Three-particle interactions from the lattice: a progress report

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Outline

• Motivations for studying 3 (or more) particles
• Status of formalism for (2 & 3) particles
  • Examples of implementations
• Alternative derivation & new form of three-particle quantization condition (QC3)
• Equivalence of different QC3s
• Conclusions & outlook
3-particle papers

Max Hansen & SRS:
“Relativistic, model-independent, three-particle quantization condition,”
arXiv:1408.5933 (PRD) [HS14]

“Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,”
arXiv:1504.04028 (PRD) [HS15]

“Perturbative results for 2- & 3-particle threshold energies in finite volume,”
arXiv:1509.07929 (PRD) [HSPT15]

“Threshold expansion of the 3-particle quantization condition,”
arXiv:1602.00324 (PRD) [HSTH15]

“Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,”
arXiv: 1609.04317 (PRD) [HSBS16]

“Lattice QCD and three-particle decays of Resonances,”
Raúl Briceño, Max Hansen & SRS:
“Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles,”
arXiv:1701.07465 (PRD) [BHS17]
“Numerical study of the relativistic three-body quantization condition in the isotropic approximation,”
arXiv:1803.04169 (PRD) [BHS18]

SRS
“Testing the threshold expansion for three-particle energies at fourth order in \( \phi^4 \) theory,”
arXiv:1707.04279 (PRD) [SPT17]

Tyler Blanton, Fernando Romero-López & SRS:
“Implementing the three-particle quantization condition including higher partial waves,” arXiv:1901.07095 (JHEP) [BRS19]

“I=3 three-pion scattering amplitude from lattice QCD,”
arXiv:1909.02973 (PRL) [BRS-PRL19]
Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:


Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

“Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism,”
arXiv:1905.11188 (PRD)

Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

“On the Equivalence of Three-Particle Scattering Formalisms,”
arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

“Generalizing the relativistic quantization condition to include all three-pion isospin channels”, arXiv:2003.10974 (JHEP) [HRS20]
Focus for today

Tyler Blanton & SRS:

“Alternative derivation of the relativistic three-particle quantization condition,”


“Equivalence of relativistic three-particle quantization conditions,”

arXiv:2007.16190 (PRD under review) [BS20b]

Tyler Blanton, Drew Hanlon, Ben Hörz, Fernando Romero-López & SRS

“$3\pi^+ \& 3K^+$ interactions beyond leading order from lattice QCD,”

Work in progress
Motivations for studying three (or more) particles using LQCD
Determining resonance properties

- Most resonances have 3 (or more) particle decay channels
  - $\omega(782, I^G J^{PC} = 0^{-1-}) \to 3\pi$ (no subchannel resonances)
  - $a_2(1320, I^G J^{PC} = 1^{-2+}) \to \rho\pi \to 3\pi$
  - Roper: $N(1440) \to \Delta\pi \to N\pi\pi$ (branching ratio 25-50%)
  - $X(3872) \to J/\Psi\pi\pi$
  - $Z_c(3900) \to \pi J/\psi, \pi\pi\eta_c, \bar{D}D^*$ (studied by HALQCD)

- N.B. If a resonance has both 2- and 3-particle strong decays, then 2-particle methods fail—channels cannot be separated as they can in experiment
Predicting weak decay amplitudes

- Calculating weak decay amplitudes/form factors involving 3 particles, e.g. $K \rightarrow \pi\pi\pi$

- N.B. Can study weak $K \rightarrow \pi\pi$ decays independently of $K \rightarrow \pi\pi\pi$, since strong interactions do not mix these final states (in isospin-symmetric limit)

- Long-term goal is to develop methods to predict CP violation in $D \rightarrow \pi\pi, K\bar{K}, (\pi\pi\pi\pi), \ldots$ (as measured by LHCb in 2019)
Determining 3-body interactions

- Determining NN & NNN interactions
  - Input for effective field theory treatments of larger nuclei & nuclear matter
  - NNN interaction important for determining properties of neutron stars
- Similarly, $\pi\pi\pi$, $\pi K\bar{K}$, … interactions needed for study of pion/kaon condensation
Two- and three-pion finite-volume spectra at maximal isospin from lattice QCD

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(Dated: May 13, 2019)

We present the three-pion spectrum with maximum isospin in a finite volume determined from lattice QCD, including, for the first time, excited states across various irreducible representations at zero and nonzero total momentum, in addition to the ground states in these channels. The required correlation functions, from which the spectrum is extracted, are computed using a newly implemented algorithm which reduces the number of operations, and hence speeds up the computation by more than an order of magnitude. The results for the $I = 3$ three-pion and the $I = 2$ two-pion spectrum are publicly available, including all correlations, and can be used to test the available three-particle finite-volume approaches to extracting three-pion interactions.
Status of formalism for (2 & 3) particles
The fundamental issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box
- How are these related to infinite-volume scattering amplitudes (which determine resonance properties)?

\[ i \mathcal{M}_{n \rightarrow m} \]

Discrete energy spectrum

\[ E_0(L) \]
\[ E_1(L) \]
\[ E_2(L) \]

Scattering amplitudes
The fundamental issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box.
- How are these related to infinite-volume scattering amplitudes (which determine resonance properties)?

N.B. This is a finite volume QFT problem.
The fundamental issue

- Lattice simulations are done in finite volumes; experiments are not

How do we connect these?
Two particles in cubic box of size $L$ with PBC and total momentum $P$

Below inelastic threshold (4 pions if have $Z_2$ symmetry), the finite-volume spectrum $E_1, E_2, ...$ is given by solutions to an equation in partial-wave $(l,m)$ space

$$\det \left[ F_{PV}(E, P, L)^{-1} + \mathcal{K}_2(E^*) \right] = 0$$

$Lüscher 86 & 91; Rummukainen & Gottlieb 85; Kim, Sachrajda & SRS 05; ...$

$\mathcal{K}_2 \sim \tan \delta/q$ is the two-particle K-matrix, which is diagonal in $l,m$

$F_{PV}$ is a known kinematical “zeta-function”, depending on the box shape & $E$; It is off-diagonal in $l,m$, since the box violates rotation symmetry

Valid up to corrections $\sim e^{-ML}$

Generalized to arbitrary masses, spins and multiple channels
State of the art: coupled 2-body channels

\[ \text{det} \left[ (F_{PV})^{-1} + K_2 \right] = 0 \]

Same form of quantization condition holds, but matrices include extra channel index

[He, Feng, Liu 05; Meißner et al. 09-11; Briceño & Davoudi 12; Hansen & SRS 12]

Practical implementation requires truncation of \( \ell, m \) indices

[Dudek, Edwards, Thomas & Wilson 14]
Status for three particles

- Applied so far only to systems of three spin-0 particles

- Three approaches
  
  - All orders diagrammatic derivation in generic relativistic EFT (RFT)  
    [Hansen & SRS 14; Briceño, Hansen & SRS 17; …]  
    - Control all sources of $1/L^n$ volume dependence, while neglecting terms $\propto e^{-ML}$  
    - Complicated derivation, but general result: holds for all 2-particle (“dimer”) partial waves  
    - Originally derived for 3 identical particles with $\mathbb{Z}_2$ symmetry; generalized to allow $2 \leftrightarrow 3$ transitions, and nonidentical but degenerate particles (e.g. 3 pions with any allowed isospin)

  - Nonrelativistic EFT [Hammer & Rusetsky 17; …]  
    - Greatly simplified derivation; applied so far only for s-wave dimers; nonrelativistic kinematics

  - ``Finite-volume unitarity” (FVU) [Mai & Döring 17; …]  
    - Based on infinite-volume unitary representation of three-particle amplitude $\mathcal{M}_3$ in terms of R matrix (generalization of $\mathcal{K}_2$)
    - Relativistic, but obtained so far only for s-wave dimers
Refs for alternate approaches

★ NREFT approach

- M. Döring et al., 1802.03362, PRD [Numerical implementation]
- J.-Y. Pang et al., 1902.01111, PRD [large volume expansion for excited levels]

★ Finite-volume unitarity (FVU) approach

- M. Mai & M. Döring, 1709.08222, EPJA [formalism]
- M. Mai et al., 1706.06118, EPJA [unitary parametrization of M₃ involving R matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of R matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]
- M. Mai et al., 1909.05749, PRD [applying FVU approach to 3π⁺ spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating 3π⁺ spectrum and comparing with FVU predictions]

★ HALQCD approach

- T. Doi et al. (HALQCD collab.), 1106.2376, Prog.Theor.Phys. [3 nucleon potentials in NR regime]
Two-step method

2 & 3 particle Spectra from LQCD

Quantization conditions

QC2: \( \det \left[ F_2^{-1} + \mathcal{H}_2 \right] = 0 \)
QC3: \( \det \left[ F_3^{-1} + \mathcal{H}_{df,3} \right] = 0 \)

Intermediate, unphysical scattering quantity

\( \mathcal{H}_{df,3} \)

Scattering amplitude

\( \mathcal{M}_3 \)

Integral equations in infinite volume

S. Sharpe, "Three particle interactions from the lattice...," seminar at U. Maryland, 9/11/2020
\[ \det \left[ F_{PV}(E, \vec{P}, L)^{-1} + \mathcal{K}_2(E^*) \right] = 0 \]

\[ \det \left[ F_3(E, \vec{P}, L)^{-1} + \mathcal{K}_{df,3}(E^*) \right] = 0 \]

- Total momentum \((E, \vec{P})\)
- Matrix indices are \(l, m\)
- \(F_{PV}\) is a finite-volume geometric function
- \(\mathcal{K}_2\) is an infinite-volume amplitude, which is real and smooth (no threshold cusps)
- It is related algebraically to \(\mathcal{M}_2\):
  \[
  \frac{1}{\mathcal{M}_2^{(\ell)}} \equiv \frac{1}{\mathcal{K}_2^{(\ell)}} - i\rho
  \]

- Total momentum \((E, \vec{P})\)
- Matrix indices are \(k, l, m\)
- \(F_3\) depends on geometric functions \(F_{PV}\) and \(G\) and also on \(K_2\)
  - \(F_3\) is known if first solve QC2
- \(\mathcal{K}_{df,3}\) is an infinite-volume 3-particle amplitude, which is real and smooth
- It is cutoff dependent and thus unphysical
- It is related to \(\mathcal{M}_3\) via integral equations [HS15]

S. Sharpe, "Three particle interactions from the lattice…," seminar at U. Maryland, 9/11/2020
Further details of QC$_3$

- All quantities are infinite-dimensional matrices with indices $k\ell m$ describing 3 on-shell particles.

\[ F_3 = \frac{1}{2\omega L^3} \left[ \frac{F}{3} - F \frac{1}{\mathcal{K}_2^{-1} + F + G} F \right] \]

- $F_3$ contains two-particle interactions ($\mathcal{K}_2$) and kinematic functions (F & G).
Status of RFT formalism

- Original work applied to scalars with $\mathbb{Z}_2$ symmetry & no subchannel resonances or 2-particle bound states (e.g. $3\pi^+$) [HS14, HS15]

$$\det \left[ F_3^{-1} + K_{df,3} \right] = 0$$

- Generalized PV prescription allows subchannel resonances & 2-particle bound states [BBHRS19]

- Alternative more cumbersome approach given in [BHS19]
Status of RFT formalism

- [BHS17] removed G-parity constraint, allowing $2 \leftrightarrow 3$ processes (step towards $N\pi \leftrightarrow N\pi\pi$)

\[
\det \left[ \begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix}^{-1} + \begin{pmatrix} \mathcal{K}_{22} & \mathcal{K}_{23} \\ \mathcal{K}_{32} & \mathcal{K}_{df,33} \end{pmatrix} \right] = 0
\]
Status of RFT formalism

- [HRS20] generalized to distinguishable but degenerate particles (e.g. $3\pi$ with $I = 0,1,2$ in isosymmetric QCD)

$$\det[1 - K_{df,3}^{[I]}(E^*) F_3^{[I]}(E,P,L)] = 0$$

$$F_3^{[I]} = \frac{F^{[I]}}{3} + \frac{1}{1 - M_{2,L}^{[I]} G^{[I]}} M_{2,L}^{[I]} F^{[I]}$$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$F^{[I]}$</th>
<th>$K_2^{[I]}$</th>
<th>$G^{[I]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$\frac{IF}{2\omega L^3}$</td>
<td>$i[2\omega L^3]K_{(\pi\pi)_2}$</td>
<td>$i\frac{1}{2\omega L^3} G$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{IF}{2\omega L^3} \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
<td>$i[2\omega L^3] \begin{pmatrix} K_{(\pi\pi)<em>2} &amp; 0 \ 0 &amp; K</em>{\rho} \end{pmatrix}$</td>
<td>$i\frac{1}{2\omega L^3} G \begin{pmatrix} -\frac{1}{2} &amp; -\frac{\sqrt{3}}{2} \ -\frac{\sqrt{3}}{2} &amp; \frac{1}{2} \end{pmatrix}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{IF}{2\omega L^3} \begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$i[2\omega L^3] \begin{pmatrix} K_{(\pi\pi)<em>2} &amp; 0 &amp; 0 \ 0 &amp; K</em>{\rho} &amp; 0 \ 0 &amp; 0 &amp; K_{\sigma} \end{pmatrix}$</td>
<td>$i\frac{1}{2\omega L^3} G \begin{pmatrix} \frac{\sqrt{15}}{6} &amp; \frac{1}{6} &amp; \frac{\sqrt{5}}{3} \ \frac{\sqrt{5}}{3} &amp; \frac{1}{2} &amp; -\frac{1}{3} \ \frac{\sqrt{3}}{3} &amp; -\frac{1}{3} &amp; \frac{1}{3} \end{pmatrix}$</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{IF}{2\omega L^3}$</td>
<td>$i[2\omega L^3] K_{\rho}$</td>
<td>$-i\frac{1}{2\omega L^3} G$</td>
</tr>
</tbody>
</table>

- e.g. $3\pi^+$
- e.g. $a_1$
- e.g. $\omega, h_1$
Examples of implementation of (RFT) QC3
Overview

\[ \det \left( F_3^{-1} + \mathcal{K}_{df,3} \right) = 0 \]

**Integral equations**

**TOY MODELS:**

- \( E_0(L) \)
- \( E_1(L) \)
- \( E_2(L) \)

**DREAM:**

**LQCD**

- \( E_0(L) \)
- \( E_1(L) \)
- \( E_2(L) \)

- \( \mathcal{K}_{df,3} \)
- \( M_3 \)

S. Sharpe, "Three particle interactions from the lattice…," seminar at U. Maryland, 9/11/2020
3-particle bound state from d-wave attraction

$E_n^A = \frac{E_n^{A+}}{m}$

Trimer appears to remain bound as $L \to \infty$

$mL = 8.1, ma_0 = -0.1, r_0 = P_0 = \mathcal{H}_{df,3} = 0$

$a_0, a_2 < 0 \Rightarrow$ no dimers
3-particle bound state from d-wave attraction

Quantization condition is useful as tool for studying infinite-volume!

Trimer appears to remain bound as $L \to \infty$

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S. Sharpe, “Three particle interactions from the lattice…,” seminar at U. Maryland, 9/11/2020
S-wave dimer properties vs $a_0$

- Study 3 particles with only s-wave scattering length $a_0$ nonzero
  - Choose $ma_0 > 1$ so that there is a dimer (``deuteron'')
  - Look at states with $E < 3m$: dimer+particle and, possibly, trimer (``triton'')
  - Use QC2 applied to dimer+particle states to determine scattering amplitude (and, in particular, scattering length $b_0$)
Beginnings of Efimov series!

NREFT fails

\[ \frac{mb_0}{ma_0} \]

Relativistic QC3

\[ \text{NREFT } \Lambda = 0.75m \]
S-wave dimer properties vs $a_0$

Beginnings of Efimov series!

No trimer | One trimer | Two trimers

S. Sharpe, “Three particle interactions from the lattice…,” seminar at U. Maryland, 9/11/2020
**S-wave dimer properties vs $a_0$**

Beginnings of Efimov series!

[BBHRS19]

QC3 allows study of relativistic bound states

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Particle-dimer scattering length

\[
\frac{mb_0}{ma_0}
\]

- No trimer
- One trimer
- Two trimers

Relativistic QC3

NREFT $\Lambda = 0.75m$
Phillips curve in toy N+D / Tritium system

Choose $a_0$ so that $m_{\text{dimer}} : m = M_D : M$ and vary $\mathcal{K}^\text{iso}_{\text{df},3}$

Similar to curve found in potential models and chiral EFT

S. Sharpe, "Three particle interactions from the lattice…," seminar at U. Maryland, 9/11/2020
First application to LQCD

Two- and three-pion finite-volume spectra at maximal isospin from lattice QCD

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$2\pi^+ (16$ levels$)$ and $3\pi^+ (11$ levels$)$ spectra for $M_\pi = 200$ MeV
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$2\pi^+ (16 \text{ levels})$ and $3\pi^+ (11 \text{ levels})$ spectra for $M_\pi = 200 \text{ MeV}$

- [BRS-PRL19] Simultaneous fit with QC2 and QC3
- s-wave $\pi^+\pi^+$ interaction
- Isotropic (momentum-independent) $\mathcal{H}_{df,3}$ with linear dependence on $E_{\text{CM}}^2$
- Predict $\mathcal{H}_{df,3}$ using leading order chiral perturbation theory

S. Sharpe, “Three particle interactions from the lattice…,” seminar at U. Maryland, 9/11/2020
Global fit to $2\pi^+$ and $3\pi^+$ spectra

**Figure S3:** Two- and three-pion spectra from Ref. [1] (blue) compared to the predictions from the global fit 5 (orange). Hollow orange points above the inelastic thresholds have not been included in the fit, but are shown for comparison. Dashed lines show the non-interacting energy levels.

Inelastic threshold

Global fit

$$\chi^2 = 26.04$$

d.o.f. = (22 - 4)

Similar fit obtained using FVU QC3 [Mai et al, 19]

3-particle spectrum primarily determined by 2-particle interactions

S. Sharpe, "Three particle interactions from the lattice...," seminar at U. Maryland, 9/11/2020
Evidence for nonzero $\mathcal{K}_{df,3}$

\begin{equation}
\mathcal{K}_{df,3} = \mathcal{K}_{iso,0}^{df,3} + \frac{E_{CM}^2 - 9M^2}{9M^2} \mathcal{K}_{iso,1}^{df,3}
\end{equation}

Global fit

\[ \frac{\chi^2}{d.o.f.} = \frac{26.04}{(22-4)} \]
Alternative derivation and new form of QC3


See also talk by Tyler Blanton at APLAT 2020:
https://conference-indico.kek.jp/event/113/contributions/2070/
Why a new RFT derivation?

- To simplify generalization of QC3 to nondegenerate particles with spin (e.g., \( N\pi\pi \)), and to QC4, ...
- Original RFT derivation is long, complicated and does not give explicit results for all quantities (e.g., \( H_{df,3} \))

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The second feature is that the three-particle scattering quantity is nonunitary—it is not simply related to the divergence-free part of the physical scattering amplitude. This is because it is defined using the PV pole prescription, and also because of the disentanglements explained in Sec. IV E. We strongly suspect, however, that a relation to the physical amplitude exists. In particular, we know from Ref. [13] that the finite-volume spectrum in a nonrelativistic theory can be determined only in terms of physical amplitudes, and the same is true in the approximations adopted in Ref. [14]. We are actively investigating this issue.

The three-particle quantization condition involves a determinant over a larger space than that required for two particles. Nevertheless, as explained in Secs. III, because the three-particle quantity that remains has a sufficiently convergent partial-wave expansion, one can make a consistent truncation of the quantization condition so that it involves only a finite number of parameters. This opens the way to practical application of the formalism.

We have provided in this paper two mild consistency checks: on the formalism—that it correctly reproduces the known results if one particle is noninteracting (see Sec. IV A), and that the number of solutions to the quantization condition in the isotropic approximation is as expected (see Appendix C). We have also worked out a more detailed check by computing our result close to the three-particle threshold \( E \approx 37 \) to be some obtained using nonrelativistic quantum mechanics [27,28]. Here our one has expansions in powers of \( L^{-1} \), and we have checked that the results agree for the first four nontrivial orders. This provides, in particular, a nontrivial check of the form factor \( F \) of Eq. (19), and allows us to relate \( L \gamma_{\text{physical}} \) to physical quantities in nonrelativistic limit. We will present this analysis elsewhere [29].

In future work, we would like to understand in detail the relations of our formulation and quantization condition to those obtained in Ref. [13,14]. One of the primary reasons for this effort is to ascertain how best to use it in practice.

**ACKNOWLEDGMENTS**

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**APPENDIX A: SUM-MINUS-INTEGRAL IDENTITIES**

In the appendix we derive the sum-minus-integral identity that plays a central role in the main text. This identity is
Comparison of strategies

• Both consider generic EFT for identical particles with a $\mathbb{Z}_2$ symmetry

• Work diagrammatically to all orders in PT; no power-counting needed

• Both analyze three-particle, finite-volume correlator for $E_{\text{CM}} < 5M$

$$C_{3,L}(E, \vec{P}) = \int_L d^4x \ e^{i(Ex^0 - \vec{P} \cdot x)} \langle 0 | T\sigma(x)\sigma^\dagger(0) | 0 \rangle$$

• Both use time-ordered PT (TOPT) to argue that only 3-particle cuts lead to singularities and thus power-law volume dependence; all others can be integrated

• [HS14] use skeleton expansion in terms of Feynman diagrams

Infinite-volume Bethe-Salpeter kernels

Momentum sums rather than integrals

operator destroying 3 particles
Comparison of strategies: old

- [HS14] use skeleton expansion in terms of Feynman diagrams

\[ C_L(E, P) = \sum_{3,} \text{terms} \]

- Replace sums with integral (PV regulated) + (sum-integral) wherever possible

- Reshuffle contributions of antiparticle poles at 3-particle cuts—which are nonsingular—into infinite-volume quantities

- Expend great effort recombining terms so that final expression is written in terms of a \( \mathcal{H}_{df,3} \) that is symmetric under particle exchange

- Define \( \mathcal{H}_{df,3} \) constructively, rather than explicitly
Comparison of strategies: new

- [BS20a] use skeleton expansion in terms of TOPT diagrams, ordered by the number of "relevant cuts" (3-particle cuts), e.g.

- Loops with only irrelevant cuts can be integrated; build up TOPT kernels $\hat{A}', \bar{B}_{2,L} = 2\omega L^3 \bar{B}_2, \bar{B}_3, \hat{A}$

- The infinite-volume objects $\hat{A}', \bar{B}_2, \bar{B}_3, \hat{A}$ contain Feynman-diagram antiparticle contributions to 3-particle cuts

- $\bar{B}_{2,L}$ is intrinsically asymmetric, since it picks out a "spectator"
Comparison of strategies: new

- Two types of relevant cuts: F- and G-like

\[
[iD_F]_{ka;pr} \equiv \delta_{kp} \delta_{ar} \frac{iD_{ka}}{2!}, \quad [iD_G]_{ka;pr} \equiv \delta_{kr} \delta_{ap} iD_{kp},
\]

\[
iD_{ka} \equiv \frac{1}{2\omega_k L^3} \frac{1}{2\omega_b (E - \omega_k - \omega_a - \omega_b) 2\omega_a L^3}
\]

Kernels depend on two 3-momenta and are off shell
Comparison of strategies: new

- **Do all orders summation before dealing with momentum sums in relevant cuts**

\[
\begin{align*}
C_{3,L}^{(1)}(E, \vec{P}) &= \hat{A}'i(D_F + D_G)\hat{A} \\
C_{3,L}^{(2)}(E, \vec{P}) &= \hat{A}'i(D_F + D_G)i(\bar{B}_{2,L} + B_3)i(D_F + D_G)\hat{A} \\
C_{3,L}^{(3)}(E, \vec{P}) &= \hat{A}'i(D_F + D_G)i(\bar{B}_{2,L} + B_3)i(D_F + D_G)i(\bar{B}_{2,L} + B_3)i(D_F + D_G)\hat{A} \\
&\quad \vdots \\
C_{3,L}^{(n)}(E, \vec{P}) &= \hat{A}'i(D_F + D_G)\left[i(\bar{B}_{2,L} + B_3)i(D_F + D_G)\right]^{n-1} \hat{A}
\end{align*}
\]

\[
\Rightarrow \quad C_{3,L}(E, \vec{P}) = C_{3,\infty}^{(0)}(E, \vec{P}) + \hat{A}'i(D_F + D_G)\frac{1}{1 - i(\bar{B}_{2,L} + B_3)i(D_F + D_G)}\hat{A}
\]

- **Simple, explicit expression!**

- **However, \( \hat{A}' \), \( \bar{B}_2 \), \( \bar{B}_3 \), \( \hat{A} \) are off shell**
Comparison of strategies: new

- **Do on-shell projection of F- and G-cuts in parallel** (using methods from [HS14])

\[
[iD_F]_{ka;pr} \equiv \delta_{kp}\delta_{ar}\frac{iD_{ka}}{2!} , \quad [iD_G]_{ka;pr} \equiv \delta_{kr}\delta_{ap} iD_{kp} ,
\]

\[
iD_{ka} \equiv \frac{1}{2\omega_k L^3} \frac{1}{2\omega_b (E - \omega_k - \omega_a - \omega_b)}
\]

\[
\frac{1}{L^6} \sum_k \sum_a = \int_k \text{PV} \int_a \frac{1}{L^3} \left[ \frac{1}{L^3} \sum_a - \text{PV} \int_a \right] \sim \tilde{F}
\]

\[
\Rightarrow D_F = \tilde{I}_F + \tilde{F}
\]

Integral operator that sews kernels together

- Decompose adjacent kernels into on-shell part and residue

\[
\Rightarrow D_G = \tilde{G} + \delta \tilde{G}
\]

Integral operator that sews kernels together

Project adjacent kernels on shell:
\[
\{k, a\} \rightarrow \{k\ell m\}\]
Comparison of strategies: new

- Implement on shell projection without symmetrizing K matrices

\[
C_{3,L}(E, \tilde{P}) = C_{3,\infty}^{(0)}(E, \tilde{P}) + \tilde{A}' i(\tilde{F} + \tilde{G} + \tilde{I}_F + \delta \tilde{G}) \frac{1}{1 - i(\tilde{B}_{2,L} + B_3)i(\tilde{F} + \tilde{G} + \tilde{I}_F + \delta \tilde{G})} \tilde{A}
\]

\[
= \tilde{C}_{3,\infty}(E, \tilde{P}) + \tilde{A}'^{(u)} i(\tilde{F} + \tilde{G}) \frac{1}{1 - i \left(2\omega L^3 \mathcal{K}_2 + \mathcal{K}_{df,3}^{(u,u)}\right)i(\tilde{F} + \tilde{G})} \tilde{A}^{(u)}
\]

where

\[
i \left(2\omega L^3 \mathcal{K}_2 + \mathcal{K}_{df,3}^{(u,u)}\right) = \frac{1}{1 - i(\tilde{B}_{2,L} + B_3)i(\tilde{I}_F + \delta \tilde{G})} \frac{i(\tilde{B}_{2,L} + B_3)}{i(\tilde{B}_{2,L} + B_3)}
\]

- Simplicity of expression is due to combining 2- and 3-particle K matrices
- \(\tilde{A}^{(u)}, \mathcal{K}_2, \mathcal{K}_{df,3}^{(u,u)}, \) & \(\tilde{A}^{(u)}\) are on-shell, infinite-volume quantities
- “(u)” & “(u,u)” indicate asymmetry due to factors of \(B_{2,L}\)
2ωL^3 \mathcal{K}_2 + \mathcal{K}^{(u,u)}_{\text{df,3}} = \ell_m B_2 \ell_{m'} p + \ell_m B_2 \propto B_2 \ell_{m'} p + \ell_m B_2 \propto p + \ell_m B_3 \ell_{m'} p + \ell_m B_3 \propto B_3 \ell_{m'} p + \ell_m B_3 \propto p + \ldots

- Momenta k, p spectate if external interaction involves two particles

On-shell kernels shown by flat ends

Meaning of asymmetry
New form of QC$_3$

\[ C_{3,L} - \tilde{C}_{3,\infty} = \tilde{A}^{t(u)} i(\tilde{F} + \tilde{G}) \frac{1}{1 - i \left( 2\omega L^3 \mathcal{K}_2 + \tilde{\mathcal{K}}_{df,3}^{(u,u)} \right) i(\tilde{F} + \tilde{G})} \tilde{A}^{(u)} \]

- Spectrum determined by poles in $C_{3,L}(E, P)$

\[ \Rightarrow \quad \det \left[ 1 + \left( 2\omega L^3 \mathcal{K}_2 + \tilde{\mathcal{K}}_{df,3}^{(u,u)} \right) \left( \tilde{F} + \tilde{G} \right) \right] = 0 \]

- $\tilde{\mathcal{K}}_{df,3}^{(u,u)}$ related to $\mathcal{M}_3$ by known integral equations
What has been gained?

[HS14, HS15]

\[
\det[1 + F_3\mathcal{K}_{df,3}] = 0
\]

\[
F_3 = \tilde{F}\left[\frac{1}{3} - \frac{1}{1/(2\omega L^3\mathcal{K}_2) + \tilde{F} + \tilde{G}}\right]
\]

☐ Complicated derivation, hard to generalize

☐ Implicit, constructive definitions

☑ \(\mathcal{K}_{df,3}\) is Lorentz invariant

☑ \(\mathcal{K}_{df,3}\) is symmetric under particle exchange, so easier to parametrize

[BS20a]

\[
\det[1 + (2\omega L^3\mathcal{K}_2 + \mathcal{K}^{(u,u)}_{df,3})(\tilde{F} + \tilde{G})] = 0
\]

☑ Greatly simplified derivation, easy to generalize

☑ Explicit expressions for all quantities

☑ Clean separation of infinite- and finite-volume quantities

☐ \(\mathcal{K}^{(u,u)}_{df,3}\) is not Lorentz invariant (because we used TOPT)

☐ Asymmetry of \(\mathcal{K}^{(u,u)}_{df,3}\) implies that description requires additional parameters
Best of both worlds?

\[
det[1 + F_3 \mathcal{H}_{df,3}] = 0
\]

\[
F_3 = \tilde{F} \left[ \frac{1}{3} - \frac{1}{1/(2\omega L^3 \mathcal{K}_2) + \tilde{F} + \tilde{G}} \tilde{F} \right]
\]

can asymmetrize to new form

\[
det[1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}'_{(u,u)}(df,3))(\tilde{F} + \tilde{G})] = 0
\]

- \(\mathcal{K}'_{(u,u)}\) given by an implicit definition
- \(\mathcal{K}'_{(u,u)}\) is Lorentz invariant
- However, \(\mathcal{K}'_{df,3} \neq \mathcal{K}^{(u,u)}_{df,3}\), due to ambiguity in asymmetrization

\[
det[1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{H}_{df,3})(\tilde{F} + \tilde{G})] = 0
\]

can symmetrize to original form

\[
det[1 + F_3 \mathcal{H}'_{df,3}] = 0
\]

- \(\mathcal{K}'_{df,3}\) obtained from \(\mathcal{K}^{(u,u)}_{df,3}\) by solving an integral equation and symmetrizing
- Can show that \(\mathcal{K}'_{df,3} = \mathcal{K}_{df,3}\) so obtain exactly the original [HS14] QC3

S. Sharpe, "Three particle interactions from the lattice…," seminar at U. Maryland, 9/11/2020
Equivalence of relativistic QC3s

Relativistic QC3 landscape

RFT = generic relativistic EFT

\[ \mathcal{H} \rightarrow \mathcal{M} \]

Integral eqs.

Equivalent representations of \( \mathcal{M} \) (infinite volume)

FVU = finite-volume unitarity

Unitary representation of \( \mathcal{M} \) in terms of \( \mathcal{R}^{(u,u)} \)

Mai et al., 17; Jackura et al., 18

QC3 (all \( \ell \))

\[ \det [F^{-1}_3 + \mathcal{H}_{df,3}] = 0 \]

[HS14]

“algebra”

Alternate form of QC3

\[ \det \left[ 1 + (2\omega L^3 \mathcal{H}^2 + \mathcal{H}^{(u,u)}_{df,3})(\mathcal{F} + \mathcal{G}) \right] = 0 \]

[BS20a]

Jackura, Dawid, Fernández-Ramírez, Mathieu, Mikhasenko, Pilloni, SRS, Szczepaniak, 19

\[ \tilde{H}_s = \tilde{F}_s + \tilde{G}_s + 1/(2\omega L^3 \mathcal{H}_2^s) \]

\[ \tilde{C}_s^{(u,u)} = \mathcal{R}_s^{(u,u)} \]

[HSREV19]
Relativistic QC3 landscape

RFT = generic relativistic EFT

\[ \mathcal{H}_{df,3} \rightarrow \mathcal{M}_3 \]

Integral eqs.

[HS15]

\[ L \rightarrow \infty \]

QC3 (all \( \ell \))

\[ \text{det} \left[ F_3^{-1} + \mathcal{H}_{df,3} \right] = 0 \]

[HS14]

“algebra”

Alternate form of QC3

\[ \text{det} \left[ 1 + (2\omega L^3 \mathcal{H}_2 + \mathcal{H}^{(u,u)}_{df,3})(\widetilde{F} + \widetilde{G}) \right] = 0 \]

[BS20a]

FVU = finite-volume unitarity

Equivalent representations of \( \mathcal{M}_3 \) (infinite volume)

Unitary representation of \( \mathcal{M}_3 \) in terms of \( \mathcal{R}^{(u,u)} \)

Mai et al., 17; Jackura et al., 18

QC3 (\( \ell = 0 \))

\[ \text{det} \left[ \widetilde{H}_s - \frac{1}{2\omega L^3} \widetilde{C}^{(u,u)}_s \frac{1}{2\omega L^3} \right] = 0 \]

Mai & Döring, 17

Is there a relation?

\[ \widetilde{H}_s = \widetilde{F}_s + \widetilde{G}_s + 1/(2\omega L^3 \mathcal{H}_2) \]

\[ \widetilde{C}^{(u,u)}_s = \mathcal{R}^{(u,u)}_s \]

[HSREV19]

S. Sharpe, “Three particle interactions from the lattice…,” seminar at U. Maryland, 9/11/2020
Relativistic QC3 landscape

RFT = generic relativistic EFT

\[ \mathcal{H}_{df,3} \rightarrow \mathcal{M}_3 \]

Integral eqs.

[HS15]

\[ L \rightarrow \infty \]

QC3 (all \( \ell \))

\[ \det \left[ F_3^{-1} + \mathcal{H}_{df,3} \right] = 0 \]

[HS14]

“algebra”

Alternate form of QC3

\[ \det \left[ 1 + (2\omega L^3 \mathcal{H}_2 + \mathcal{H}'_{df,3})(\tilde{F} + \tilde{G}) \right] = 0 \]

[BS20a]

\[ \tilde{H} = \tilde{F} + \tilde{G} + 1/(2\omega L^3 \mathcal{H}_2) \]

“algebra”

Alternate form of QC3

\[ \det \left[ \tilde{H} - \frac{1}{2\omega L^3} \mathcal{R}^{(u,u)} \frac{1}{2\omega L^3} \right] = 0 \]

[BS20b]

FVU = finite-volume unitarity

Unitary representation of \( \mathcal{M}_3 \) in terms of \( \mathcal{R}^{(u,u)} \)

Mai et al., 17; Jackura et al., 18

 QC3 (\( \ell = 0 \))

\[ \det \left[ \tilde{H}_s - \frac{1}{2\omega L^3} \mathcal{R}^{(u,u)} \frac{1}{2\omega L^3} \right] = 0 \]

Mai & Döring, 17

Form from [HSREV19]

Alternate form of QC3

\[ \det \left[ 1 + (2\omega L^3 \mathcal{H}_2 + \mathcal{H}'_{df,3})(\tilde{F} + \tilde{G}) \right] = 0 \]

[BS20b]
Relativistic QC3 landscape

RFT = generic relativistic EFT

\[ \mathcal{H}_{df,3} \rightarrow \mathcal{M}_3 \]
Integral eqs.

[HS15]

\[ L \rightarrow \infty \]

QC3 (all \( \ell \))

\[ \det \left[ F_3^{-1} + \mathcal{H}_{df,3} \right] = 0 \]

[HS14]

“algebra”

Alternate form of QC3

\[ \det \left[ 1 + (2\omega L^3 \mathcal{H}_2 + \mathcal{H}^{(u,u)}_{df,3})(\overline{F} + \overline{G}) \right] = 0 \]

[BS20a]

FVU = finite-volume unitarity

Unitary representation of \( \mathcal{M}_3 \) in terms of \( \mathcal{R}^{(u,u)} \)

Mai et al., 17; Jackura et al., 18

QC3 (\( \ell = 0 \))

\[ \det \left[ \tilde{H}_s - \frac{1}{2\omega L^3} \mathcal{R}^{(u,u)} \frac{1}{2\omega L^3} \right] = 0 \]

Mai & Döring, 17

• Provides derivation of FVU QC3
• Generalizes result to all \( \ell \)
• Shows equivalence of approaches

Alternate form of QC3

\[ \det \left[ \tilde{H} - \frac{1}{2\omega L^3} \mathcal{R}^{(u,u)} \frac{1}{2\omega L^3} \right] = 0 \]

[BS20b]
Step 1: equate forms for $M_{3}^{(u,u)}$

- R-matrix form for asymmetric, infinite-volume three-particle amplitudes

$$M_{3}^{R,(u,u)}$$

- If symmetrize, get unitary $M_{3}$

- Asymmetry not defined in terms of diagrams (Feynman or TOPT)

- If $R^{(u,u)}$ is Lorentz invariant, then so is $M_{3}^{R,(u,u)}$
Step 1: equate forms for $\mathcal{M}^{(u,u)}_3$

- RFT form for Feynman-diagram-based asymmetric amplitude [BS20a]

\[
\mathcal{M}_3^{(u,u)} = \lim_{L \to \infty} \mathcal{M}_{3,L}^{(u,u)} = \mathcal{M}_{df,3,L}^{(u,u)} + D_L^{(u,u)}
\]

\[
\mathcal{M}_{df,3,L}^{(u,u)} = \frac{1}{1 + \mathcal{K}_{2,L}(\tilde{F} + \tilde{G})} \mathcal{K}_{df,3}^{(u,u)}
\]

\[
D_L^{(u,u)} = -\mathcal{M}_{2,L} \tilde{G} \mathcal{M}_{2,L} \frac{1}{1 + \tilde{G} \mathcal{M}_{2,L}}
\]

- If symmetrize, get unitary $\mathcal{M}_3$

- Asymmetry defined in terms of Feynman diagrams

- $\mathcal{K}_{df,3}^{(u,u)}$ is Lorentz invariant
Step 1: equate forms for $\mathcal{M}_{3}^{(u,u)}$

- RFT form for Feynman-diagram-based asymmetric amplitude [BS20a]

$$
\mathcal{M}_{3}^{(u,u)} = \lim_{L \to \infty} \mathcal{M}_{3,L}^{(u,u)}
$$

$$
\mathcal{M}_{3,L}^{(u,u)} = \mathcal{M}_{df,3,L}^{(u,u)} + \mathcal{D}_{L}^{(u,u)}
$$

$$
\mathcal{M}_{df,3,L}^{(u,u)} = \frac{1}{1 + \tilde{K}_{2,L}(\tilde{F} + \tilde{G})} \mathcal{K}_{df,3}^{(u,u)}
$$

$$
\mathcal{D}_{L}^{(u,u)} = -\mathcal{M}_{2,L} \tilde{G} \mathcal{M}_{2,L} \frac{1}{1 + \tilde{G} \mathcal{M}_{2,L}}
$$

- If symmetrize, get unitary $\mathcal{M}_{3}$
- Asymmetry defined in terms of Feynman diagrams
- $\mathcal{K}_{df,3}^{(u,u)}$ is Lorentz invariant

Equating $\mathcal{M}_{3}^{\tilde{K}_{df,3},(u,u)}$ and $\mathcal{M}_{3}^{(u,u)}$ gives integral equation relating $\tilde{K}_{df,3}^{(u,u)}$ and $\mathcal{K}_{df,3}^{(u,u)}$
Step 2: rewrite asymmetric QC₃

- Algebraic manipulations

\[
\det \left[ 1 + (2\omega L^3 \mathcal{H}_2 + \mathcal{H}_{df,3}'(u,u))(\widetilde{F} + \widetilde{G}) \right] = 0 \quad \Rightarrow \quad \det \left[ \widetilde{H} - X^{(u,u)} \right] = 0
\]

\[
\widetilde{H} = \widetilde{F} + \widetilde{G} + \mathcal{H}_{2,L}^{-1}
\]

\[
\mathcal{H}_{2,L} = (2\omega L^3) \mathcal{H}_2
\]

\[
X^{(u,u)} = \mathcal{H}_{2,L}^{-1} \mathcal{H}_{df,3}'(u,u) \mathcal{H}_{2,L}^{-1} \frac{1}{1 + \mathcal{H}_{df,3}'(u,u) \mathcal{H}_{2,L}^{-1}}
\]
Step 3: combine

$$\det \left[ \widetilde{H} - X^{(u,u)} \right] = 0$$

$$X^{(u,u)} = \mathcal{H}_2,L^{-1} \mathcal{K}'_{df,3}(u,u) \mathcal{K}_2,L^{-1} \frac{1}{1 + \mathcal{K}'_{df,3}(u,u) \mathcal{K}_2,L^{-1}}$$

- Using integral equation relating $\mathcal{R}^{(u,u)}$ and $\mathcal{K}'_{df,3}(u,u)$, can show that

$$\left[ (2\omega L^3) X^{(u,u)} (2\omega L^3) \right]_{k'l'm';l'm'} = \left[ \mathcal{R}^{(u,u)} \right]_{k'l'm';l'm'} + \mathcal{O}(e^{-mL})$$

- Substituting gives claimed result

$$\det \left[ \widetilde{H} - (2\omega L^3)^{-1} \mathcal{R}^{(u,u)} (2\omega L^3)^{-1} \right] = 0$$

- Derivation valid only if use smooth cutoff function, and appropriate form for $\widetilde{G}$

- We expect that, if we take the NR limit, we will obtain the NREFT form of the QC3 generalized to all $\ell$
Conclusions & Outlook
Summary

- Formalism is ready to use for phenomenologically interesting case of 3 pions with all allowed isospins
  - Also provides numerical method for studying properties of relativistic 2- and 3-particle bound states
- First comparisons with LQCD data for $3\pi^+$ show evidence for 3-particle quasilocal interaction $\mathcal{K}_{df,3}$
  - Several similar studies have recently appeared: [Culver et al., 1911.09047 (GWU), Fischer et al., 2008.03035 (ETMC), Hansen et al., 2009.04931 (Hadspec)]
- Simplified derivation of QC3 using TOPT
- Equivalence of RFT and FVU forms demonstrated, and FVU form generalized
Relative merits of forms of QC$_3$

\[ \det \left[ F_3^{-1} + K_{df,3} \right] = 0 \]

- Symmetric three-particle K matrix
- Threshold expansion requires fewer parameters
- Little intuition in presence of three-particle resonances
- $K_{df,3}$ depends on PV pole prescription

\[ \det \left[ \tilde{F} + \tilde{G} + \frac{1}{2\omega L^3} K_2 - \frac{1}{2\omega L^3} R^{(u,u)} \frac{1}{2\omega L^3} \right] = 0 \]

- Asymmetric three-particle R matrix
- Threshold expansion requires more (redundant) parameters
- Some experience and intuition from JPAC studies of fitting amplitudes to experimental data
- $R^{(u,u)}$ independent of pole prescription
To-do list for QC3s

- Generalize formalism to broaden applications
  - Nondegenerate particles, e.g. $K^+K^+\pi^+$ [Pang et al., 2008.13014 (NREFT)]
  - Spin for, e.g., $N\pi\pi$
  - Determination of Lellouch-Lüscher factors to allow application to $K\rightarrow 3\pi$ etc

- Develop physics-based parametrizations of $\mathcal{H}_{df,3}$ to describe resonances
  - Need to learn how to solve integral equations relating $K_{df,3}$ to $M_3$ above threshold [Jackura, INT talk, 8/20; Hansen et al., 2009.04931]
  - Understand appearance of unphysical solutions (wrong residue) for some values of parameters—observed in [BHS18; BRS19]
    - May be due to truncation, or due to exponentially suppressed effects, or both
  - Move on to QC4 !?
A taste of the future?
[Blanton, Hanlon, Hörz, Romero-López, SRS, in progress]

- Many precise $3\pi^+$ & $3K^+$ levels in all irreps in 8 frames for three choices of quark masses
- Should allow extraction of s- and d-wave parameters of 2- and 3-particle interactions

$m_\pi = 345$ MeV, $m_K = 440$ MeV
Thank you!
Questions?
Backup slides
\( F_3 \) collects 2-particle interactions

\[
F_3 = \frac{1}{2\omega L^3} \left[ \frac{F}{3} - F \frac{1}{\mathcal{K}^{-1} + F + G} \right]
\]

- \( F \) & \( G \) are known geometrical functions, containing cutoff function \( H \)

\[
F_{p\ell' \ell'; k\ell m} = \delta_{p k} \ H(\hat{k}) \ F_{PV, \ell' \ell'; \ell m}(E - \omega_k, \hat{P} - \hat{k}, L)
\]

\[
G_{p\ell' \ell'; k\ell m} = \left( \frac{k^*}{q_p^*} \right)^{\ell'} \frac{4\pi Y_{\ell' \ell m}(\hat{k}^*) H(\hat{p}) H(\hat{k}) Y_{\ell m}^*(\hat{p}^*)}{(P - k - p)^2 - m^2} \left( \frac{p^*}{q_k^*} \right)^{\ell} \frac{1}{2\omega L^3}
\]

Relativistic form introduced in [BHS17]

\[
F_{PV, \ell' \ell'; \ell m}(E, \hat{P}, L) = \frac{1}{2} \left( \frac{1}{L^3} \sum_{\hat{k}} \right) - \text{PV} \ \int \frac{d^3k}{(2\pi)^3} \frac{4\pi Y_{\ell' \ell m}(\hat{k}^*) Y_{\ell m}^*(\hat{k}^*) h(\hat{k})}{2\omega_L 2\omega_{P-k} (E - \omega_k - \omega_{P-k})}
\]

Relativistic form equivalent up to exponentially-suppressed terms

\[
Y_{\ell m}(\hat{k}^*) = \sqrt{4\pi} \left( \frac{k^*}{q^*} \right)^{\ell} \ Y_{\ell m}(\hat{k}^*)
\]
Evidence for trimer bound by $a_2$

Binding caused by d-wave attraction! Relevant for atomic physics?

$m a_0 = -0.1$, $m a_2 = -1.3$, $r_0 = P_0 = \mathcal{H}_{df,3} = 0$
Definitions of asymmetric kernels

- In original RFT approach (using Feynman diagrams & Bethe-Salpeter kernels)

\[
\left[ M_3^{(u,u)} \right]_{ka:pr} = a \begin{array}{c}
\begin{array}{c}
 B_2 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 p
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_2 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 B_2 \\
 p
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_2 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 B_2 \\
 r
\end{array}
\end{array} + \\
+ a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 p
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 r
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 p
\end{array}
\end{array} + \\
+ a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_3 \\
 r
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_3 \\
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\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
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 p
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
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 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
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 r
\end{array}
\end{array} + \\
+ a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_3 \\
 p
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_3 \\
 B_2 \\
 r
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_3 \\
 B_2 \\
 p
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_3 \\
 B_3 \\
 r
\end{array}
\end{array} + \ldots
\]

- In our approach (using time-ordered perturbation theory)

\[
\left[ \tilde{M}_3^{(u,u)} \right]_{ka:pr} = a \begin{array}{c}
\begin{array}{c}
 B_2 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 p
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_2 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 B_2 \\
 p
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_2 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 B_2 \\
 r
\end{array}
\end{array} + \\
+ a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 p
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 r
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 p
\end{array}
\end{array} + \\
+ a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_3 \\
 r
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 B_3 \\
 p
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_2 \\
 B_3 \\
 r
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_3 \\
 B_2 \\
 p
\end{array}
\end{array} + a \begin{array}{c}
\begin{array}{c}
 B_3 \\
 k
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 B_3 \\
 B_3 \\
 r
\end{array}
\end{array} + \ldots
\]

k & p assigned to spectators

Cuts in time-ordered PT

TOPT kernels (no 3-particle cuts)
Asymmetric kernels differ!

- Consider a particular Feynman diagram

\[
\begin{align*}
&\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)},scale=0.8]
    \node (a) at (0,0) {a};
    \node (p) at (2,0) {p};
    \node (k) at (0,-1) {k};
    \node (r) at (2,-1) {r};
    \draw (a) to [bend right] (p);
    \draw (k) to [bend right] (r);
\end{tikzpicture}
\quad \in \quad
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)},scale=0.8]
    \node (a) at (0,0) {a};
    \node (p) at (2,0) {p};
    \node (k) at (0,-1) {k};
    \node (r) at (2,-1) {r};
    \draw (a) to [bend right] (p);
    \draw (k) to [bend right] (r);
    \node (B) at (1,0) {$B_2$};
\end{tikzpicture}
\end{align*}
\]

- In TOPT the two time orderings are put into different terms—one being symmetrized

\[
\begin{align*}
&\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)},scale=0.8]
    \node (a) at (0,0) {a};
    \node (p) at (2,0) {p};
    \node (k) at (0,-1) {k};
    \node (r) at (2,-1) {r};
    \draw (a) to [bend right] (p);
    \draw (k) to [bend right] (r);
    \draw[red,dotted] (a) to (p);
\end{tikzpicture}
\quad \in \quad
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)},scale=0.8]
    \node (a) at (0,0) {a};
    \node (p) at (2,0) {p};
    \node (k) at (0,-1) {k};
    \node (r) at (2,-1) {r};
    \draw (a) to [bend right] (p);
    \draw (k) to [bend right] (r);
    \node (B) at (1,0) {$B_2$};
\end{tikzpicture}
\end{align*}
\]

\[
\begin{align*}
&\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)},scale=0.8]
    \node (a) at (0,0) {a};
    \node (p) at (2,0) {p};
    \node (k) at (0,-1) {k};
    \node (r) at (2,-1) {r};
    \draw (a) to (p);
    \node (B) at (1,0) {$B_3$};
\end{tikzpicture}
\quad \in \quad
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)},scale=0.8]
    \node (a) at (0,0) {a};
    \node (p) at (2,0) {p};
    \node (k) at (0,-1) {k};
    \node (r) at (2,-1) {r};
    \draw (a) to (p);
    \node (B) at (1,0) {$B_3$};
\end{tikzpicture}
\end{align*}
\]

- Thus \( \mathcal{M}_3^{(u,u)} \neq \widetilde{\mathcal{M}}_3^{(u,u)} \), although both symmetrize to \( \mathcal{M}_3 \)
Asymmetric kernels ⇒ redundancy

- E.g., asymmetric form of QC3 holds with (at least) two different kernels

\[
\det\left[ 1 + (2\omega L^3 \mathcal{H}_2 + \mathcal{H}^{(u,u)}_{df,3})(\mathcal{F} + \mathcal{G}) \right] = 0
\]

\[
\det\left[ 1 + (2\omega L^3 \mathcal{H}_2 + \overline{\mathcal{H}}^{(u,u)}_{df,3})(\overline{\mathcal{F}} + \overline{\mathcal{G}}) \right] = 0
\]

- R matrix representation of \( \mathcal{M}^{(u,u)}_3 \) holds for all choices of asymmetry

\[\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{p'} \\
\text{p}
\end{array} & = & \begin{array}{c}
\begin{array}{c}
\text{B}
\end{array} \\
\text{p'} & \text{p}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{B}
\end{array} \\
\text{p'} & \text{k} & \text{p}
\end{array}
\end{array}
\end{align*}\]

\[\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{B}
\end{array} \\
\text{p'} & \text{p}
\end{array} & = & \begin{array}{c}
\begin{array}{c}
\text{p'}
\end{array} \\
\text{p}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{R}
\end{array} \\
\text{p'} & \text{p}
\end{array}
\end{array}
\]

Formula from Mai et al., 17 & Jackura et al., 18

Figures from Jackura et al., 19

We call it \( \mathcal{R}^{(u,u)} \) to emphasize its asymmetry

Can set this equal to either \( \mathcal{M}^{(u,u)}_3 \) or \( \overline{\mathcal{M}}^{(u,u)}_3 \) : leads to different, equally valid, \( \mathcal{R}^{(u,u)} \)
Sketch of derivation of 3-particle quantization condition

Derivation

- Generic relativistic EFT, working to all orders
  - Do not need a power-counting scheme
  - To simplify analysis: impose a global $Z_2$ symmetry (G parity) & consider identical scalars
- Obtain spectrum from poles in finite-volume correlator
  - Consider $E_{CM} < 5m$ so on-shell states involve only 3 particles

\[
C_L(E, \vec{P}) = \sum + \sum + \sum + \cdots + \cdots
\]

Momentum sums rather than integrals

Infinite-volume Bethe-Salpeter kernels

Arbitrary operator creating 3 particles

S. Sharpe, “Three particle interactions from the lattice…,” seminar at U. Maryland, 9/11/2020
Derivation

Replace sums with integrals plus sum-integral differences to extent possible

- If summand has pole or cusp then difference $\sim 1/L^n$ and must keep (Lüscher zeta function)
- If summand is smooth then difference $\sim \exp(-mL)$ and drop

Avoid cusps by using PV prescription—leads to generalized 3-particle $K$ matrix

Subtract above-threshold divergences of 3-particle $K$ matrix—leads to $K_{df,3}$
• Reorganize, resum, … to separate infinite-volume on-shell relativistically-invariant non-singular scattering quantities ($K_2, K_{df,3}$) from known finite-volume functions ($F$ [Lüscher zeta function] & $G$ [“switch function”])

$$\Rightarrow \quad \det \left[ F_3^{-1} + \mathcal{K}_{df,3} \right] = 0$$
• Relate $K_{df,3}$ to $M_3$ by taking infinite-volume limit of finite-volume scattering amplitude
  
  • Leads to infinite-volume integral equations involving $M_2$ & cut-off function $H$
  
  • Can formally invert equations to show that $K_{df,3}$ (while unphysical) is relativistically invariant and has same properties under discrete symmetries $(P,T)$ as $M_3$

\[
\begin{align*}
  iM_{L,3\rightarrow 3} &= iD_L + S \left[ L_{L} \frac{1}{1 - iF_3 \frac{1}{iK_{df,3\rightarrow 3}}} \right] \\
  iM_{3\rightarrow 3} &= \lim_{L \to \infty} \left| i \epsilon \right| \frac{1}{iM_{L,3\rightarrow 3}} \\
\end{align*}
\]

Sums over $k$ go over to integrals with $i\epsilon$ pole prescription