $B \to K^* \, l \, \nu \to K \, \pi \, l \, \nu$ decay amplitude

- Issues associated with 2 particles ($1/L^n$ finite-volume effects,…) are theoretically understood [Briceño, Hansen & Walker-Loud, …]

Present frontier

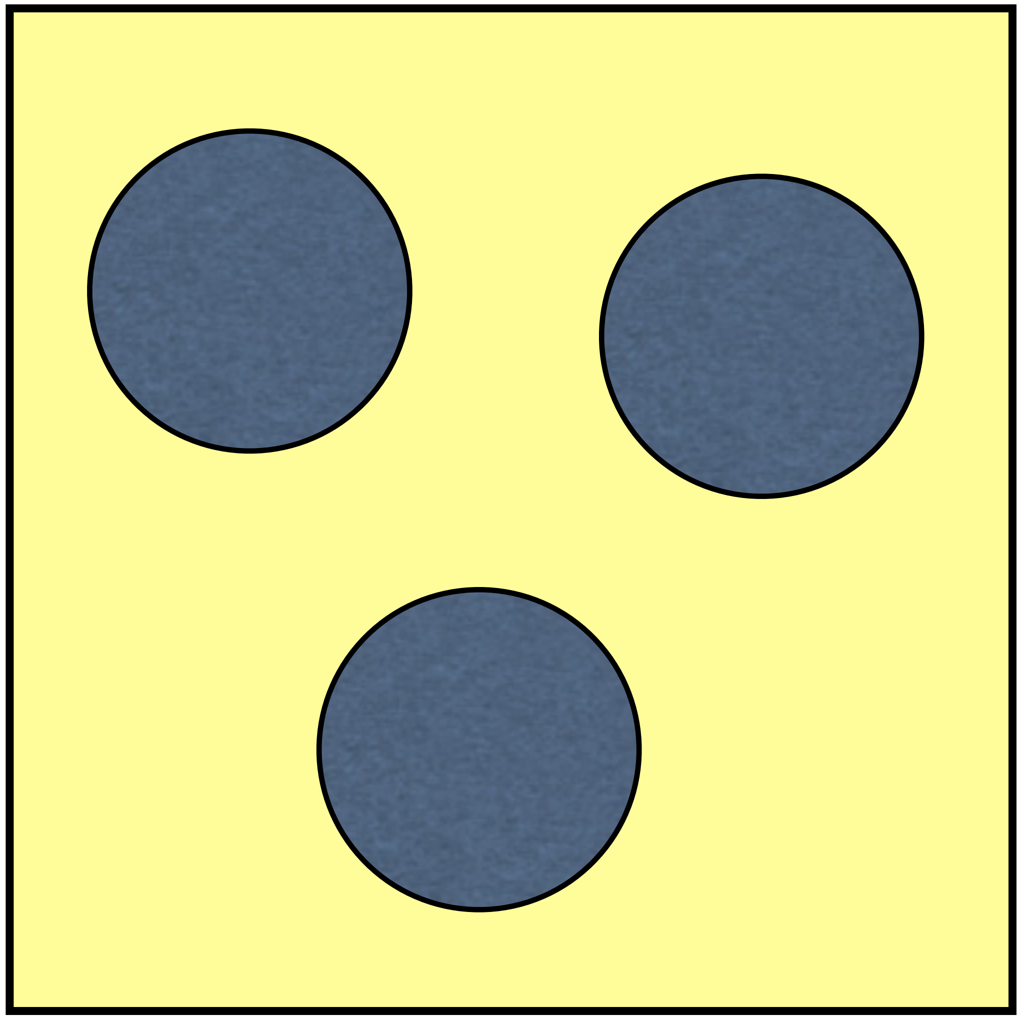


e.g. ``ρ" form factor

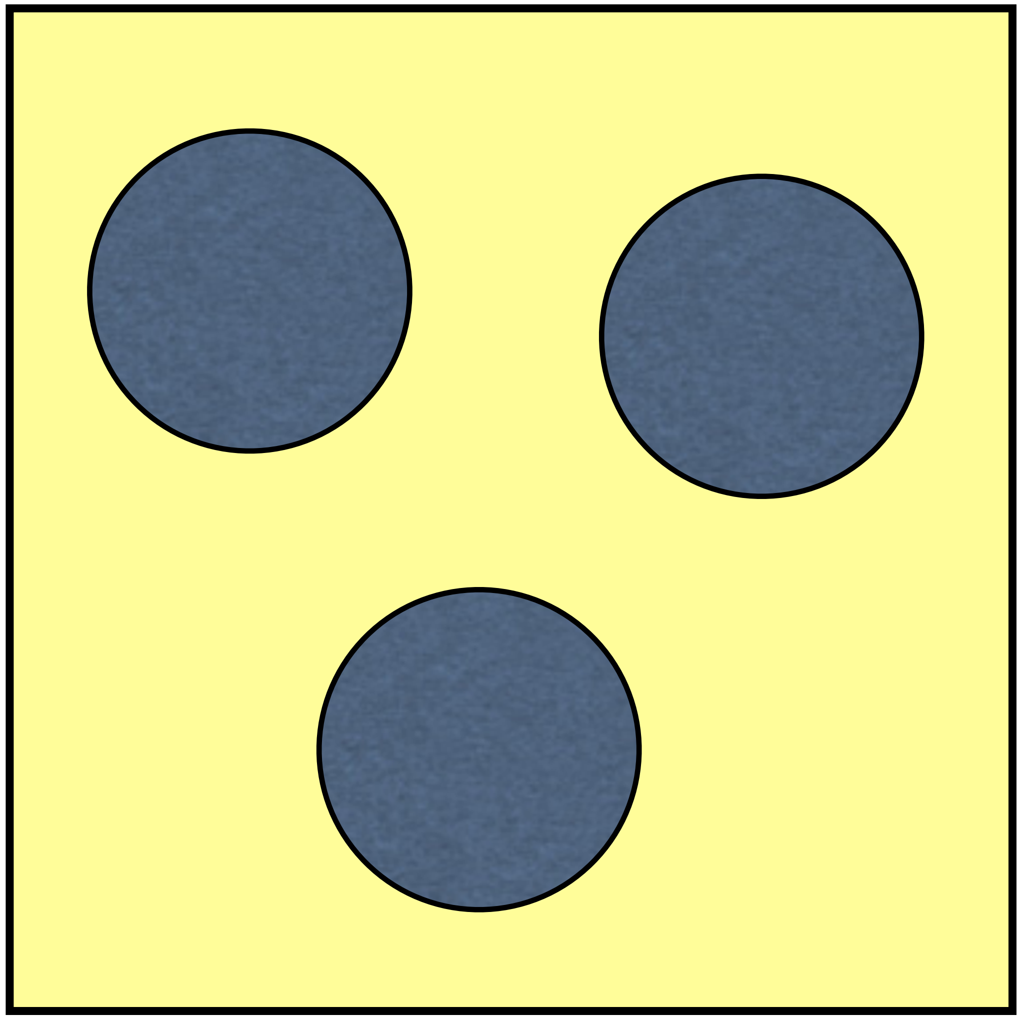
- Issues associated with 2 particles ($1/L^n$ finite-volume effects,…) are theoretically understood [Bernard et al.,Briceño & Hansen]
- Not yet implemented in simulations

Just beyond the frontier

Just beyond the frontier



Just beyond the frontier



- Simulations access the three-particle region of the spectrum
- What can we learn from them?
- Why do we care?

Motivation(s) for studying three (or more) particles

Resonances

- Studying resonances with three particle decay channels

e.g. $\omega(782) \to \pi\pi\pi$ $\qquad N(1440) \to N\pi\pi$

Resonances

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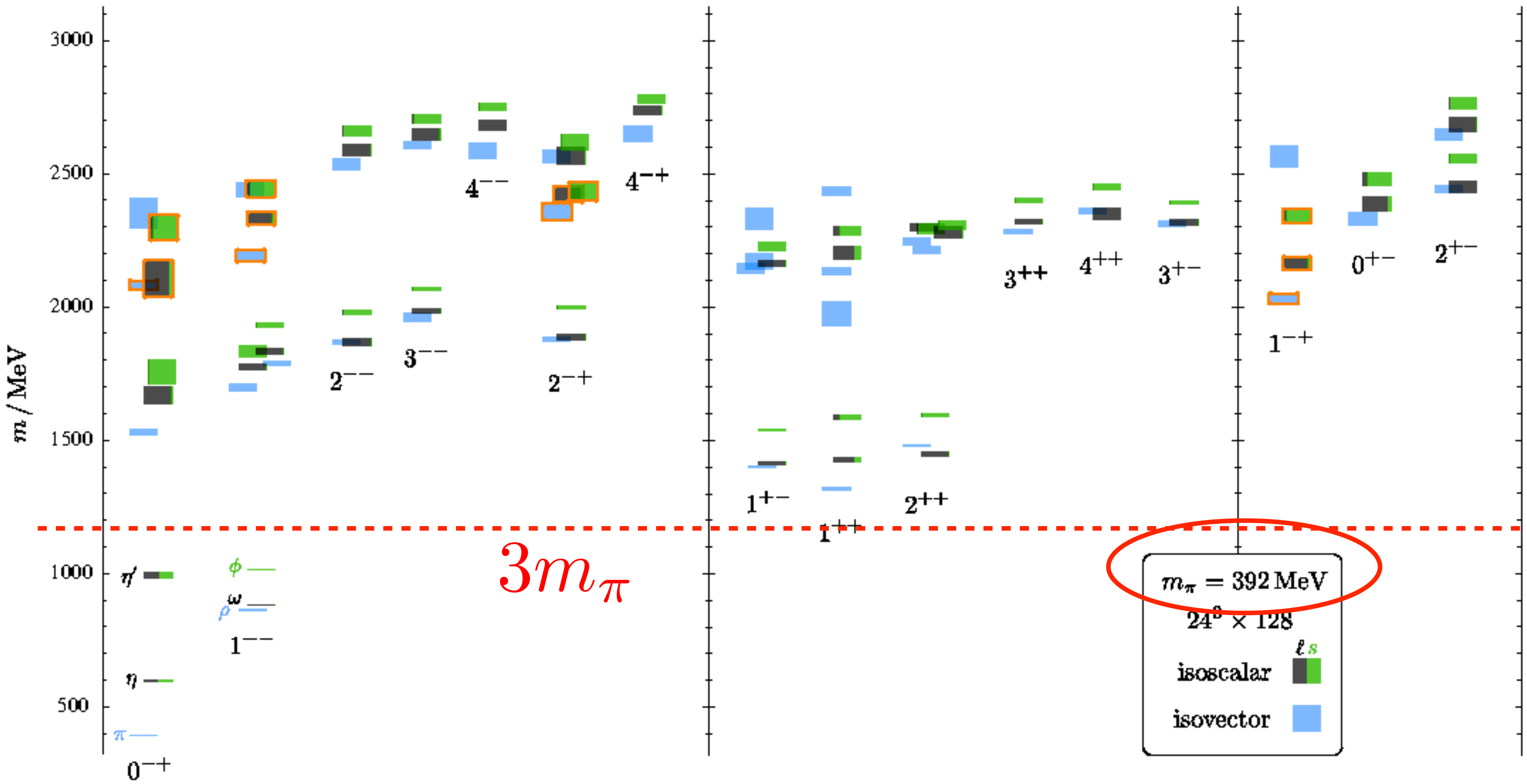
  e.g. $\omega(782) \to \pi\pi\pi$ $\quad N(1440) \to N\pi\pi$

- N.B. If a resonance has both 2- and 3-particle strong decays, then 2-particle methods fail—channels cannot be separated as they can in an experiment

Resonances



Dudek, Edwards, Guo & C.Thomas [HadSpec], arXiv:1309.2608

Weak decays

Weak decays

- Calculating weak decay amplitudes/form factors involving 3 particles, e.g. $K \to \pi\pi\pi$

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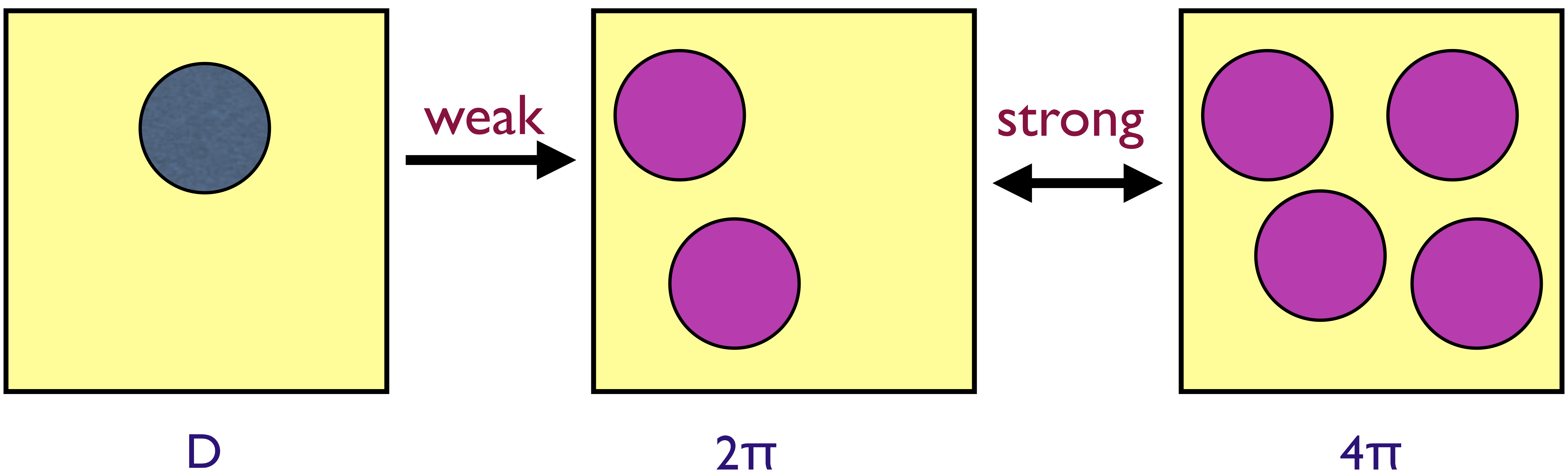
- N.B. Can study weak $K \rightarrow 2\pi$ decays independently of $K \rightarrow 3\pi$, since strong interactions do not mix these final states (in isospin-symmetric limit)

A more distant motivation

- Calculating CP-violation in $D\to\pi\pi, K\overline{K}$ in the Standard Model
- Finite-volume state is a mix of $2\pi, K\overline{K}, \eta\eta, 4\pi, 6\pi, \ldots$
- Need 4 (or more) particles in the box!



3-body interactions

3-body interactions

- Determining NNN interaction
  - Input for effective field theory treatments of larger nuclei & nuclear matter
- Similarly, $\pi\pi\pi$, $\pi K\bar{K}$, … interactions needed for study of pion/kaon condensation

Inclusive decays

Inclusive decays

- $B \to X_u l \nu$, $B \to X_c l \nu$, etc. involve many channels containing multiple strongly-interacting particles
  - Extending Lellouch-Lüscher approach seems impossibly complicated
  - Alternative approaches using smearing of Euclidean-time correlators are promising
    - Hansen, Meyer & Robaina [arXiv:1704.08993]—see later today
    - Optical potential [Agadjanov *et* al., arXiv: 1603.07205]
    - Optical theorem at subthreshold kinematics [Hashimoto, arXiv: 1703.01881]
    - Shape function [Aglietti *et al.*, hep-ph/9804416]
    - Long distance contributions to ΔM_K [Christ, Feng, Martinelli & Sachrajda, 1504.01170]
- Related ideas apply to light-cone wave functions and structure functions

Overview of theoretical issues for 2 and 3 particles

The fundamental issue

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- Lattice simulations are done in finite volumes

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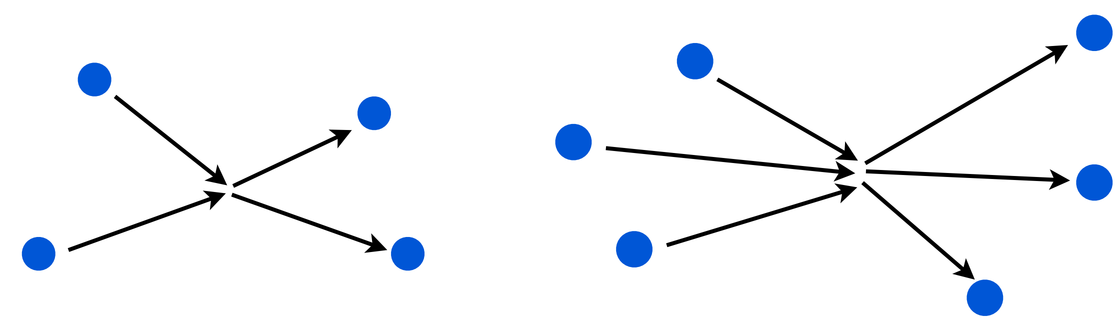
- Experiments are not



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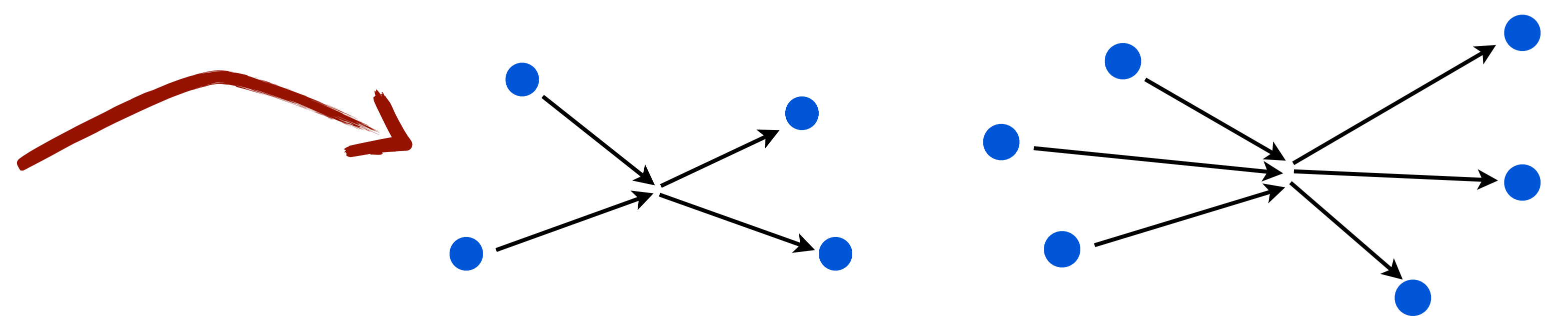
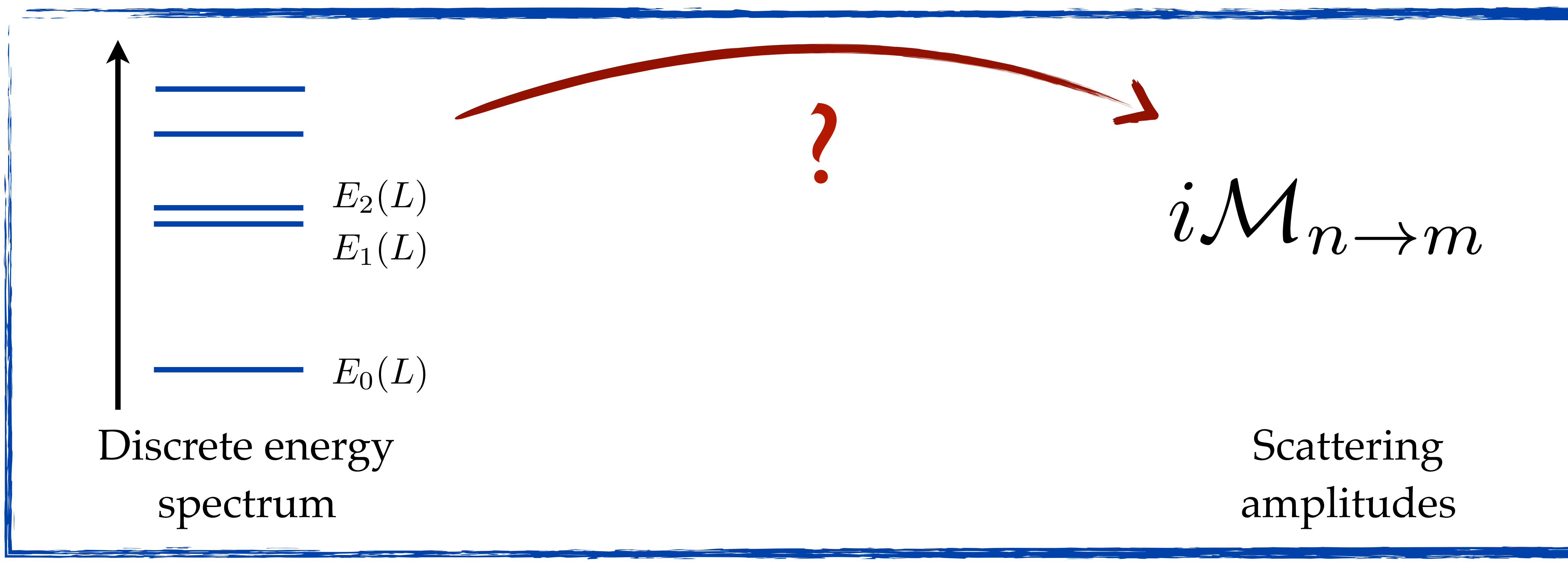


How do we connect these?

The fundamental issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box
- How are these related to infinite-volume scattering amplitudes (which determine resonance properties)?



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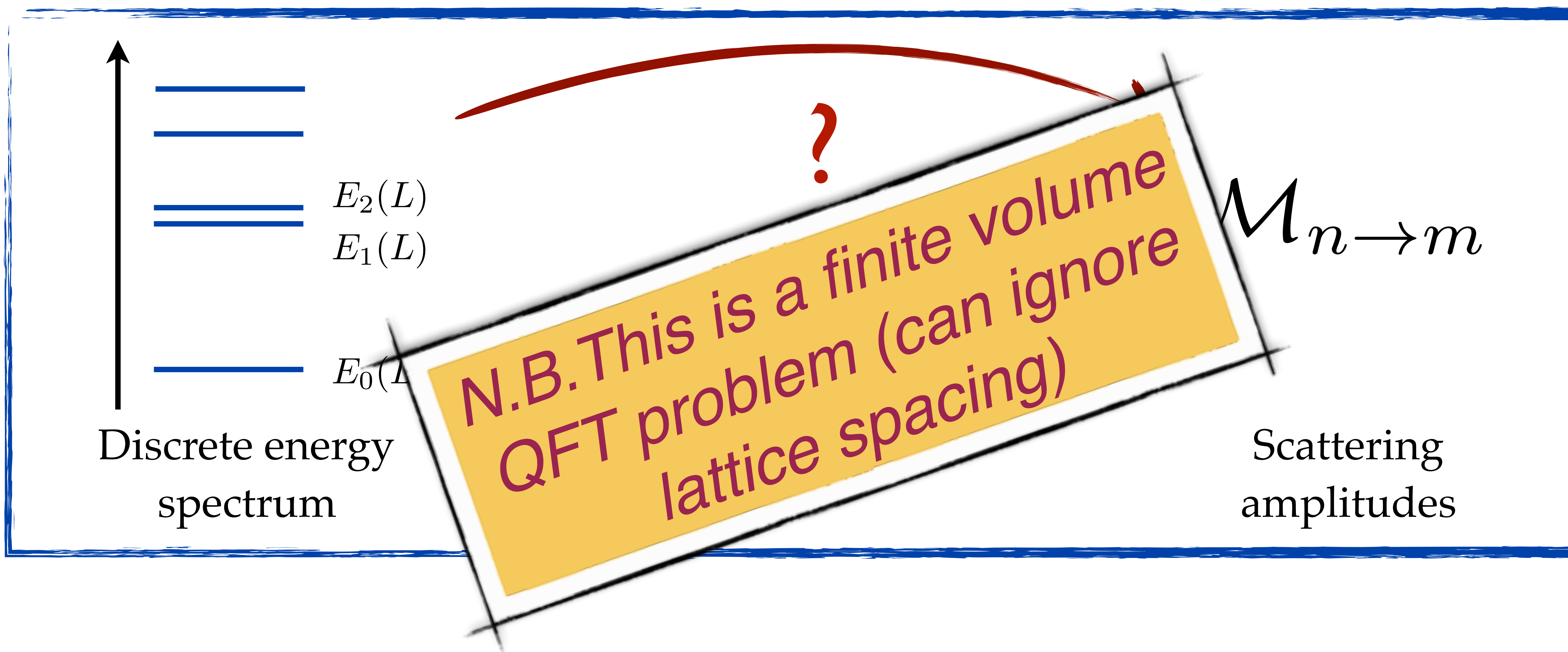
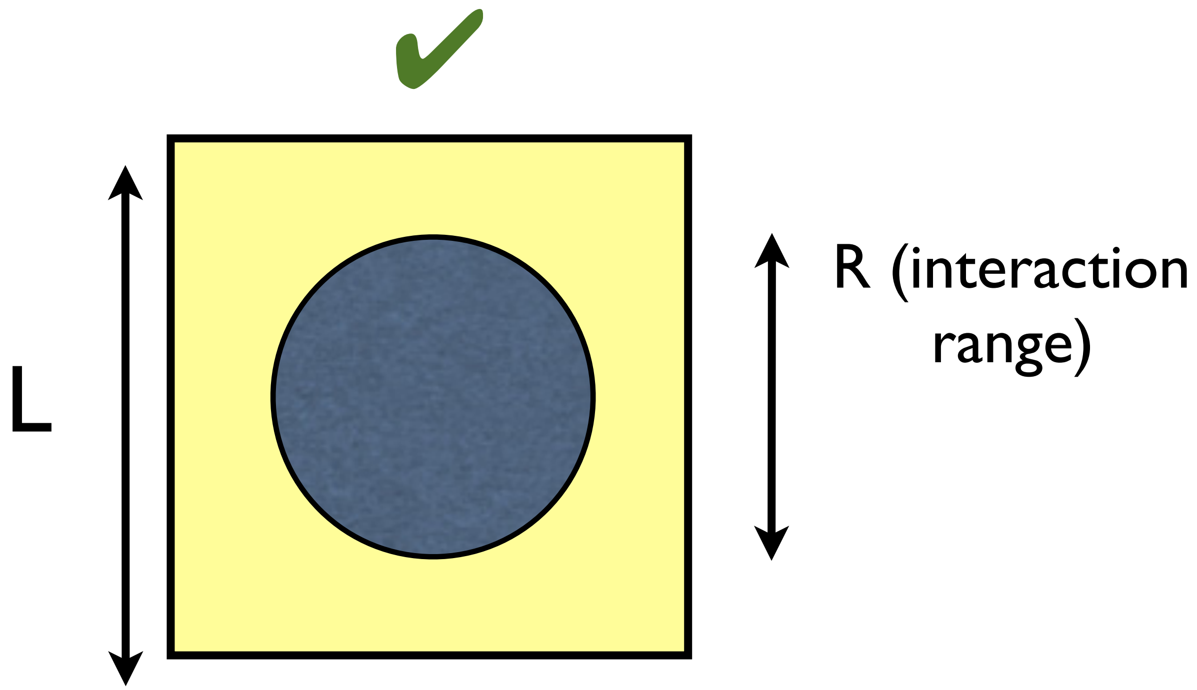


When is the spectrum related to scattering amplitudes?



Single (stable) particle with L>R
Particle not "squeezed"
Spectrum same as in infinite volume up to corrections proportional to $e^{-M_\pi L}$
[Lüscher]

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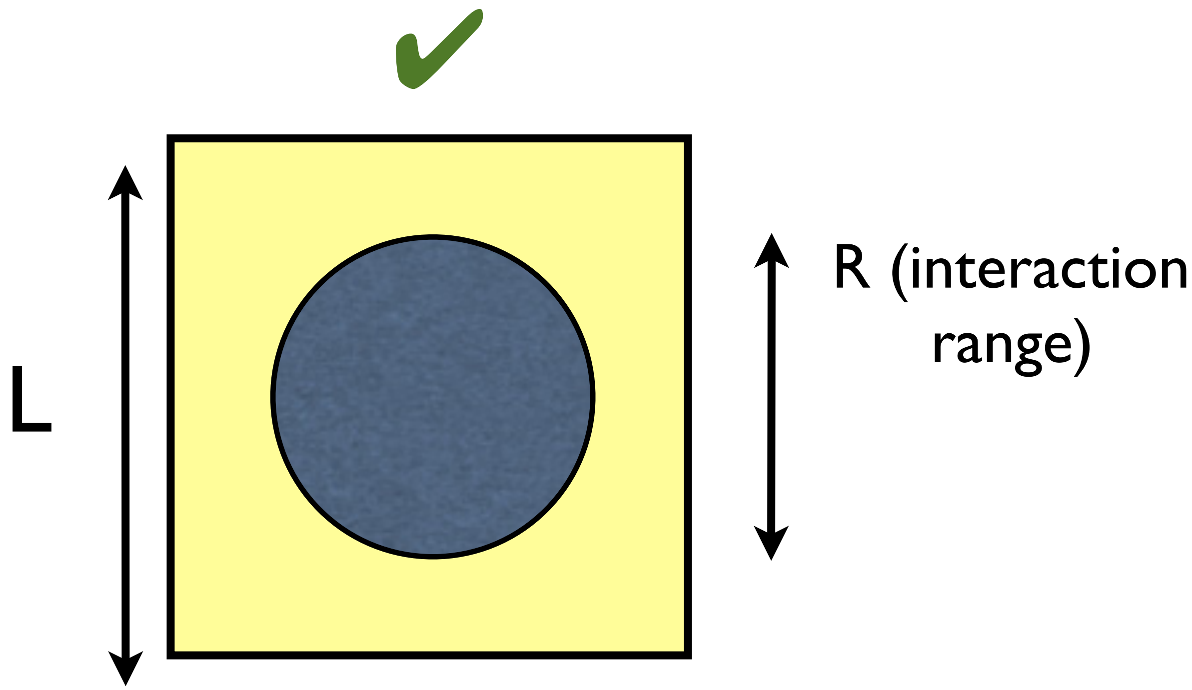
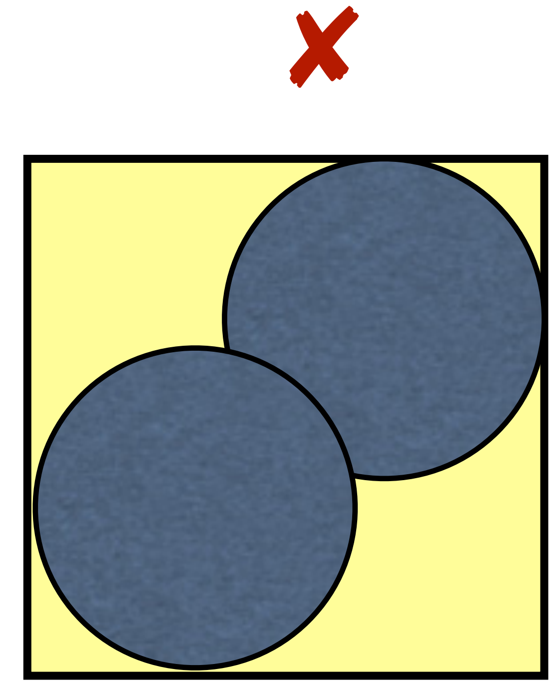
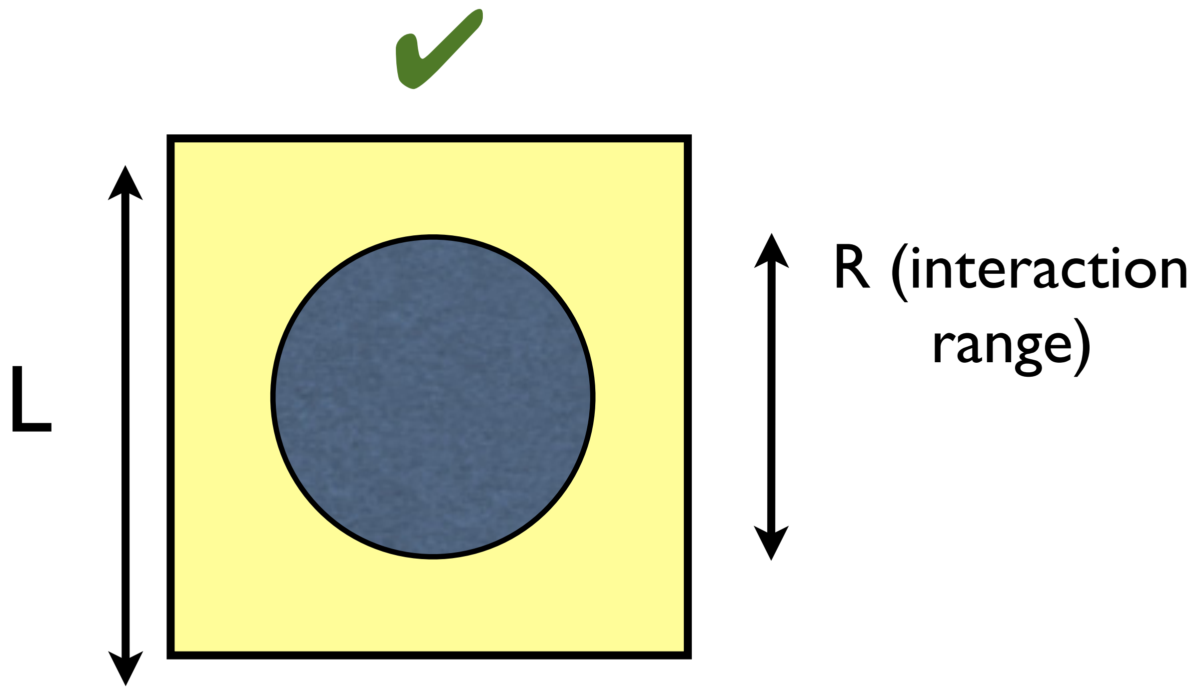




Single (stable) particle with L>R
Particle not "squeezed"
Spectrum same as in infinite volume up to corrections proportional to $e^{-M_\pi L}$
[Lüscher]

L<2R
No "outside" region.
Spectrum NOT related to scatt. amps.
Depends on finite-density properties

When is the spectrum related to scattering amplitudes?



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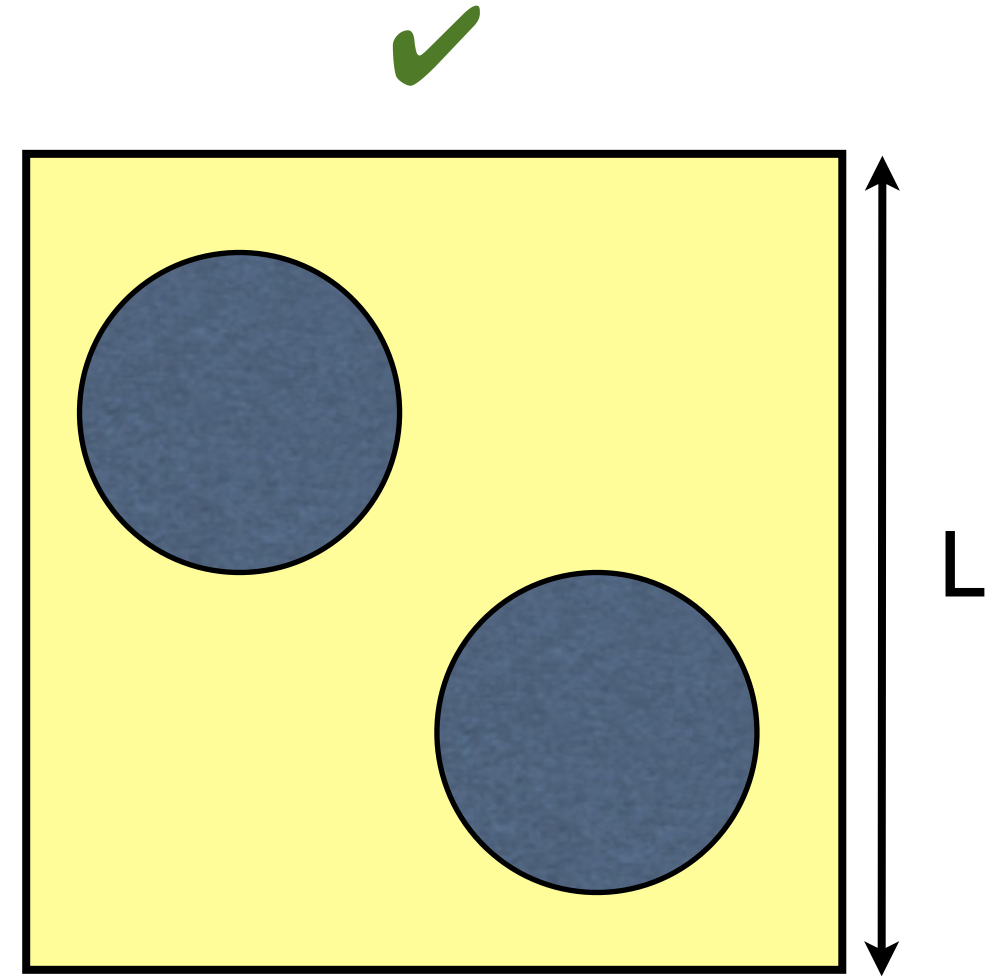
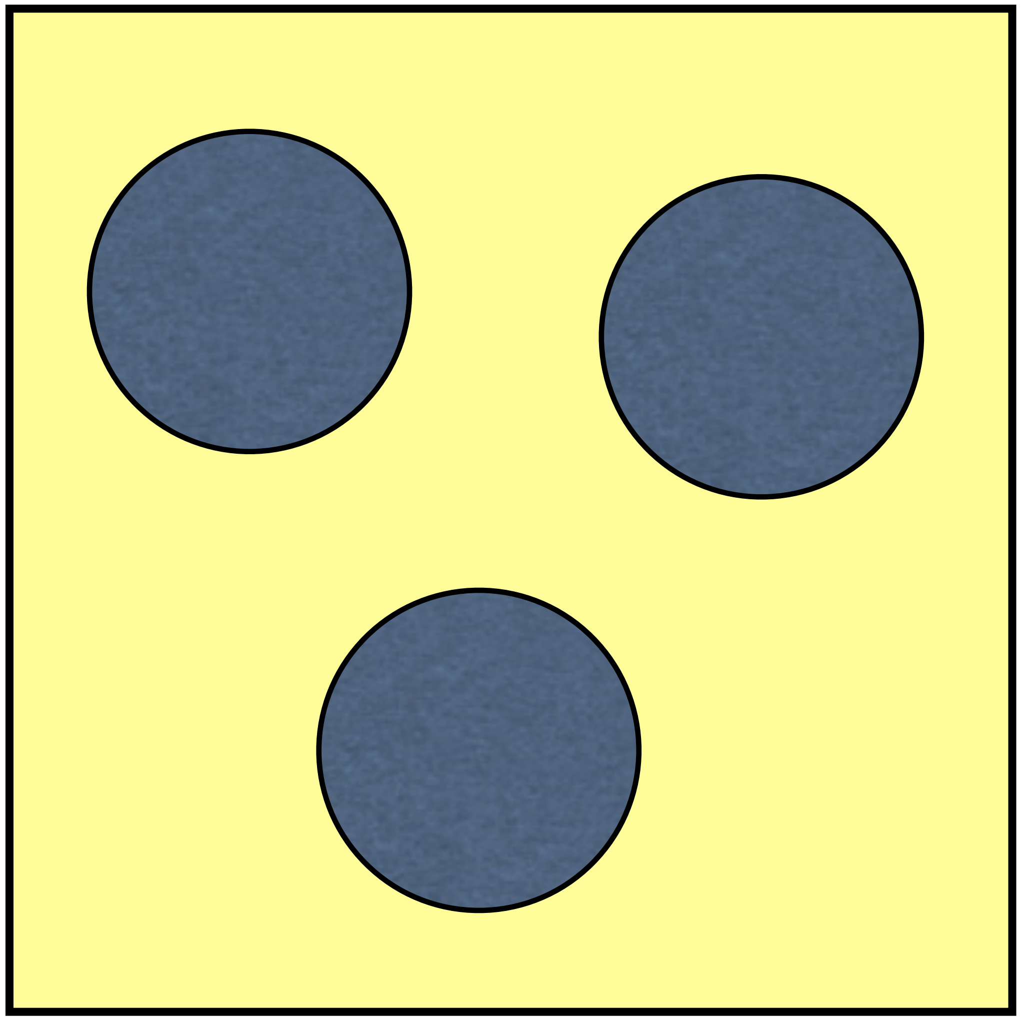


L>2R
There is an "outside" region.
Spectrum IS related to scatt. amps. up to corrections proportional to $e^{-M_\pi L}$
[Lüscher]
Theoretically understood;
numerical implementations mature.

...and for 3 particles?



- Spectrum IS related to 2→2, 2→3 & 3→3 scattering amplitudes up to corrections proportional to e^{-ML} [Polejaeva & Rusetsky]
- General relativistic formalism developed in various cases [Hansen & SRS, Briceño, Hansen & SRS]
- Formalism based on NREFT recently proposed [Hammer, Pang & Rusetsky]
- Practical applicability under investigation

HALQCD method

- Alternative approach, followed by the HALQCD collaboration [Aoki et al.], using the Bethe-Salpeter wave-function calculated with lattice QCD to determine potentials and from these, by solving the Schrödinger equation, scattering amplitudes
- Extended from 2-particle to 3- (and higher) particle case in non-relativistic domain
- Potentially more powerful than the Lüscher-like methods I discuss today, but based on certain assumptions

Two-particle results

Single-channel 2-particle quantization condition

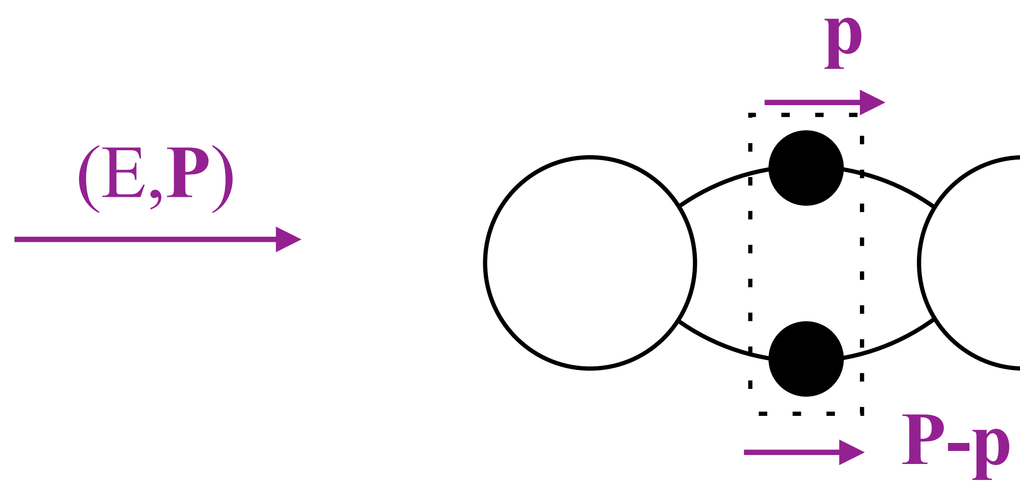
[Lüscher 86 & 91; Rummukainen & Gottlieb 85; Kim, Sachrajda & SRS 05; …]

- Two particles (say pions) in cubic box of size L with PBC and total momentum **P**
- Below inelastic threshold (4 pions), the finite-volume spectrum $E_1, E_2, \ldots$ is given by solutions to a secular equation in partial-wave *(l,m)* space (up to exponentially suppressed corrections)

$$\boxed{\det\left[F_{\rm PV} + \mathcal{K}_2^{-1}\right] = 0}$$

- $\mathcal{K}_2 \sim \tan\delta/q$ is the K-matrix, which is diagonal in *l,m* space
- $F_{\rm PV}$ is a known kinematical "zeta-function", depending on the box shape & E; It is an off-diagonal matrix in *l,m*, since the box violates rotation symmetry

Finite-volume function



$$F_{2;\ell' m';\ell m}(E,\vec{P}) \equiv \frac{1}{2}\left[\frac{1}{L^3}\sum_{\vec{p}} - \mathrm{PV}\int \frac{d^3p}{(2\pi)^3}\right] \frac{4\pi Y_{\ell' m'}(\hat{p}^*) Y^*_{\ell,m}(\hat{p}^*)}{2\omega_p 2\omega_{Pp}(E-\omega_p-\omega_{Pp})} \left(\frac{p^*}{q^*}\right)^{\ell+\ell'} h(\vec{p})$$

$$\propto \left(\frac{2\pi}{L}\right)^{1+\ell+\ell'} \mathcal{Z}_{\ell',m';\ell,m}(x^2,\, \mathbf{n_P})$$

q^* = on-shell CM momentum

$x=q^*L/(2\pi)$

$\mathbf{n_P} = \mathbf{P}L/(2\pi)$

"Zeta-functions"

$Z_{4,0}$ & $Z_{6,0}$ for **P=0** [Luu & Savage, `11]

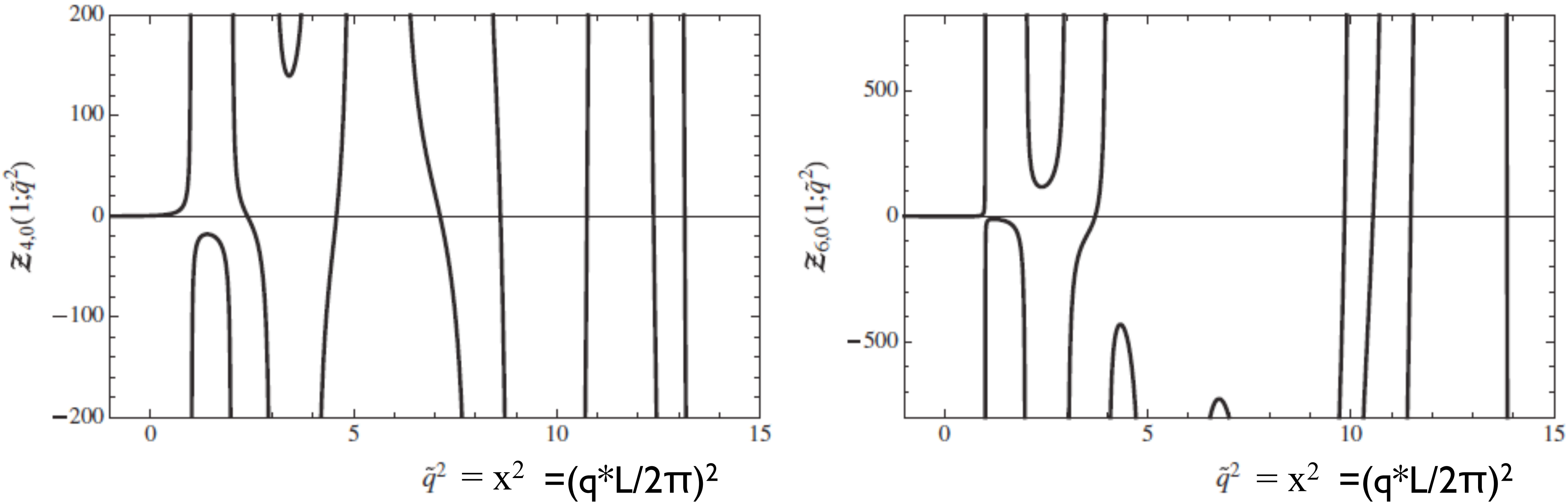


FIG. 29. The functions $Z_{4,0}(1;\tilde{q}^2)$ (left panel) and $Z_{6,0}(1;\tilde{q}^2)$ (right panel).

Single-channel 2-particle quantization condition

$$\boxed{\det\left[(F_{\rm PV})^{-1} + \mathcal{K}_2\right] = 0}$$

- Infinite-dimensional determinant must be truncated to be practical; truncate by assuming that $\mathcal{K}_2$ vanishes above l_{max}

- If $l_{max}=0$, obtain one-to-one relation between energy levels and $\mathcal{K}_2$

$E_n^* = \sqrt{E_n^2 - \vec{P}^2}$

CM energy

$$\boxed{\mathcal{K}_{2,s}(E_n^*) = -\frac{1}{F_{{\rm PV};00;00}(E_n, \vec{P}, L)}}$$

"measured" energy-level

Equivalent to: $\tan[\delta(q^*)] = -\tan[\phi^P(q^*)],$

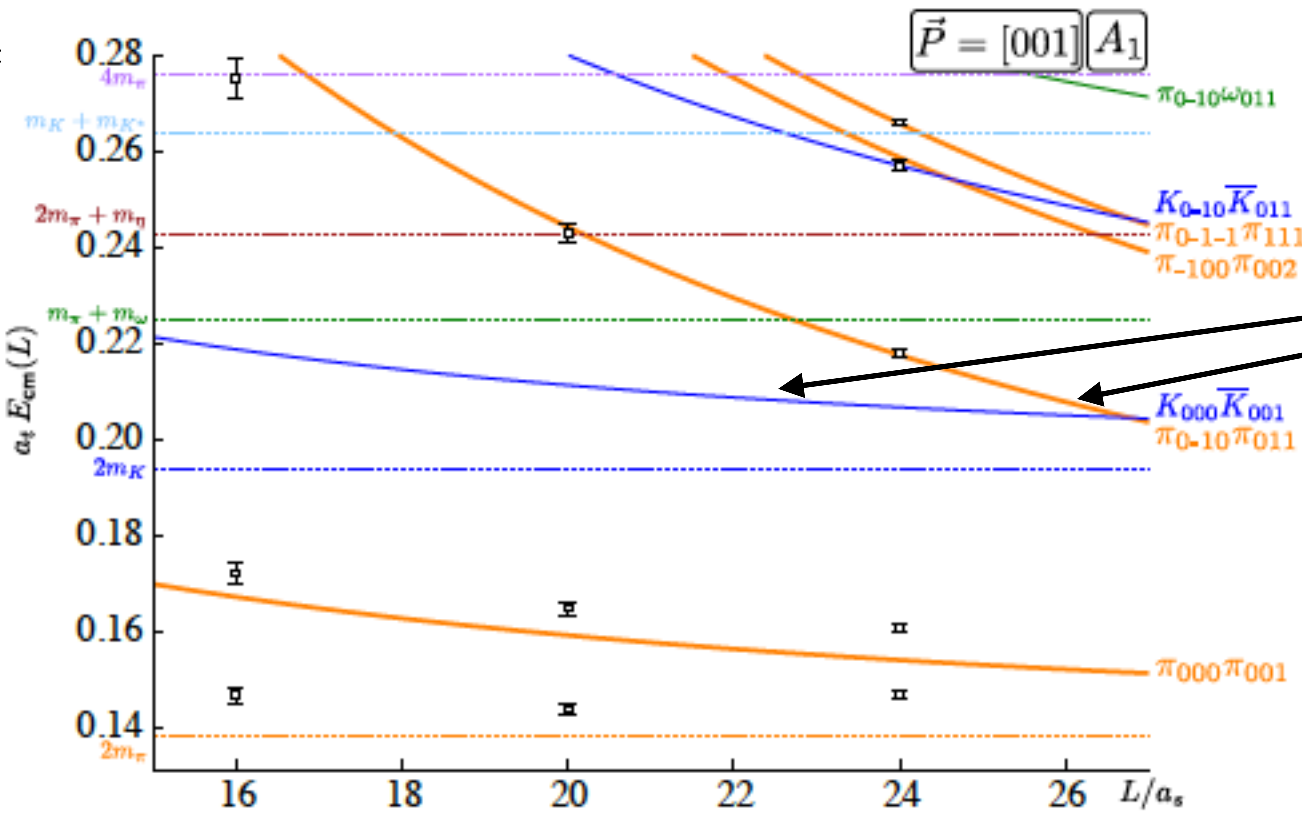
Application to ρ meson

[Dudek, Edwards & Thomas, 1212.0830]

- Proof of principle calculation with $M_\pi \sim 400$ MeV, several **P**, many spectral levels

P=(0,0,1)2π/L

a E*



L/a

free-particle energies

2 M_K

2 M_π

Elastic region where quantization condition applies

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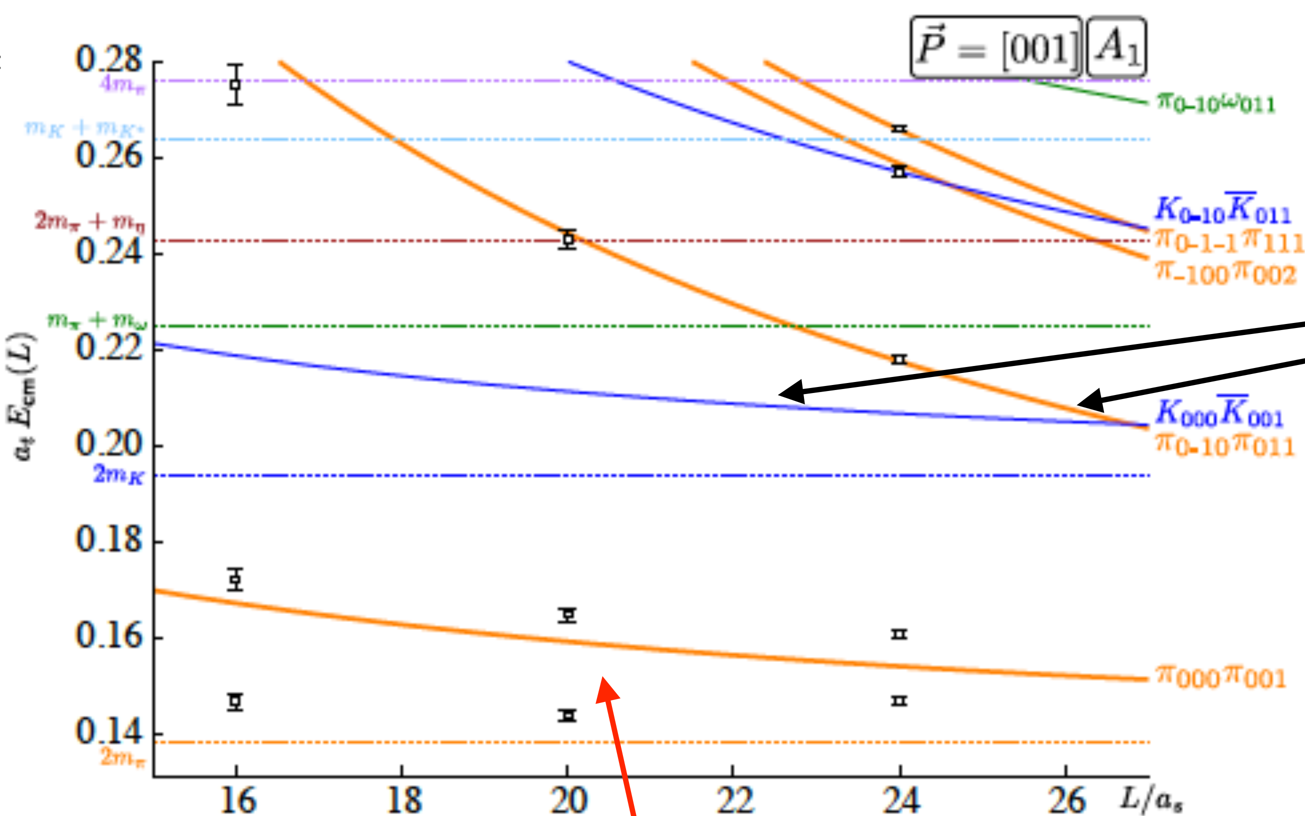


L/a

free-particle energies

2 M_K

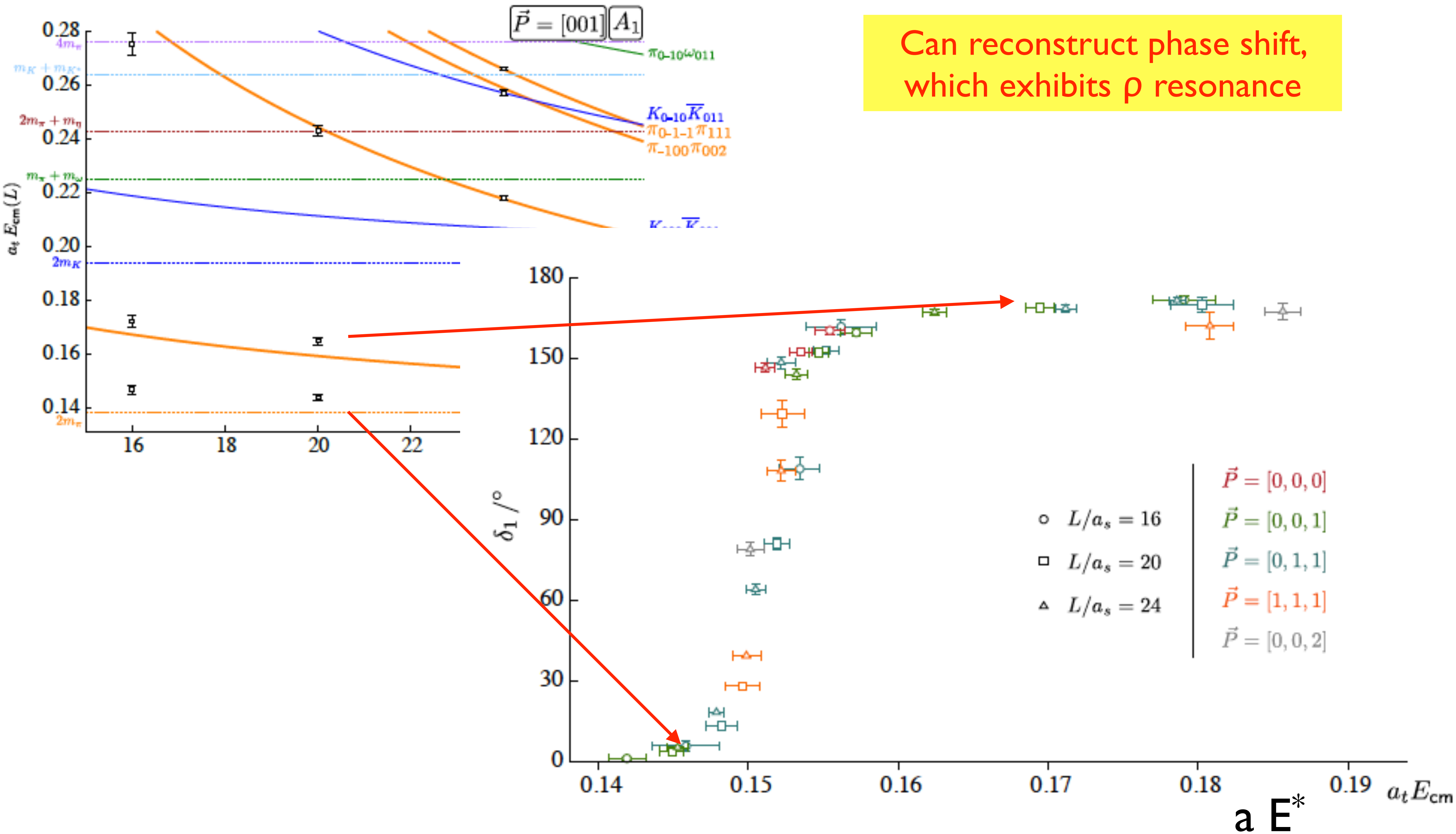
2 M_π

Elastic region where quantization condition applies

KEY POINT: there are "extra" levels here, and neither are close to the free levels

Application to ρ meson

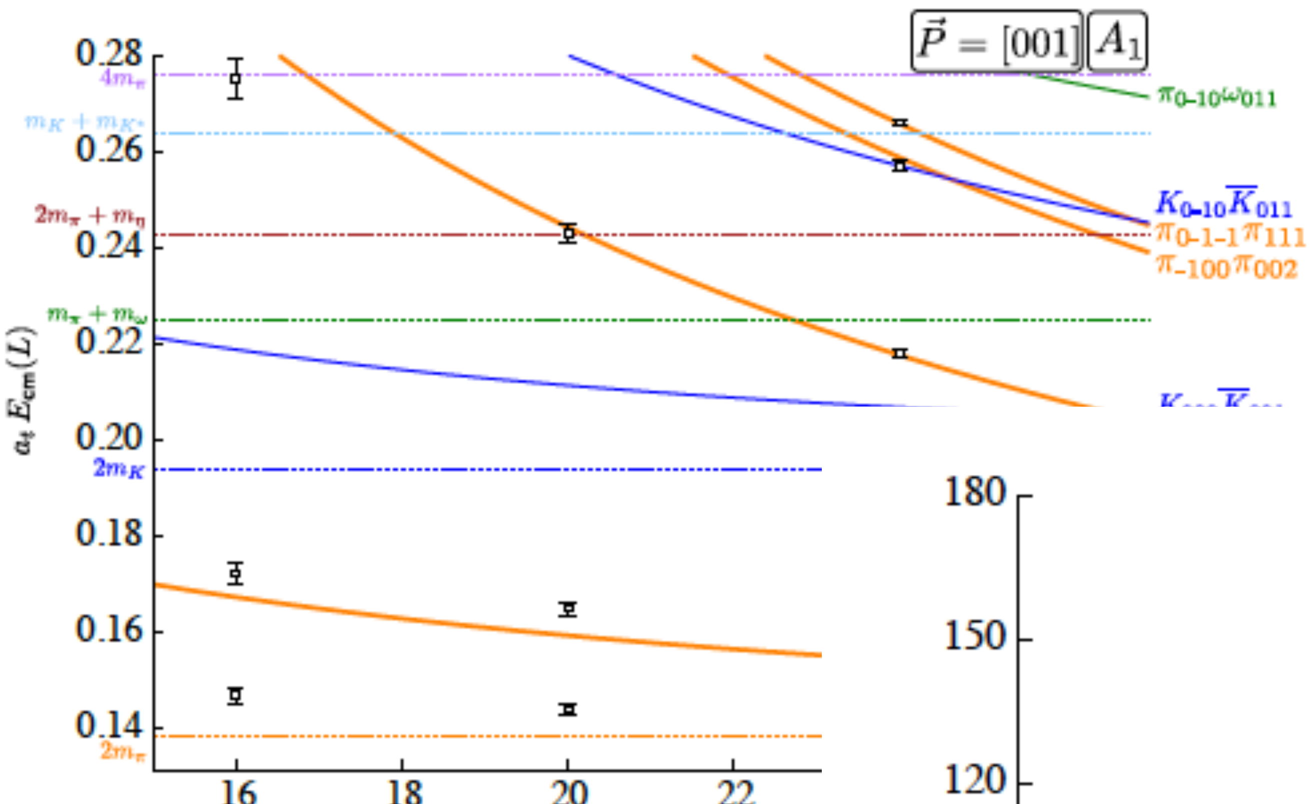
[Dudek, Edwards & Thomas, 1212.0830]



Can reconstruct phase shift, which exhibits ρ resonance

Application to ρ meson

[Dudek, Edwards & Thomas, 1212.0830]



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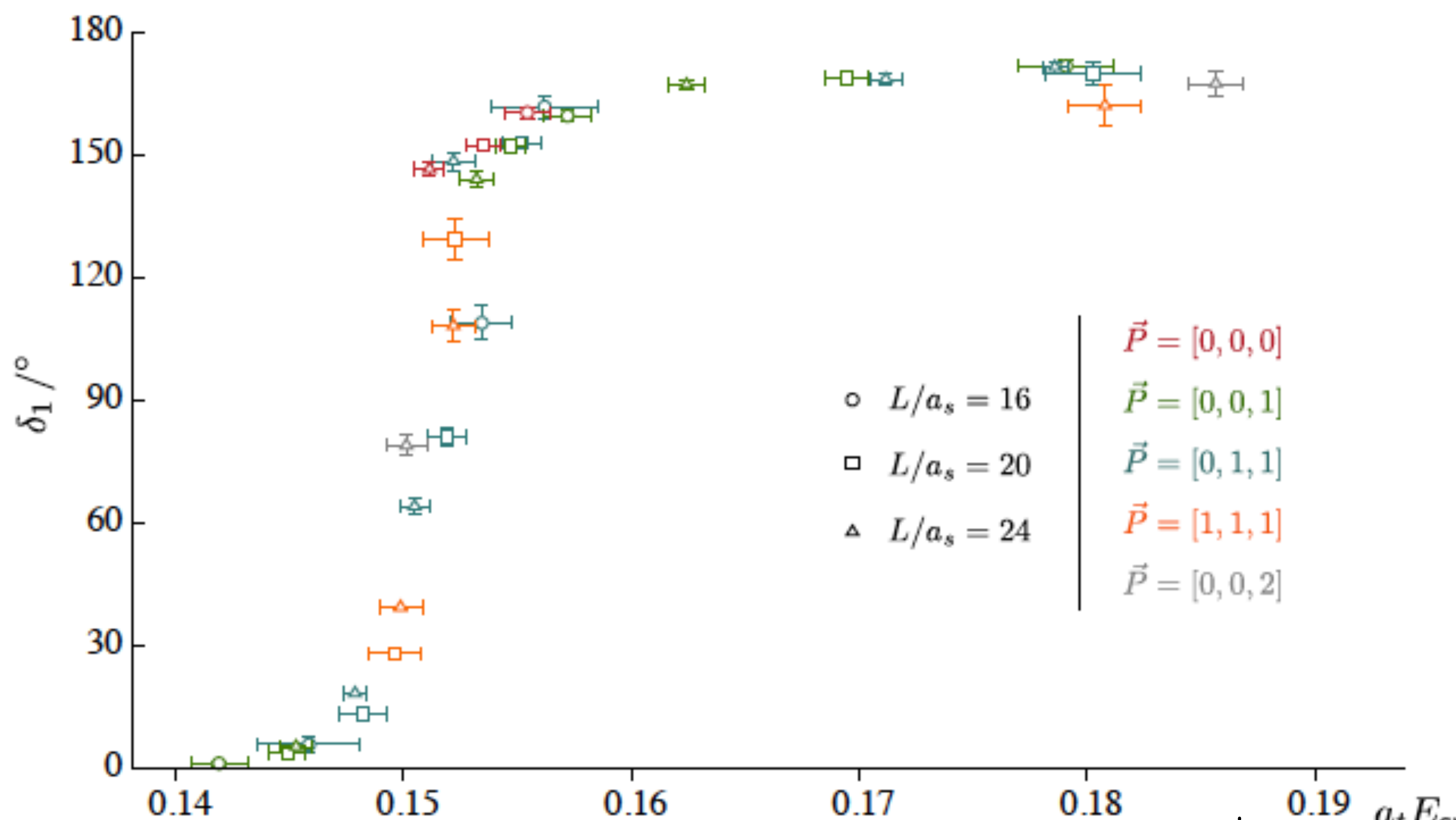
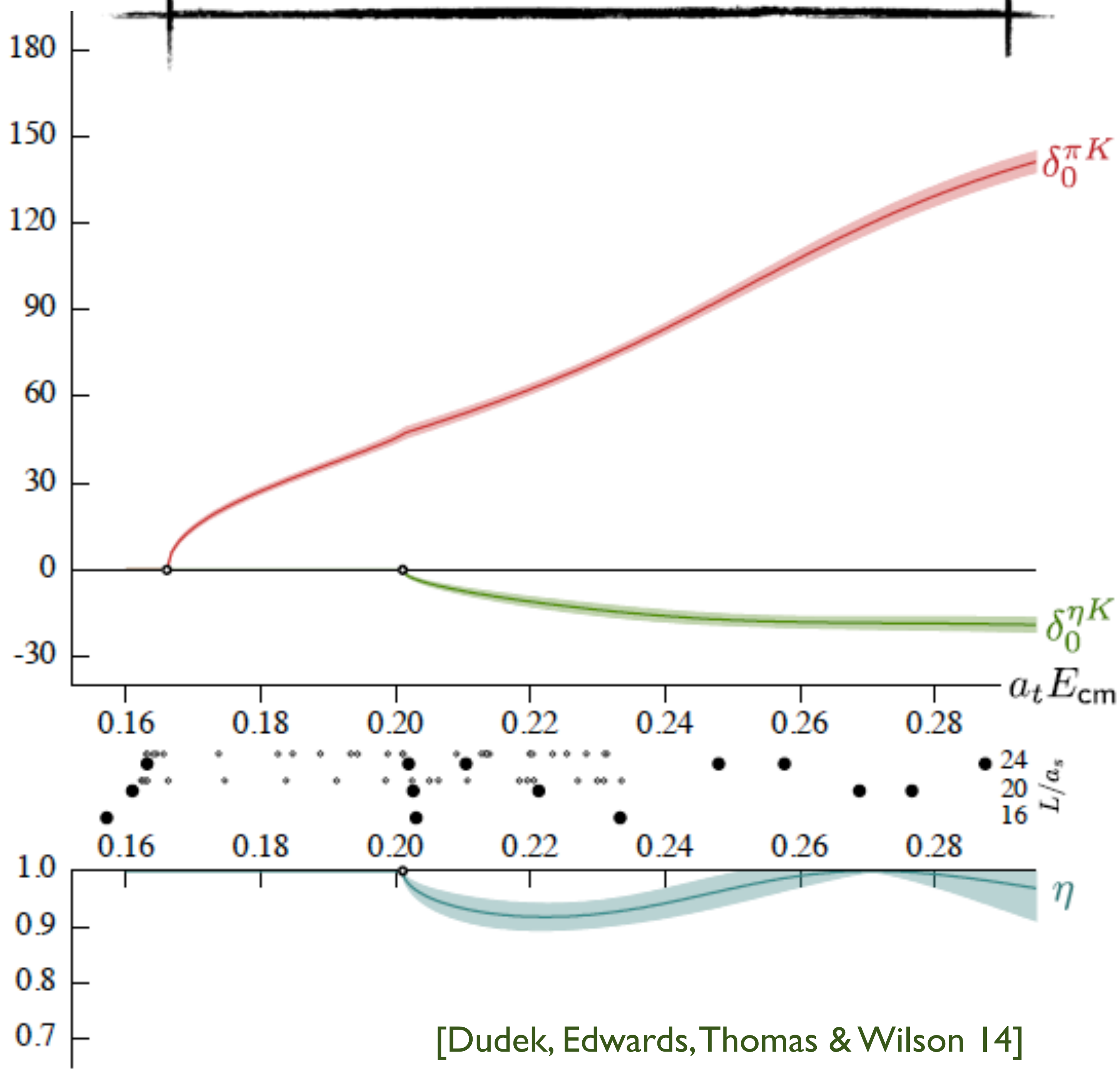


a E^*

State of the art: coupled 2-body channels

$$\det\left[(F_{\rm PV})^{-1} + \mathcal{K}_2\right] = 0$$



[Dudek, Edwards, Thomas & Wilson 14]

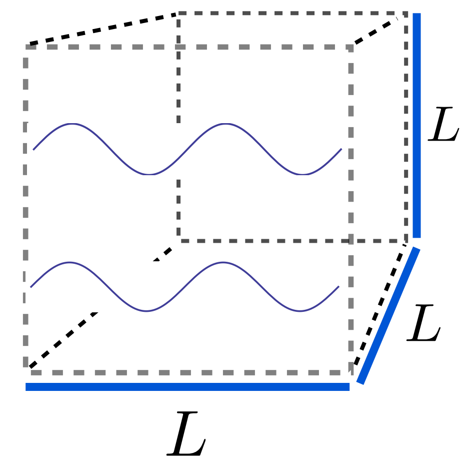
Same form of quantization condition holds, but matrices include extra channel index

[Meißner et al., Briceño & Davoudi, Hansen & SRS]

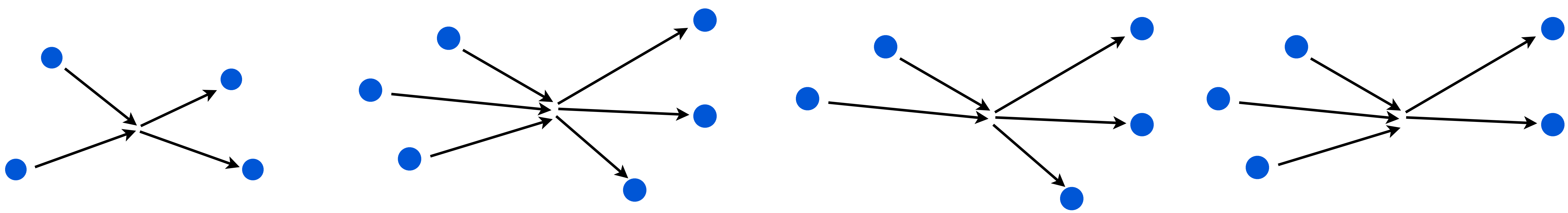
Three-particle results using relativistic formalism

3-particle analysis [Hansen & SRS, Briceño, Hansen & SRS]

- Work in continuum (assume that LQCD can control discretization errors)



- Cubic box of size L with periodic BC, and infinite (Minkowski) time
  - Spatial loops are sums: $\frac{1}{L^3}\sum_{\vec{k}}$ $\qquad \vec{k} = \frac{2\pi}{L}\vec{n}$
- Consider identical scalar particles with physical mass m, interacting *arbitrarily* in a general relativistic effective field theory



3-particle analysis [Hansen & SRS, Briceño, Hansen & SRS]

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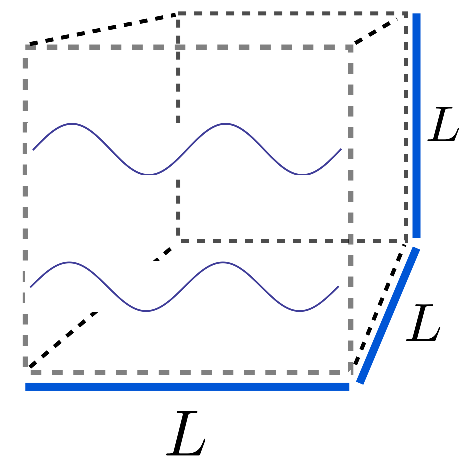
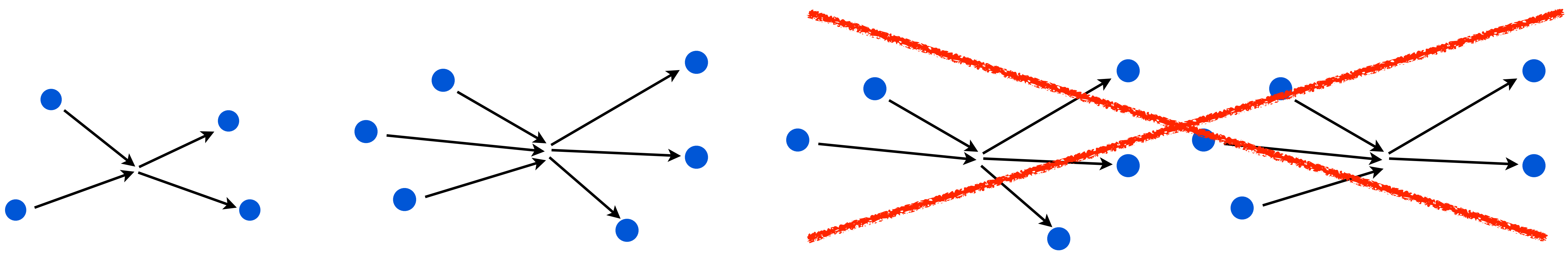


- Cubic box of size L with periodic BC, and infinite (Minkowski) time
  - Spatial loops are sums: $\frac{1}{L^3}\sum_{\vec{k}}$ $\quad \vec{k} = \frac{2\pi}{L}\vec{n}$
- Consider identical scalar particles with physical mass m, interacting *arbitrarily* in a general relativistic effective field theory



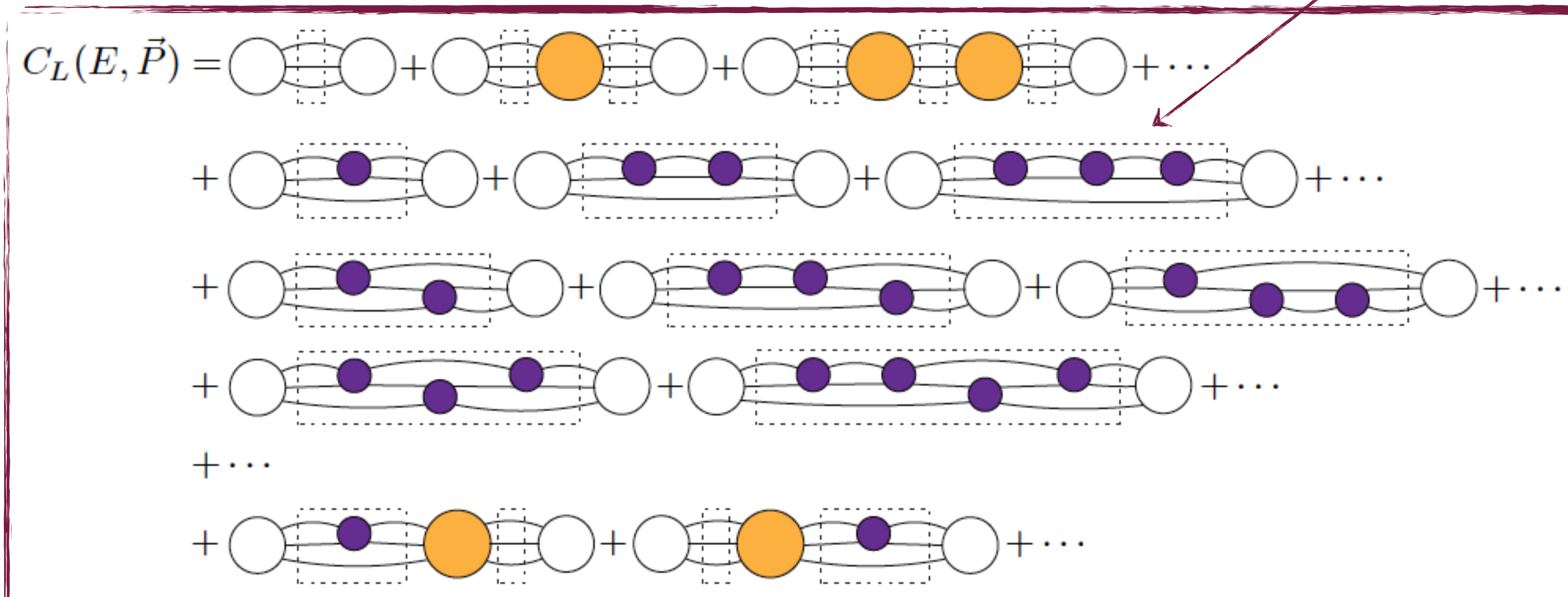
For simplicity, first show the result with Z_2 symmetric theory with even-legged vertices

Methodology

Momentum sums rather than integrals

- Obtain spectrum from poles in 3→3 correlation

  ⇒ Skeleton expansion in terms of Bethe-Salpeter kernels



- On-shell cuts or cusps imply sum-integral differences have $1/L^n$ difference

  ⇒ Keep track of cuts to all orders, and remove cusps with PV pole prescription

  ⇒ Subtract above-threshold divergences of 3-particle scattering amplitude

3-particle quantization condition with Z_2 symmetry

[Hansen & SRS, arXiv:1408.5933]

3-particle quantization condition with Z_2 symmetry

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$$\boxed{\det\left[F_3^{-1} + \mathcal{K}_{3,\mathrm{df}}\right] = 0}$$

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Matrices in the space describing 3-particle on-shell kinematics

Known kinematical quantity: essentially the same as F_{PV} in 2-particle analysis

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G is known kinematical quantity containing cut-off function H

- Superficially similar to 2-particle form …

$$\det\left[F_{\mathrm{PV}} + \mathcal{K}_2^{-1}\right] = 0$$

- … but F_3 contains both kinematical, finite-volume quantities (F_{PV} & G) and the dynamical, infinite-volume quantity $\mathcal{K}_2$

3-particle quantization condition with Z_2 symmetry

$$\det\left[F_3^{-1} + \mathcal{K}_{3,\mathrm{df}}\right] = 0$$

- All quantities are (infinite-dimensional) matrices, e.g. $(F_3)_{\mathbf{k}\,lm;\,\mathbf{p}\,l'm'}$, with indices

[finite volume "spectator" momentum: **k**=2π**n**/L] x [2-particle CM angular momentum: *l*,*m*]

Three on-shell particles with total energy-momentum (E, **P**)

- For large **k** other two particles are below threshold; must include such configurations by analytic continuation up to a cut-off at k~m [provided by H(**k**)]

3-particle quantization condition with Z_2 symmetry

$$\boxed{\det\left[F_3^{-1} + \mathcal{K}_{3,\mathrm{df}}\right] = 0}$$

- Important limitation: our present derivation requires that $\mathcal{K}_2$ in all two-particle channels has no poles (above or below threshold)
  - Why? Such poles lead to additional finite-volume dependence not accounted for in the derivation
  - Implies that two-particle bound states or resonances are not allowed
  - We are working on eliminating this limitation

Truncation in 3 particle case

$$\boxed{\det\left[F_3^{-1} + \mathcal{K}_{3,\mathrm{df}}\right] = 0}$$

$$F_3 = \frac{F_{\widetilde{\mathrm{PV}}}}{2\omega L^3}\left[-\frac{2}{3} + \frac{1}{1 + (1+\mathcal{K}_2 G)^{-1}\mathcal{K}_2 F_{\widetilde{\mathrm{PV}}}}\right]$$

- For fixed E & **P**, as spectator momentum |**k**| increases, remaining two-particle system drops below threshold
  - F_{PV} smoothly interpolates to 0 due to H factors; same holds for G
- Thus **k** sum is naturally truncated (with, say, N terms required)
  - e.g. if E=4m, **P**=0, mL=5 then N=19 (with [0,0,0], [0,0,1] & [0,1,1] **k** shells)
- l is truncated if both $\mathcal{K}_2$ and $\mathcal{K}_{\mathrm{df},3}$ vanish for $l > l_{max}$
- Yields determinant condition truncated to $[N(2l_{max}+1)]^2$ block

Truncation in 3 particle case

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- Given prior knowledge of $\mathcal{K}_2$ (e.g. from 2-particle quantization condition) each energy level E_i of the 3 particle system gives information on $\mathcal{K}_{\mathrm{df},3}$ at the corresponding 3-particle CM energy E_i^*
- Probably need to proceed by parametrizing $\mathcal{K}_{\mathrm{df},3}$, in which case one would need at least as many levels as parameters at given energy
- Given $\mathcal{K}_2$ and $\mathcal{K}_{\mathrm{df},3}$ one can reconstruct $\mathcal{M}_3$
- The locality of $\mathcal{K}_{\mathrm{df},3}$ is crucial for this program
- Clearly very challenging in practice, but there is an existence proof....

Isotropic approximation

- Assume $\mathcal{K}_{df,3}$ is pure s-wave and depends only on E*
- Also assume $\mathcal{K}_2$ only non-zero for s-wave ($\Rightarrow l_{max}=0$) and known
- Truncated [N x N] problem simplifies: $\mathcal{K}_{df,3}$ has only 1 non-zero eigenvalue, and problem collapses to a single equation:

$$\boxed{1 + F_3^{\rm iso} \mathcal{K}_{\rm df,3}^{\rm iso}(E^*) = 0}$$

Known in terms of
two particle scattering amplitude

$$F_3^{\rm iso} \equiv \sum_{\vec k, \vec p} \frac{1}{2\omega_k L^3} \left[F^s_{\widetilde{\rm PV}} \left(-\frac{2}{3} + \frac{1}{1 + [1 + \mathcal{K}_2^s G^s]^{-1} \mathcal{K}_2^s F^s_{\widetilde{\rm PV}}} \right) \right]_{k,p}$$

- Numerical exploration underway [5 slide talk]

Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

- Three-particle quantization condition depends on $\mathcal{K}_{\mathrm{df},3}$ rather than the three-particle scattering amplitude $\mathcal{M}_3$
- $\mathcal{K}_{\mathrm{df},3}$ is an infinite-volume quantity (loops involve integrals) but is not physical
  - Depends on the cut-off function H
  - It was forced on us by the analysis, and is a local vertex
- To complete the quantization condition we must relate $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

Involve only $\mathcal{M}_2$ and G so "known"

[Hansen & SRS, arXiv:1504.04248]

$$i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \; i\mathcal{K}_{\mathrm{df},3\to3} \frac{1}{1 - iF_3 \; i\mathcal{K}_{\mathrm{df},3\to3}} \mathcal{R}_L\right]$$

$$i\mathcal{M}_{3\to3} = \lim_{L\to\infty}\Big|_{i\epsilon} i\mathcal{M}_{L,3\to3}$$

Sums over k go over to integrals with iε pole prescription

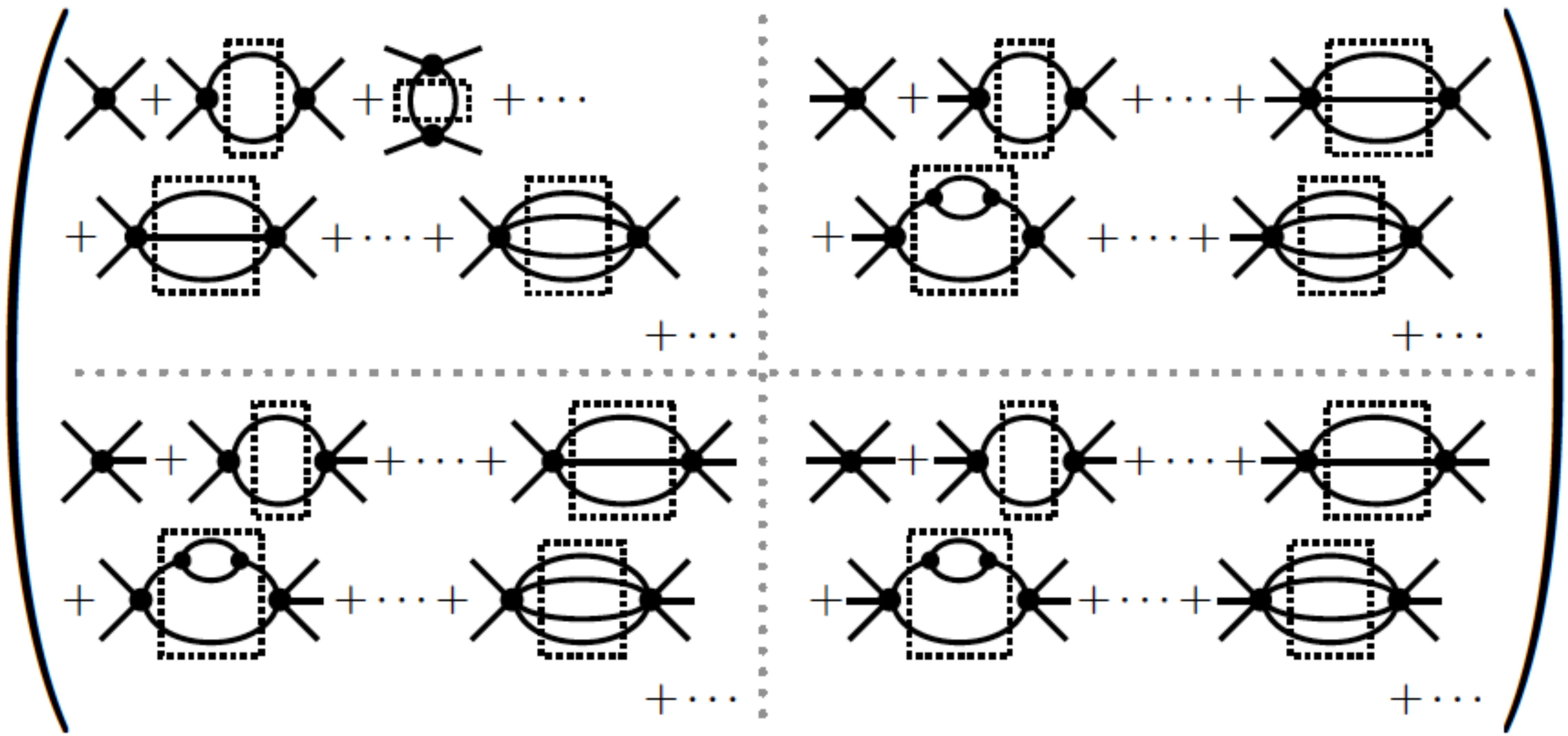
- Result is an integral equation giving $\mathcal{M}_3$ in terms of $\mathcal{K}_{\mathrm{df},3}$
- Requires knowing $\mathcal{M}_2$ (including continued below threshold)
- Completes formalism—shows that finite-volume spectrum is given by infinite-volume scattering amplitudes

Tests of formalism

- Reproduces threshold expansion [Hansen & SRS, 16]
  - Energy of state nearest threshold is given by a power series in 1/L, which can be obtained using NRQM [Beane, Detmold & Savage, 07; Tan 08] or perturbation theory [Hansen & SRS, 16; SRS 17]
- Reproduces volume dependence of Efimov-like three-particle bound state [Hansen & SRS, 16]
  - Dependence on L can be predicted by NRQM [Meißner, Rios & Rusetsky, 14]

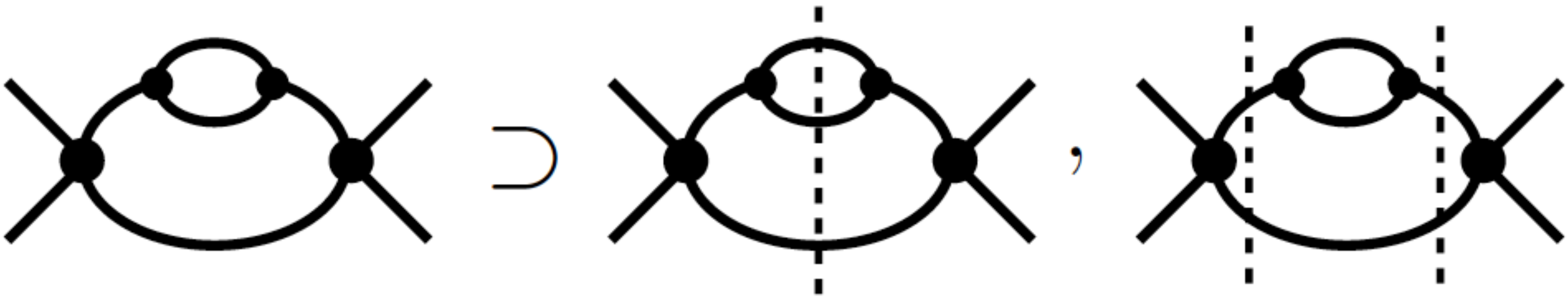
Removing the Z_2 constraint [Briceño, Hansen & SRS]

- Generalization is straightforward in principle, but keeping track of all cuts is more challenging, so we developed a somewhat different approach, based more extensively on time-ordered PT
  - Consider $3m < E^* < 4m$ where both 2- and 3-particle cuts are present
  - Work directly with finite-volume scattering amplitude

$\mathcal{M}_L =$ 

Removing the Z_2 constraint

- One of new challenges is dealing with cuts of self-energy diagrams



- Cannot use fully-dressed propagators, requiring some gymnastics to make sure cuts occur at positions of renormalized masses
- Since we continue below three-particle threshold, work is needed to avoid simultaneous two-and three-particle cuts in such diagrams

Removing the Z_2 constraint

[Briceño, Hansen & SRS, arXiv:1701.07465]

- Final result can be brought into a familiar form, with an additional channel index

$$\det\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \underbrace{\begin{pmatrix}F_2 & 0\\ 0 & F_3\end{pmatrix}}_{\text{Same old FV zeta-functions}} \underbrace{\begin{pmatrix}\mathcal{K}_{22} & \mathcal{K}_{23}\\ \mathcal{K}_{32} & \mathcal{K}_{\mathrm{df},33}\end{pmatrix}}_{\text{Infinite-volume, unphysical K-matrices}}\right] = 0$$

Same old FV zeta-functions

Infinite-volume, unphysical K-matrices Related to $\mathcal{M}_{22}$, $\mathcal{M}_{23}$, $\mathcal{M}_{32}$ & $\mathcal{M}_{33}$ by known integral equations

- Shortcoming that $\mathcal{K}_2$ cannot have poles remains

Comparison with NREFT approach

[Hammer, Park & Rusetsky, arXiv:1706.07700, 1707.02176]

NREFT approach

- Expand two and three-particle interactions in powers of p/Λ

e.g. $\mathcal{L}_3^{LO} = -\frac{D_0}{6}\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi$ $\mathcal{L}_3^{NLO} = -\frac{D_2}{12}(\psi^\dagger\psi^\dagger\nabla^2\psi^\dagger\,\psi\psi\psi + \text{h.c.})$

- Treat system as particle + dimer (technical trick from [Bedaque, Hammer & van Kolck, 1998])
- Assume Z_2 symmetry
- Spectrum given by poles in finite-volume particle-dimer scattering amplitude, resulting in

Infinite-volume quantity

$$\det\left(\delta_{ll'}\delta_{mm'}\delta_{pq} - Z_l(p,q;E)R_{lm,l'm'}(q;E)\right) = 0\,.$$

Finite-volume quantity

- Determine D_0, D_2, etc. needed to reproduce measured spectrum
- Solve infinite-volume integral equation to obtain scattering amplitudes in terms of determined D_0, D_2, …

Similarities

- Both approaches need to parametrize interactions (D_n vs $\mathcal{K}_{3,df}$), and these intermediate quantities are cutoff dependent

- Dimer field sums two-particle bubbles in finite volume in exact correspondence to what we do

- Both approaches need to solve integral equation(s) to relate intermediate quantities to scattering amplitudes

Overall, both approaches very similar—indeed, HPR argue that they can be related algebraically

Differences

- NREFT vs. relativistic EFT—mainly/totally a matter of kinematics?
- NREFT approach imposes Z_2 symmetry, so far
- HPR sum over relative momentum of particle and dimer, while we replace sum with ``sum-minus-integral + integral"
  - Advantage of HPR: do not have to worry about K-matrix poles or cusps, so derivation is simpler
  - Disadvantage of HPR: need to use a much larger cutoff on momentum sums, and test cutoff independence of final physical quantities
  - Possible advantage of HPR: integral equations in infinite-volume are simpler

Differences are mainly issues of practical implementation; need numerical tests to see which approach is better

Summary

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- Enormous progress in the two-particle sector
- Substantial progress in the three-particle sector where a major issue is how to turn the formalism into something practical
  - Extensions to higher spins, nonidentical particles and Lellouch-Lüscher factors will likely be straightforward
  - We (BHS) need to incorporate K-matrix poles in our approach and do a detailed comparison to NREFT
- Moving to 4+ particles in this fashion looks challenging but does not obviously introduce new theoretical issues
- Several interesting ideas for addressing inclusive processes

Upcoming workshops

"Multi-Hadron Systems from Lattice QCD" @ INT (Seattle)

Organizers: Raúl Briceño, Max Hansen, SRS, David Wilson

February 5—9, 2018

"Scattering Amplitudes and Resonance Properties from Lattice QCD" @ MITP (Mainz)

Organizers: Max Hansen, Sasa Prelovsek, SRS, Hartmut Wittig, Georg von Hippel

August, 27—31 2018