Three-particle scattering amplitudes from lattice QCD

Steve Sharpe
University of Washington

Outline

- Motivations
- The fundamental issue & its resolution
- Recent applications of three-particle formalism
  - Three-particle amplitudes involving pions & kaons at physical quark masses
  - $\pi(1300) \to 3\pi$ resonance
  - $DD\pi$ scattering, relevant for the T_{cc}^{+}
  - Progress on $N\pi\pi$ — towards the Roper resonance
  - Dreaming of three neutrons
- Summary & Outlook

Contributions at Lat25

Session	Title	Speaker	Room	Time
Monday a.m. plenary	Hadron spectroscopy and interactions	*Jeremy Green*	*Homi Bhabha Auditorium*	11:45 - 12:30
	Nucleon-nucleon interactions: past, present, and future	*Amy Nicholson*	*Homi Bhabha Auditorium*	12:30 - 13:00
Monday p.m.	Lattice study of ccus tetraquark channel in D(*)Ds(*) scattering	*Tanishk Shrimal*	*AG66*	17:00 - 17:20
	Importance of local tetraquark operators for the $T_{cc}(3875)^+$ spectrum	*Andres Stump*	*AG66*	17:20 - 17:40
	T_{cc} pole trajectory	*Protick Mohanta*		
Tuesday p.m.	Bayesian Analysis and Analytic Continuation of Scattering Amplitudes from Lattice QCD	*Miguel Salg*	*AG66*	17:00 - 17:20
	Towards four-pion effects in multi-hadron decays	*Rajnandini Mukherjee*	*AG66*	17:20 - 17:40
Friday p.m.	Systematic uncertainties in lattice QCD calculations of multi-hadron systems	*Robert Perry*	*AG66*	15:10 - 15:30
	QCD Predictions for Physical Multimeson Scattering Amplitudes	*Fernando Romero López*	*AG66*	16:40 - 17:00
	Three-body study of the T_{cc}^+ from lattice QCD	*Andre Baiao Raposo*	*AG66*	17:00 - 17:20
	Extending the finite volume formalism to the $N\pi\pi$ system at maximal isospin	*Stephen Sharpe*	*AG66*	17:20 - 17:40
	Emergence of the pi(1300) Resonance from Lattice QCD	*Maxim Mai*	*AG66*	17:40 - 18:00

Previous related plenaries

- Lattice 2024
  - Nilmani Mathur, "Hadron Spectroscopy from Lattice QCD"
  - Felix Erben, "From scattering towards multi-hadron weak decays"
- Lattice 2023
  - Drew Hanlon, "Hadron spectroscopy and few-body dynamics from LQCD"
- Lattice 2022
  - Fernando Romero-López, "Multi-hadron interactions from lattice QCD"
- Lattice 2021
  - Ben Hörz, "Spectroscopy and hadron interactions"
- Lattice 2019
  - Akaki Rusetsky, "Three particles on the lattice"

Topics not discussed

- Matrix elements involving resonances
  - Felipe Ortega Gama, "Two-particle matrix elements in a box"
  - Ajay Shanmuga Sakthivasan, "The EM Form Factor of the Rho Meson on the lattice"
- Electroweak decay amplitudes
  - Luka Leskovec, plenary at Lattice 2023, "Electroweak transitions involving resonances"
  - Erik Lundstrum, "Isospin-breaking contributions to ε' from lattice QCD"
  - Felix Erben, "$K\pi$ scattering as a step towards $B \to K^*\ell^+\ell^-$ from lattice QCD"
- Methods for inclusive decays
  - Marco Garofalo, "Inclusive semileptonic decays of heavy mesons from lattice QCD"
  - Francesca Margari, "The smeared R-ratio in isoQCD from first-principles lattice simulations"
  - D. Gavriel, "A novel framework for spectral density reconstruction via quadrature-based Laplace inversion"
  - Ryan Abbott, "Moment problems and spectral functions"
  - Ahmed Elgaziari, "Study on the systematic effects on $b \to c$ inclusive semileptonic decays"
  - Sarah Fields, "Nevanlinna-Pick interpolation from inexact data"
  - Gabriele Moriandi, "Towards the time-like pion form factor beyond the elastic regime using domain-wall QCD"
  - Christian Zimmermann, "The hadronic tensor from four-point functions on the lattice"
  - A. del Pino Rubio, "Hadronic contributions to $\alpha(Q^2)$ & $\sin^2\theta_W$ from spectral reconstruction on the lattice"

Topics not discussed

- Ground state energy of multiparticle systems
  - [Lee, Huang, Yang, 1957]
  - "n-Boson Energies at Finite Volume and Three-Boson Interactions", [Beane, Detmold, Savage, 0707.1670 (PRD)]
  - …
  - "Lattice quantum chromodynamics at large isospin density" (6144 pions in a lattice) [Abbott et al. (NPLQCD), 2307.15014 (PRD) & 2406.09273 (PRL)]

Motivations

Underlying motivations

- Determine properties of strong interaction resonances from QCD
  - E.g. exotics such as $T_{cc}(3875)^+ \to DD^* \to DD\pi$
  - Long-standing puzzles, e.g. Roper $N(1440) \to N\pi + N\pi\pi$
- Determine three particle "forces" for $3n,\ 3\pi,\ 3K, \ldots$
  - Needed to understand neutron star EoS, properties of large nuclei, …

Underlying motivations

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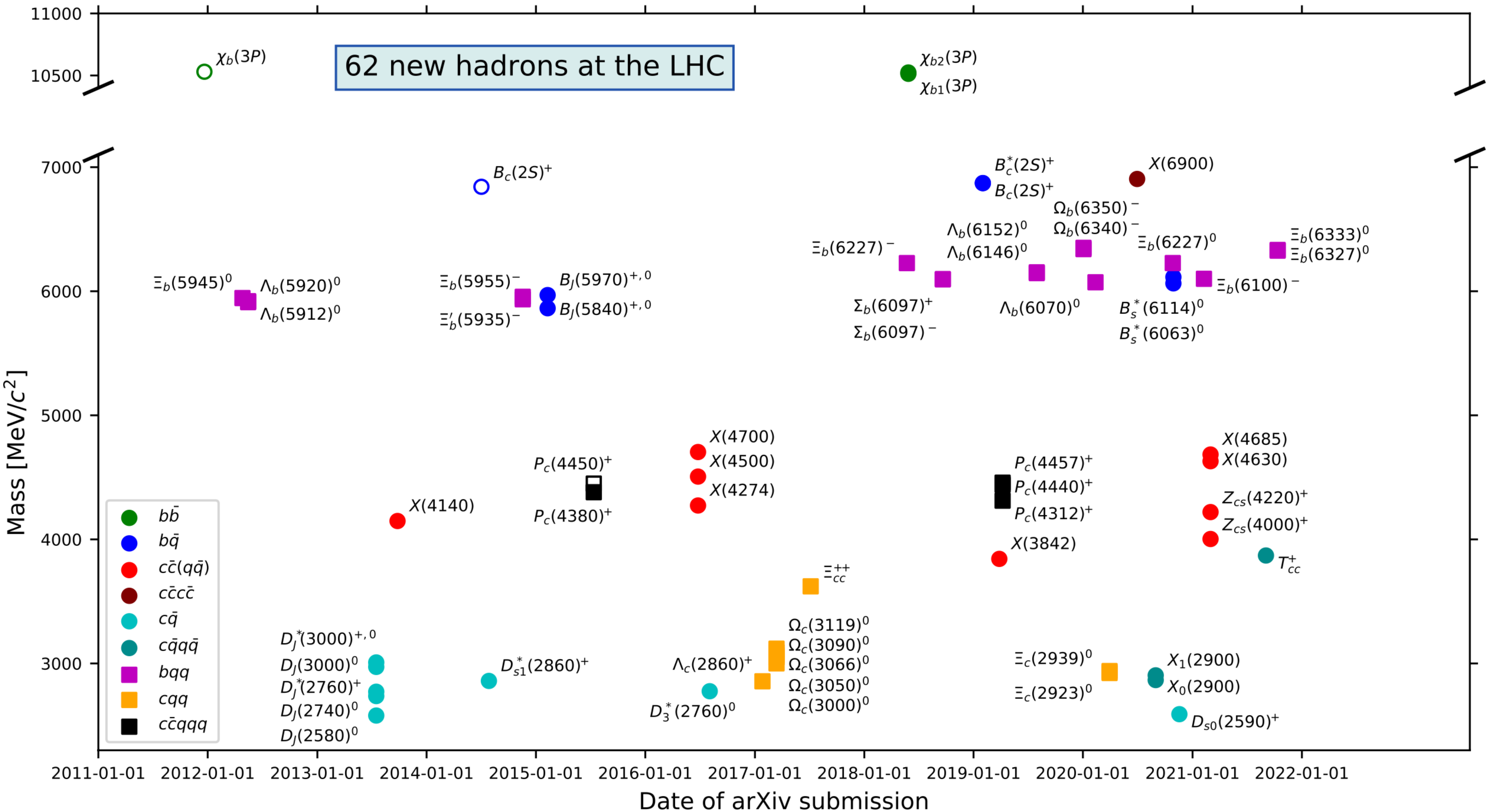
- Determine three particle "forces" for $3n,\ 3\pi,\ 3K, \ldots$
  - Needed to understand neutron star EoS, properties of large nuclei, …
- Calculate weak decay amplitudes within the Standard Model, in order to search for new physics
  - E.g. $K \to 2\pi$ (essentially done), $K \to 3\pi$ (method known), & $D \to \pi^+\pi^-,\ K^+K^-$ (open question)

Cornucopia of exotics



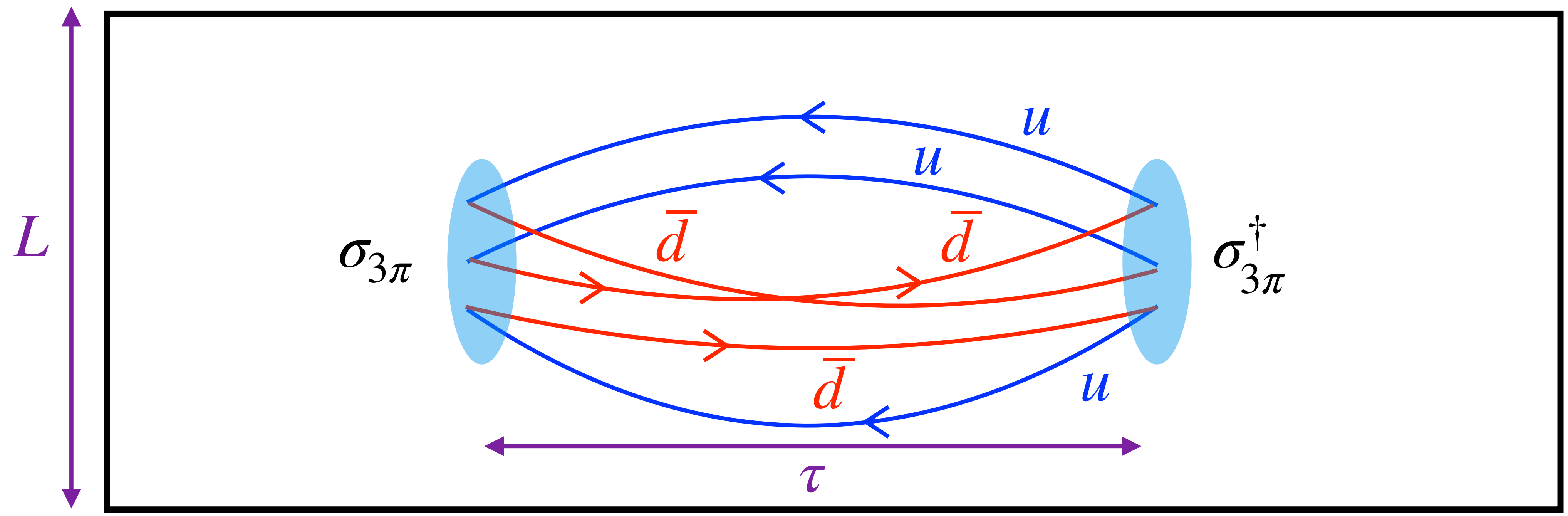
[I. Danilkin, talk at INT workshop, March 23]

+ data from Babar, Belle, COMPASS, …

The fundamental issue & its resolution

On the one hand…

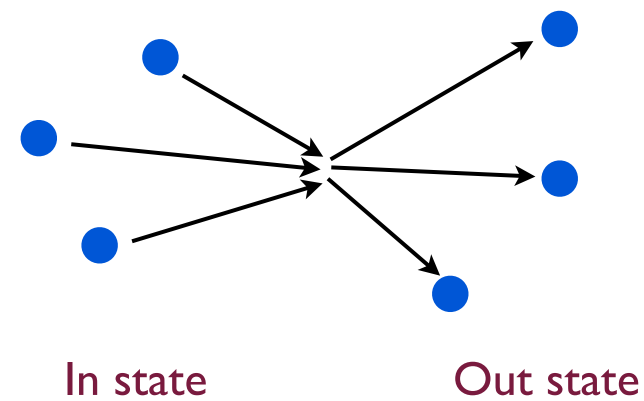
- LQCD determines energies and properties of finite-volume eigenstates
  - Obtained from Euclidean correlation functions
  - Outputs are physical quantities (with statistical & systematic errors)



…while on the other

- We want infinite-volume scattering amplitudes, e.g.

$$\mathscr{M}_3 \sim$$



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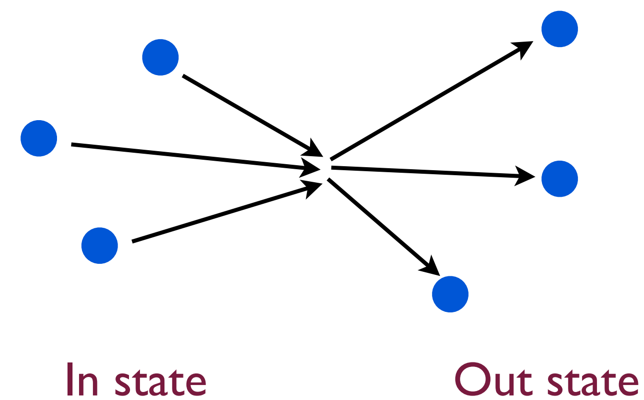
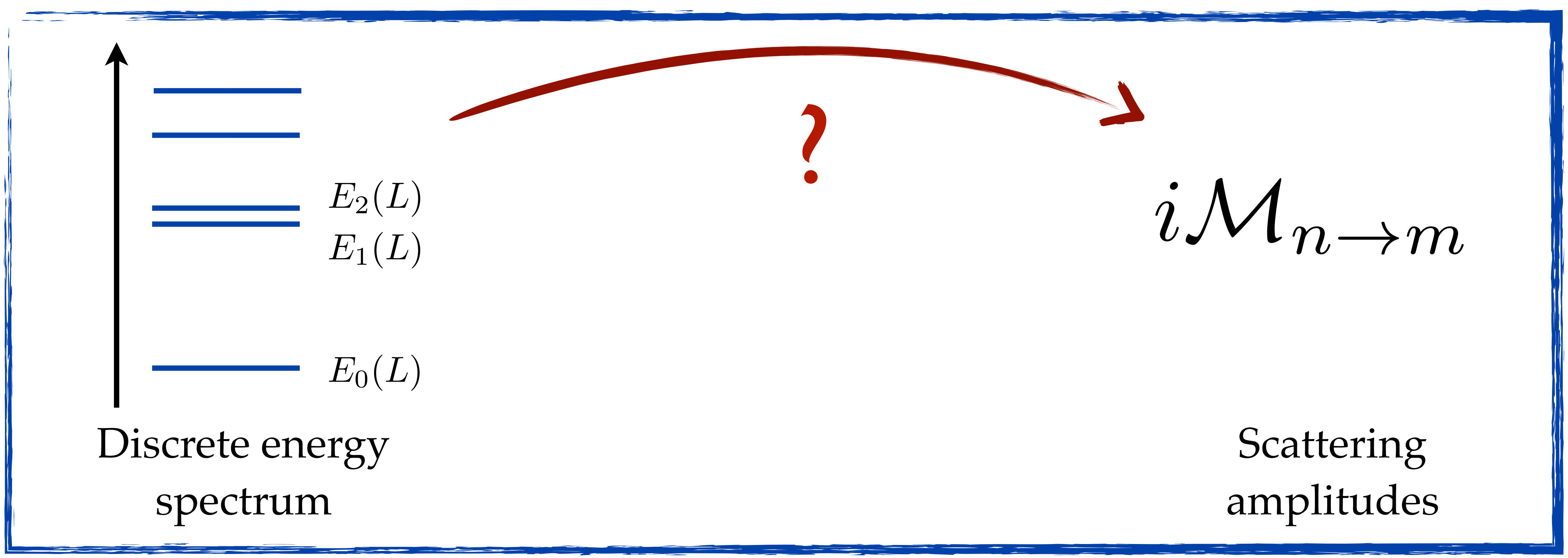


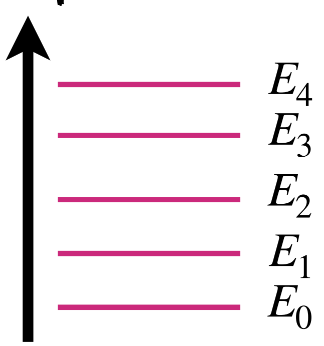
- How do we relate these? A finite-volume QFT problem.



2-particle formalism

[Lüscher, 1986-91 + many subsequent works]

Two-particle spectrum



↔

Quantization condition: QC2

$$\det_{\ell m}\left[\mathcal{K}_2 + F_2^{-1}\right] = 0$$

Known kinematical "Lüscher zeta" function

Fit →

K-matrix

$$\mathcal{K}_2$$

Unitarity relation →

$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho\mathcal{K}_2}$$

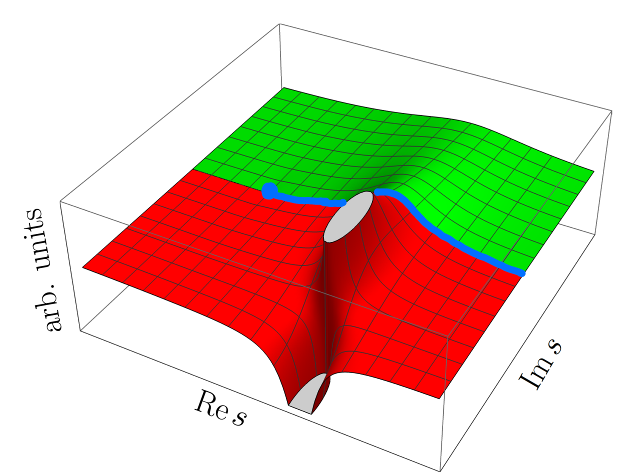
Phase space

Scattering amplitude

$$\mathcal{M}_2$$

Analytic Continuation ↓

Resonance poles, Bound states



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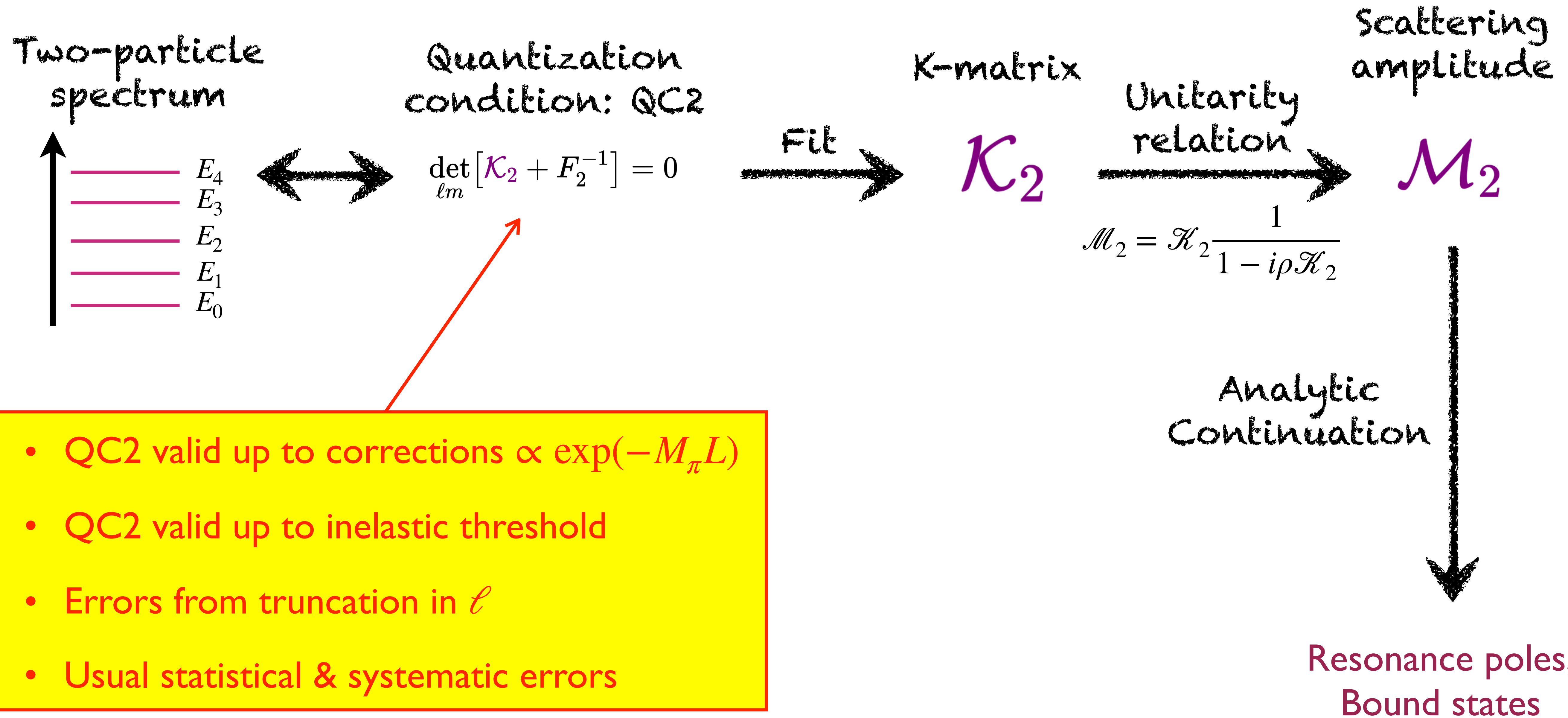
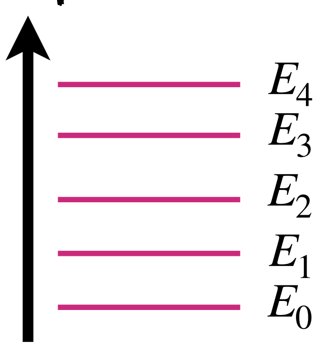


- QC2 valid up to corrections $\propto \exp(-M_\pi L)$
- QC2 valid up to inelastic threshold
- Errors from truncation in ℓ
- Usual statistical & systematic errors

2-particle formalism

[Lüscher, 1986-91 + many subsequent works]

Two-particle spectrum



↔ **Quantization condition: QC2**

$$\det_{\ell m}\left[\mathcal{K}_2 + F_2^{-1}\right] = 0$$

Fit → **K-matrix** $\mathcal{K}_2$ → **Unitarity relation** → **Scattering amplitude** $\mathcal{M}_2$

$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho\mathcal{K}_2}$$

Analytic Continuation ↓

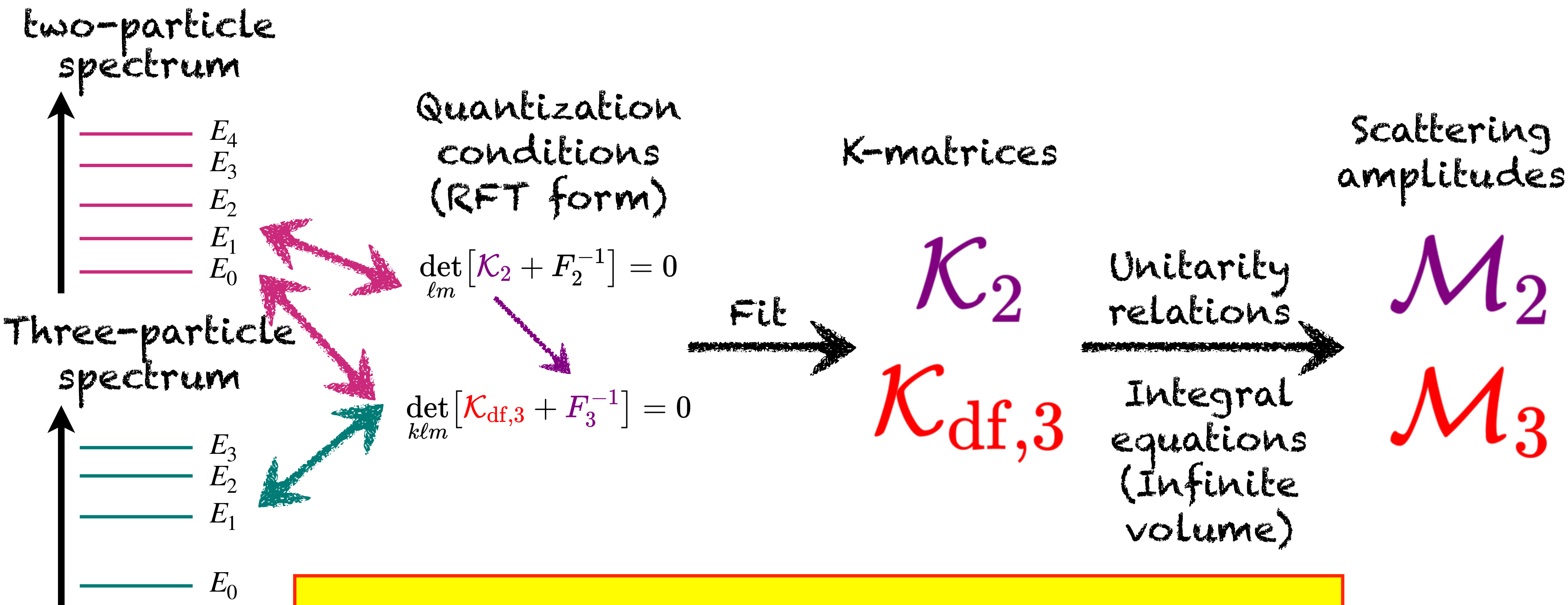
Resonance poles, Bound states

- QC2 valid up to corrections $\propto \exp(-M_\pi L)$
- QC2 valid up to inelastic threshold
- Errors from truncation in ℓ
- Usual statistical & systematic errors

- Parametrize $\mathcal{K}_2$
- Model average over "reasonable" forms
- Uncertainties increase as move into complex plane

3-particle formalism

[Hansen & SRS, 2014 & 2015 + many subsequent works]



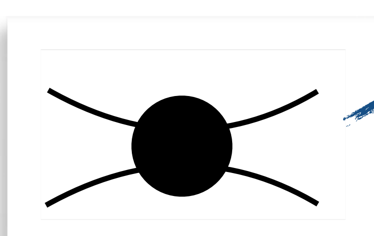
- Using more data than QC2, but fitting much more challenging
- More quantities to parametrize, with limited phenomenological knowledge for $\mathcal{K}_{\mathrm{df},3}$
- Solving integral equations involves new challenges

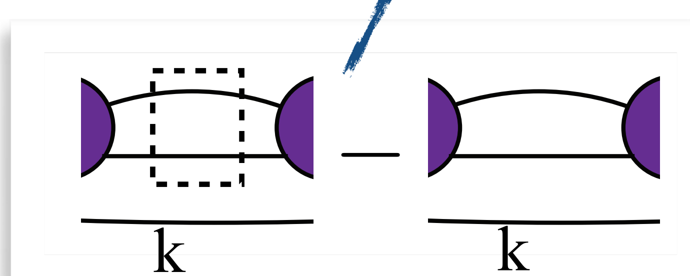
QC3

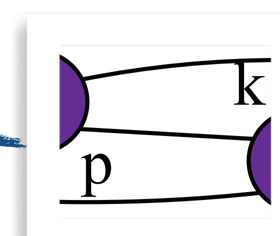
$$\det_{k\ell m}\left[F_3^{-1}+\mathcal{K}_{\mathrm{df},3}\right]=0$$

- QC3 valid up to corrections $\propto \exp(-M_\pi L)$
- QC3 valid up to first inelastic threshold, e.g. $E^* = 5M_\pi$ for 3 pion system
- Matrix indices are spectator finite-volume momentum $\boldsymbol{k}$, and pair CM-frame ℓ, m
- RFT formalism includes smooth cutoff in $\boldsymbol{k}$; must truncate "by hand" in ℓ

$$F_3=\frac{1}{2\omega L^3}\left[\frac{F}{3}-F\frac{1}{1/\mathcal{K}_{2,L}+F+G}F\right]$$



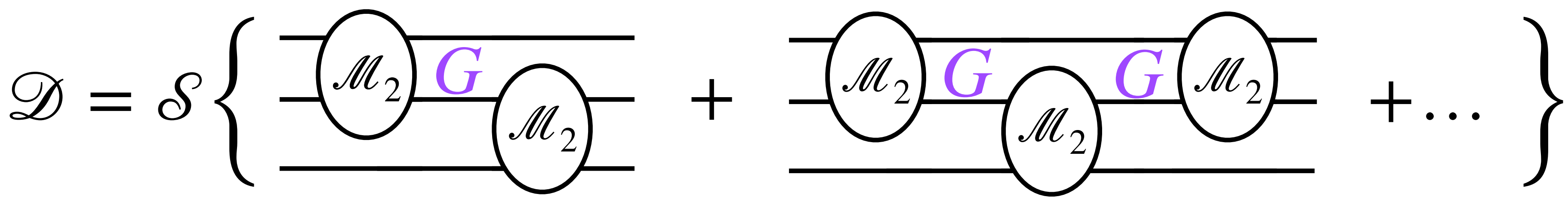




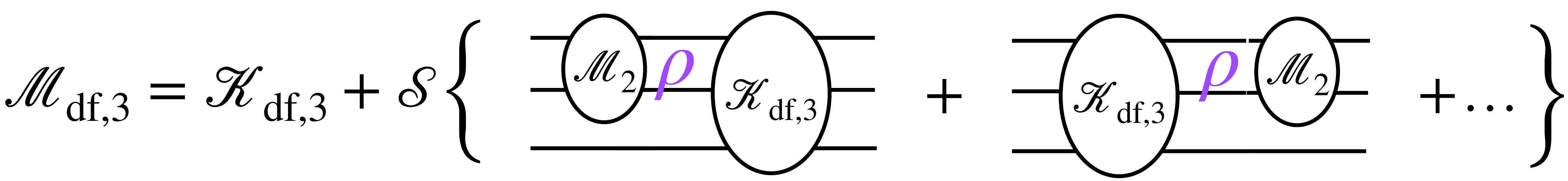
Relating $\mathcal{K}_{\text{df},3}$ to $\mathcal{M}_3$

$$\mathcal{M}_3 = \lim_{L\to\infty} \mathcal{S}\left\{\mathcal{D}_L^{(u,u)} + \mathcal{M}_{\text{df,3,L}}^{(u,u)}\right\} = \mathcal{D} + \mathcal{M}_{\text{df},3}$$

- $\mathcal{D}$ contains all divergent contributions to $\mathcal{M}_3$, but depends on cutoff function $H(\vec{k})$

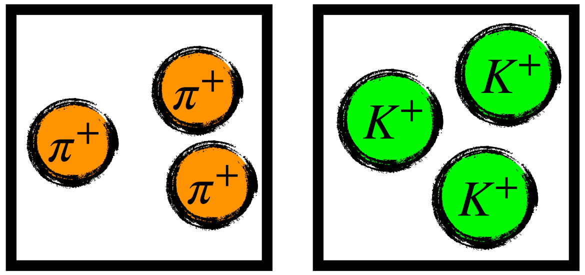


- $\mathcal{M}_{\text{df},3}$ is divergence-free, equals $\mathcal{K}_{\text{df},3}$ at leading order, and is also cutoff-dependent

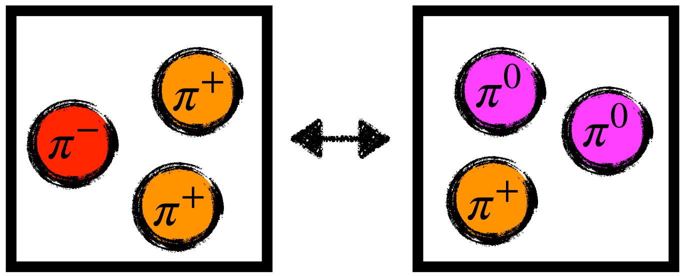


- "Decorations" ensure that $\mathcal{M}_3$ is unitary

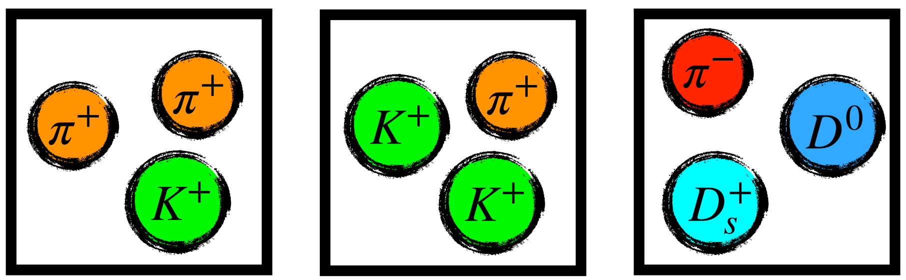
Status: formalism



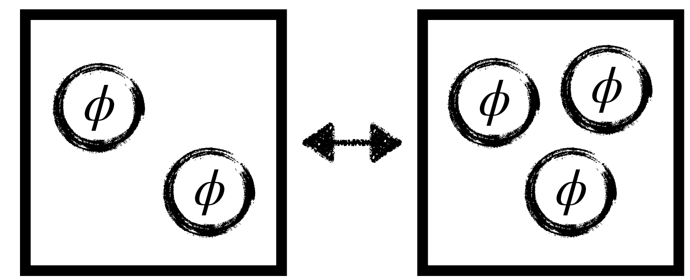
[Hansen & SRS 14,15 (RFT); Hammer, Pang, Rusetsky 17 (NREFT); Mai, Döring 17 (FVU)]
[Many subsequent refinements & tests]



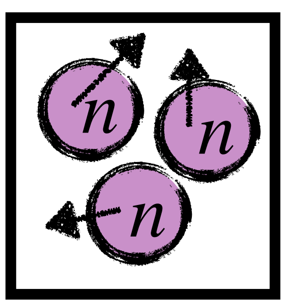
[Hansen, Romero-López, SRS 20;
Mai, Alexandru, Brett, Culver, Döring, Lee, Sadasivan (FVU) 21]



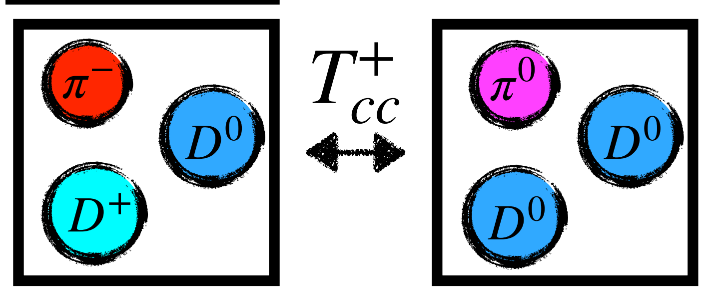
[Blanton, SRS 20, 21]



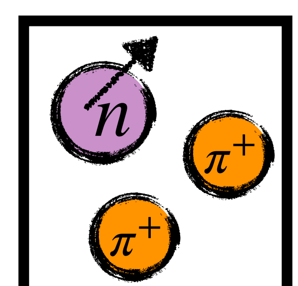
[Briceño, Hansen, SRS 17]



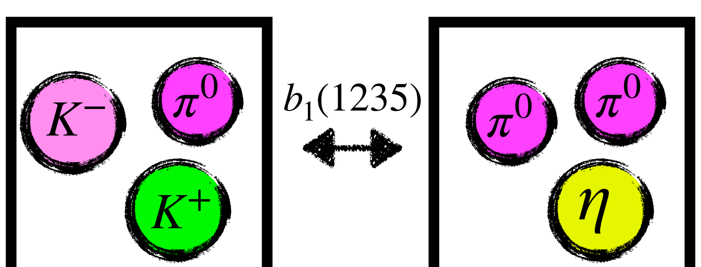
[Draper, Hansen, Romero-López, SRS 23]



[Hansen, Romero-López, SRS 24]

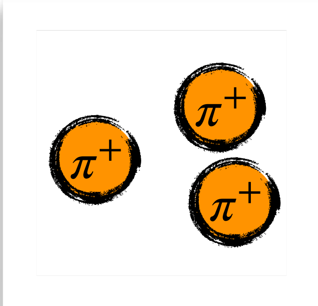


[Hansen, Romero-López, SRS 25]



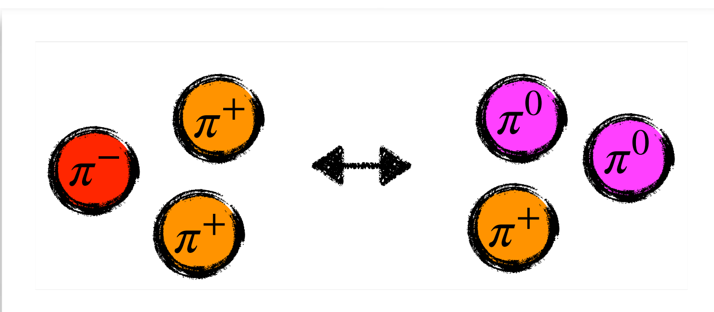
[Draper, SRS 24]

Status: solving integral equations

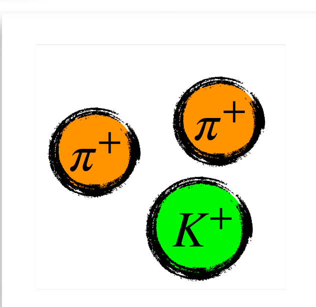


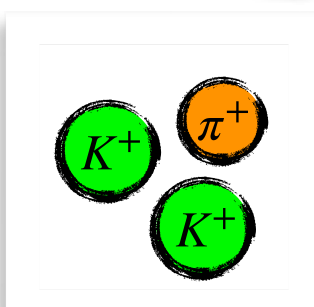
[Hansen, Briceño, Edwards, Thomas, Wilson, 20]

[Jackura, Briceño, Dawid, Islam, McCarty, 20;
Dawid, Islam, Briceño, 23;
Dawid, Islam, Briceño, Jackura, 23]

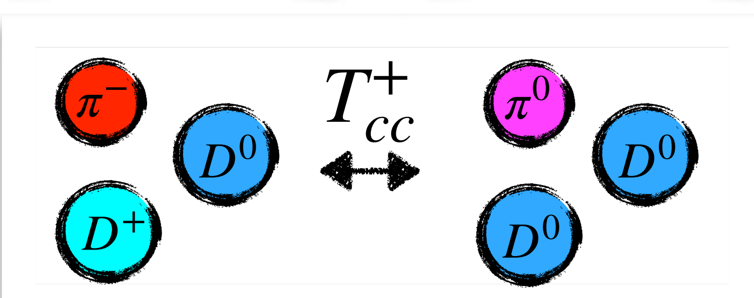


[Mai, Alexandru, Brett, Culver, Döring, Lee, Sadasivan (IVU) 21;
Jackura, Briceño, 23; Briceño, Costa, Jackura, 24; Jackura, Chambers, Briceño, 25;
Yan, Mai, Garofalo, Feng, Meißner, Liu, Liu, Urbach (IVU) 24;
Yan, Mai, Garofalo, Feng, Döring, Liu, Liu, Meißner, Urbach (IVU) 25]





[Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López, SRS, Skinner, 25]



[Dawid, Romero-López, SRS 24, 25]

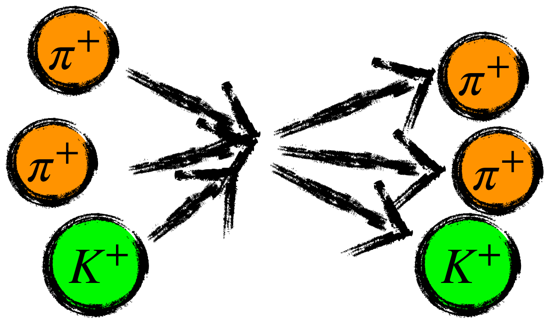
Recent applications of three-particle formalism

QCD predictions for physical multimeson scattering amplitudes

Sebastian M. Dawid,[1] Zachary T. Draper,[1] Andrew D. Hanlon,[2] Ben Hörz,[3] Colin Morningstar,[4] Fernando Romero-López,[5,6] Stephen R. Sharpe,[1] and Sarah Skinner[4]

[2502.14348 (PRL) & 2502.17976 (PRD)
Talk by Fernando Romero-López]

Three PGBs at maximal isospin



$$3\pi^+, \quad 3K^+, \quad \pi^+\pi^+K^+, \quad K^+K^+\pi^+$$

- Benchmark system, with simple repulsive dynamics
  - First calculation at physical quark masses
  - Study $J^P = 0^-, 1^+, 2^-$, including subchannels with $\ell = 0,1,2$
  - Compare to expectations from ChPT
- Use CLS ensembles ($\mathcal{O}(a)$ improved Wilson) + GEVP

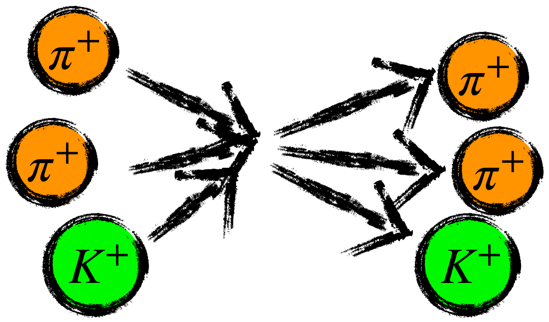
	$(L/a)^3 \times (T/a)$	M_π[MeV]	M_K[MeV]	$N_{\rm cfg}$	$M_\pi L$
N203	$48^3 \times 128$	340	440	771	5.41
N200	$48^3 \times 128$	280	460	1712	4.42
D200	$64^3 \times 128$	200	480	2000	4.20
E250	$96^3 \times 192$	130	500	505	4.05

[Blanton, SRS, et al., PRL 2020 & JHEP 2021] [Draper, SRS, et al., JHEP 2023] (N203, N200, D200)

New work (E250)

$$a \simeq 0.063 \text{ fm}$$

Three PGBs at maximal isospin



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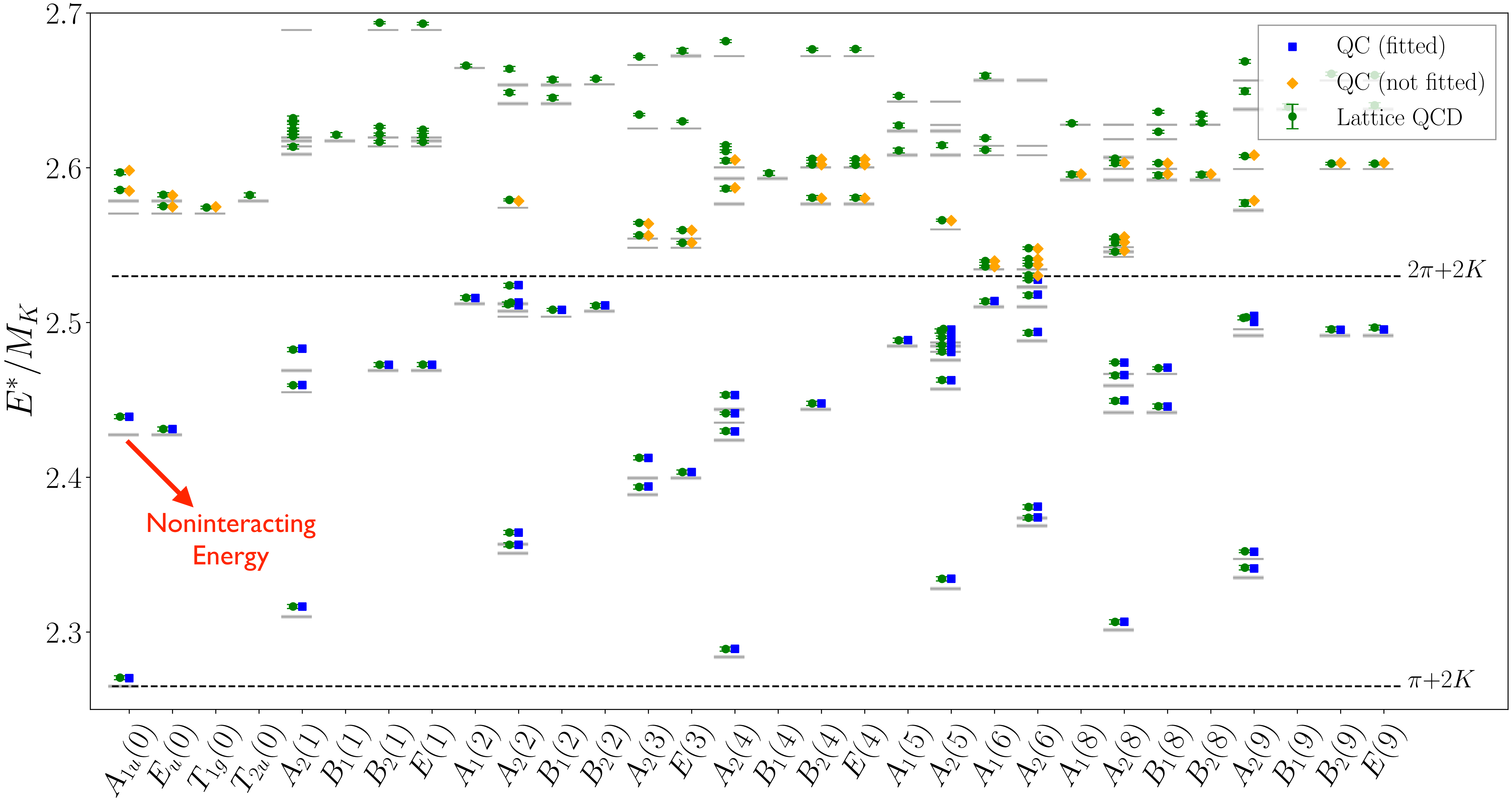
[Blanton, SRS, et al., PRL 2020 & JHEP 2021] [Draper, SRS, et al., JHEP 2023]

New work

$$a \simeq 0.063 \text{ fm}$$

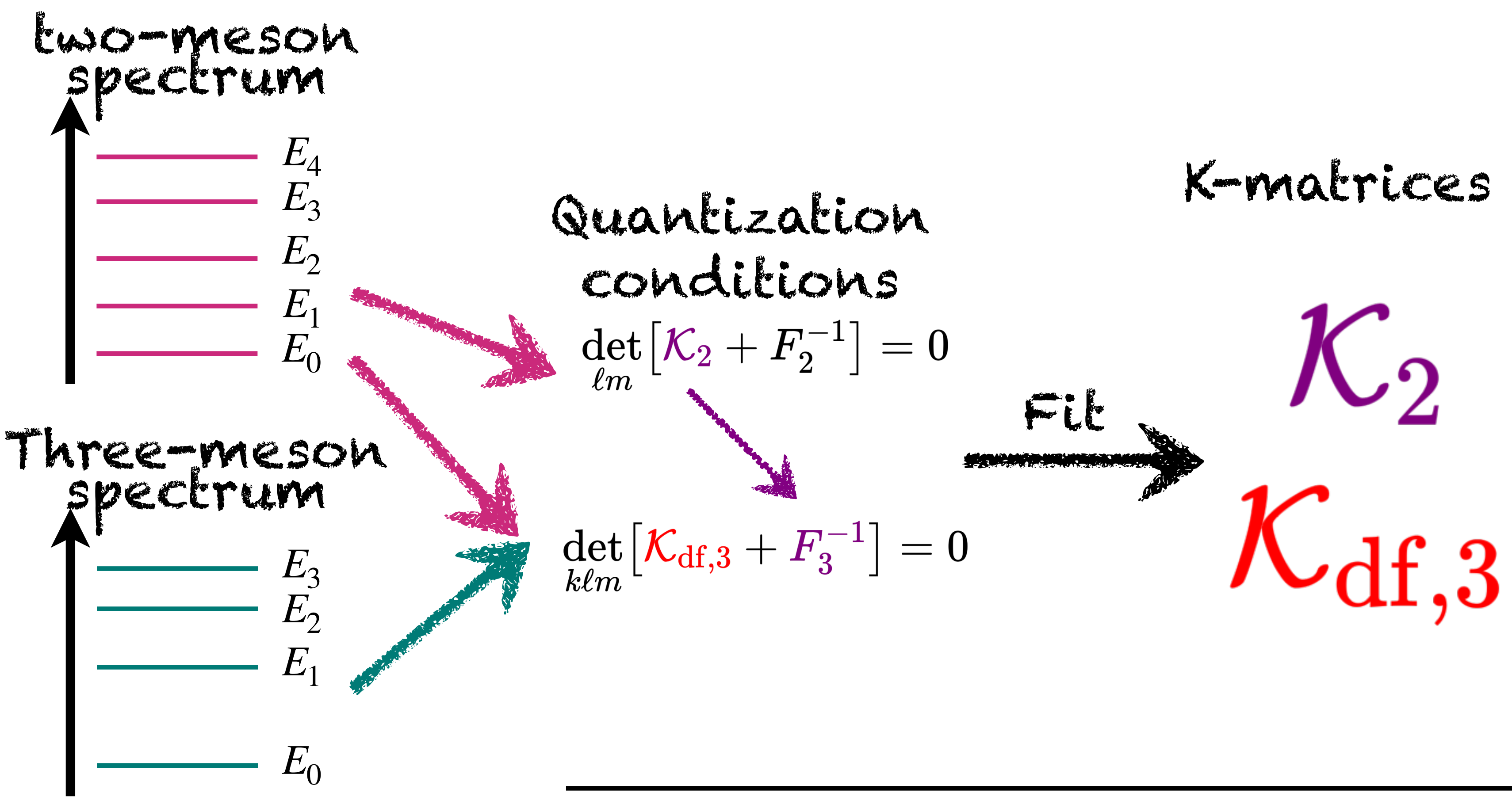
$$\text{tr } m_q = 2m_{ud} + m_s \simeq \text{const}$$

Example of spectrum



E250: $K^+K^+\pi^+$

Fits to spectra



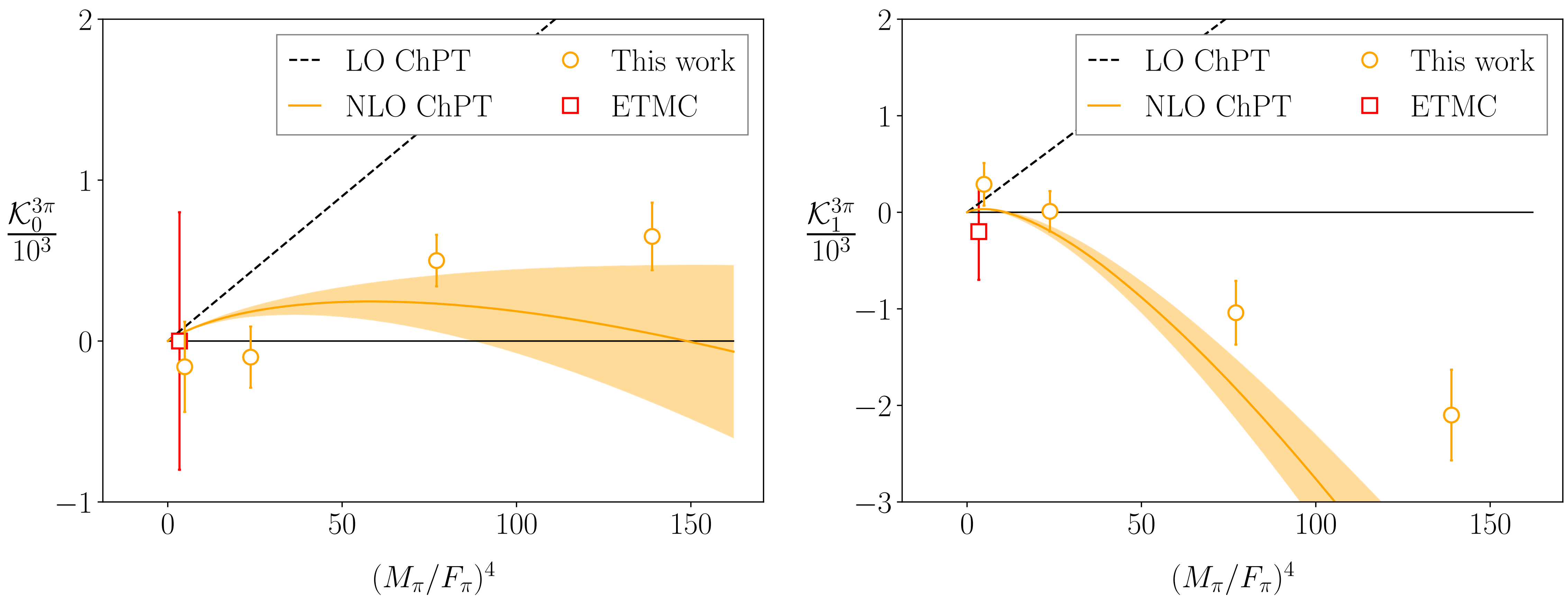
channels	# levels	# parameters	χ^2/DOF
$2\pi/3\pi$	$34+32$	6	1.14
$2\pi/\pi K/2\pi K$	$24+25+23$	9	1.23
$2K/\pi K/2K\pi$	$25+40+50$	10	1.95
$2K/3K$	$40+53$	6	1.49

Parameters in $\mathcal{K}_2$ and $\mathcal{K}_3$: use threshold expansions

Comparing $\mathscr{K}_{\text{df},3}$ to ChPT

- Parametrize $\mathscr{K}_{\text{df},3}$ in a threshold expansion, keeping first two terms
- From fits to spectra, we find non vanishing results for these terms at heavier quark masses
- Compare to NLO Chiral Perturbation theory calculation of $\mathscr{K}_{\text{df},3}$ for $3\pi \to 3\pi$

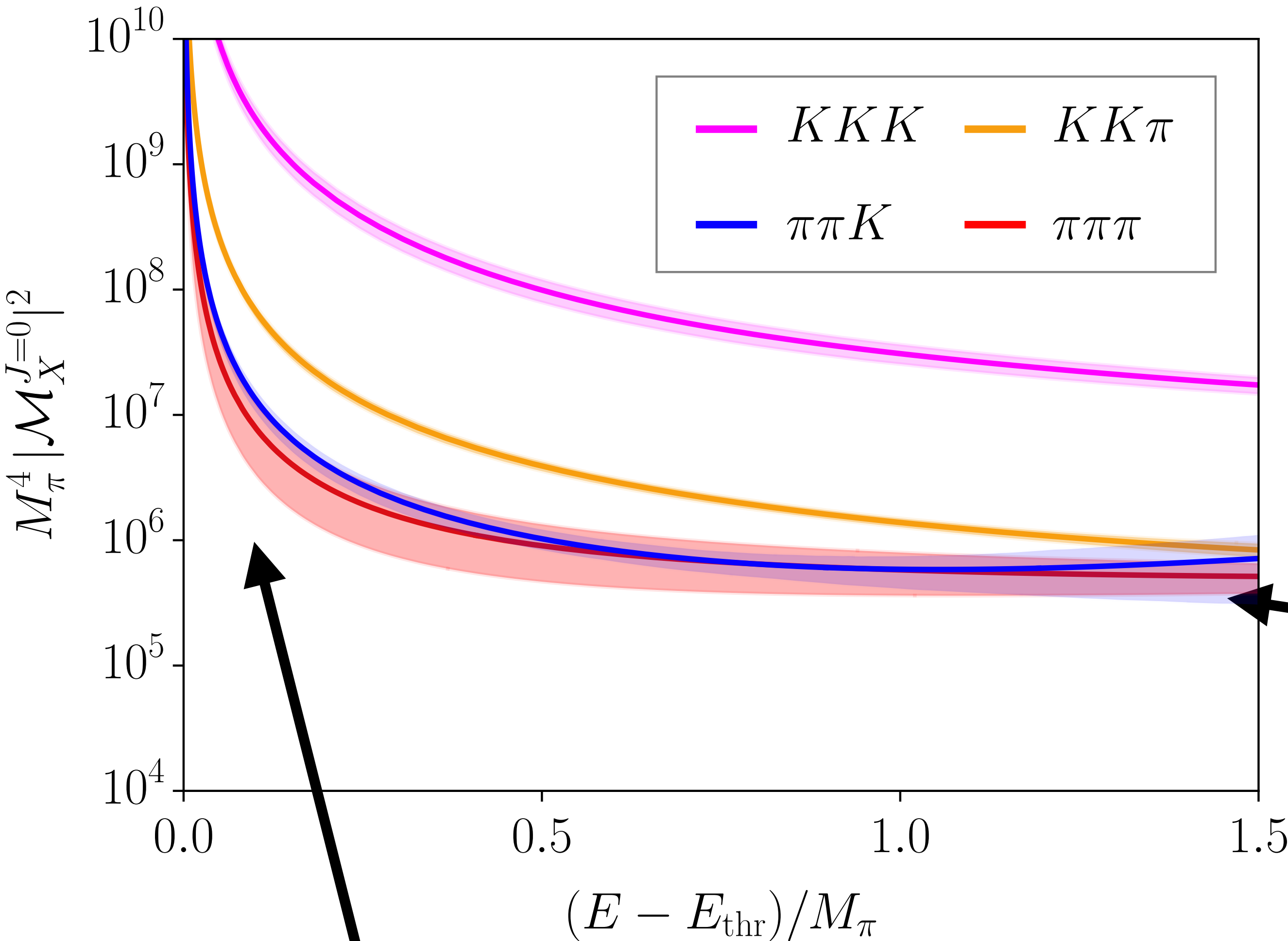
[Baeza-Ballesteros, Bijnens, Husek, Romero-López, SRS, Sjö, 2303.13206 (JHEP) & 2401.14293 (JHEP)]



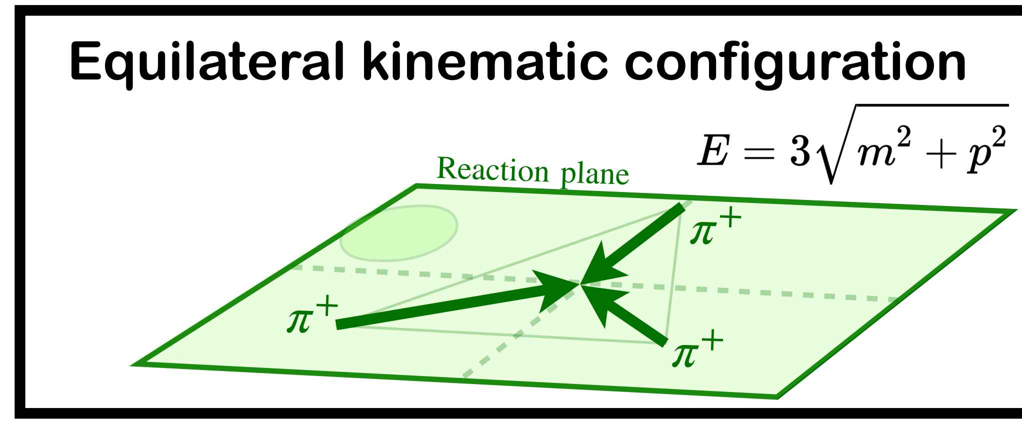
Large NLO corrections in ChPT resolve LO inconsistency

Physical 3-particle amplitudes

$M_\pi^4 |\text{Amplitude}|^2$

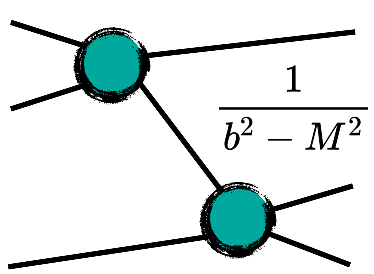


$J^P = 0^-$

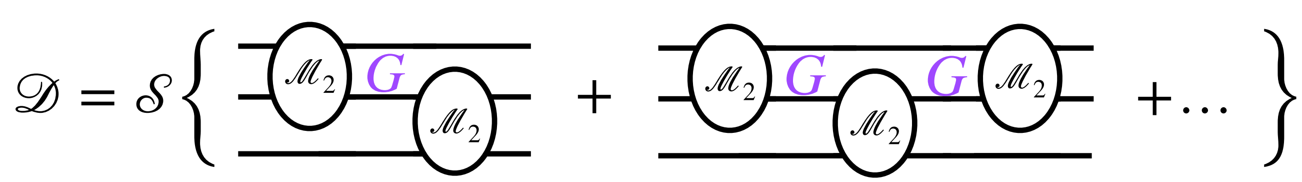


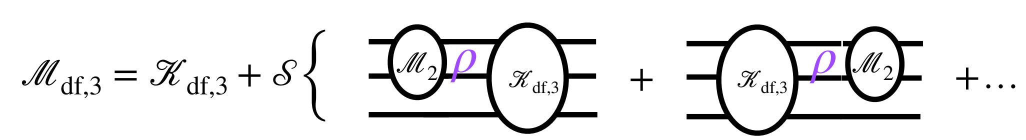
Pion interactions are chirally suppressed

Divergent at threshold



$$\mathcal{M}_3 = \mathcal{D} + \mathcal{M}_{\rm df,3}$$

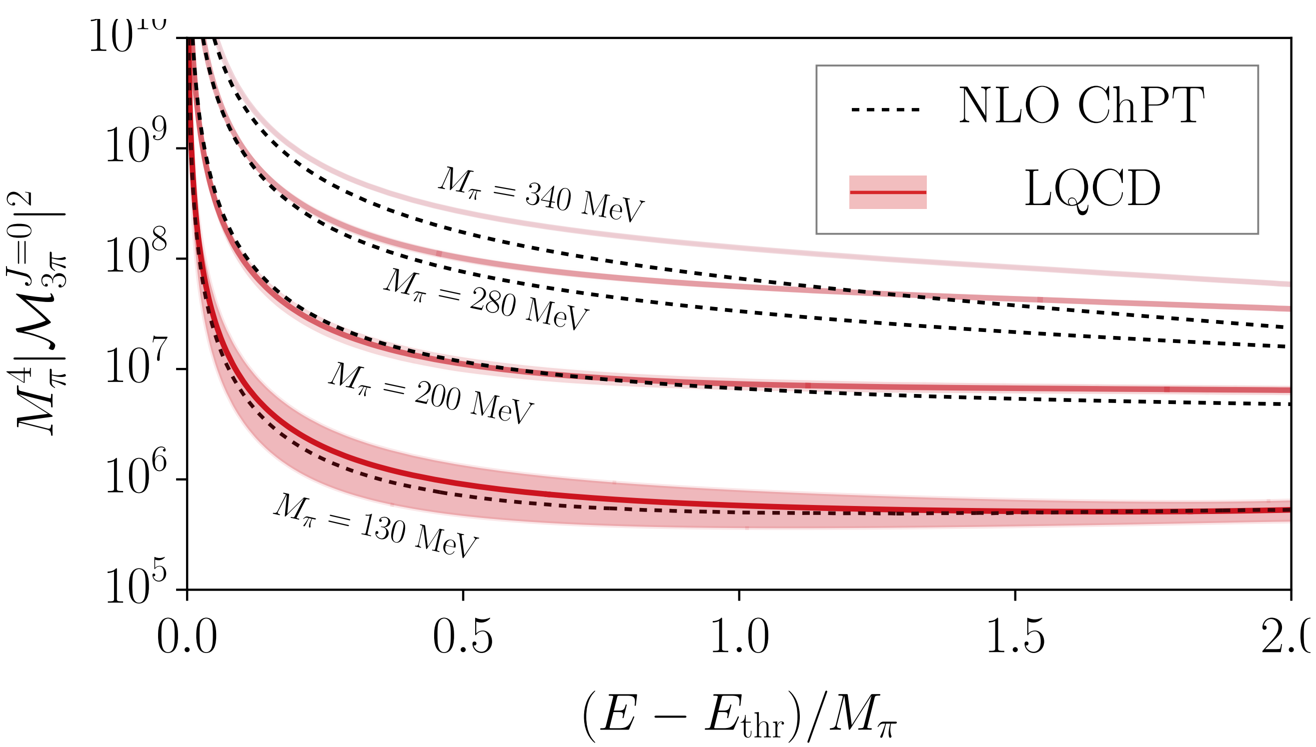


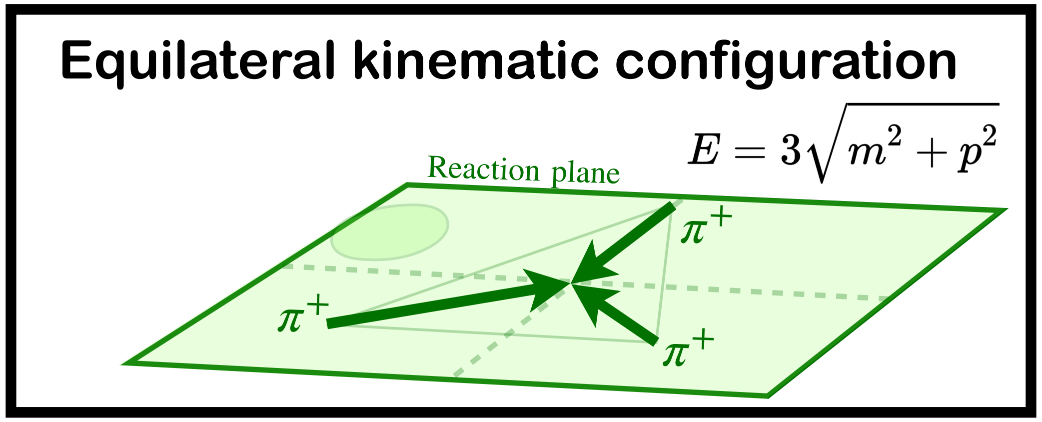


Comparison with ChPT

$M_\pi^4 |\text{Amplitude}|^2$

$J^P = 0^-$





Observe expected breakdown of ChPT as M_π or p_π increase

3-particle resonances

Emergence of the $\pi(1300)$ Resonance from Lattice QCD

Haobo Yan (燕浩波),[1,2,*] Maxim Mai,[3,4,†] Marco Garofalo,[2,‡] Yuchuan Feng,[4,§] Michael Döring,[4,¶] Chuan Liu (刘川),[1,5,6,**] Liuming Liu (刘柳明),[7,8,††] Ulf-G. Meißner,[2,9,10,‡‡] and Carsten Urbach[2,§§]

[2510.09476, talk by Maxim Mai]

ω Meson from lattice QCD

Haobo Yan (燕浩波),[1,2,*] Maxim Mai,[2,3,4,†] Marco Garofalo,[2,‡] Ulf-G. Meißner,[2,5,6,§] Chuan Liu,[1,7,8,¶] Liuming Liu,[9,10,**] and Carsten Urbach[2,††]

[2407.16659, PRL]

Natural candidate for QC3?

$\pi(1300)$ $I^G(J^{PC}) = 1^-(0^{-+})$ [PDGlive]

$\pi(1300)$ MASS	[1]	1300 ± 100 MeV
$\pi(1300)$ WIDTH	[1]	200 to 600 MeV

$\pi(1300)$ **DECAY MODES**

Mode		Fraction (Γ_i / Γ)	Scale Factor/ Conf. Level	P(MeV/c)
Γ_1	$\rho\pi$	seen		404
Γ_2	$\pi(\pi\pi)_{S-\text{wave}}$	seen		
Γ_3	$\gamma\gamma$			650

[1] Our estimate. See the Particle Listings for details.

Pro: s-wave decay to 3π implies strong 3-particle coupling

Con: Lies well above the inelastic threshold for QC3

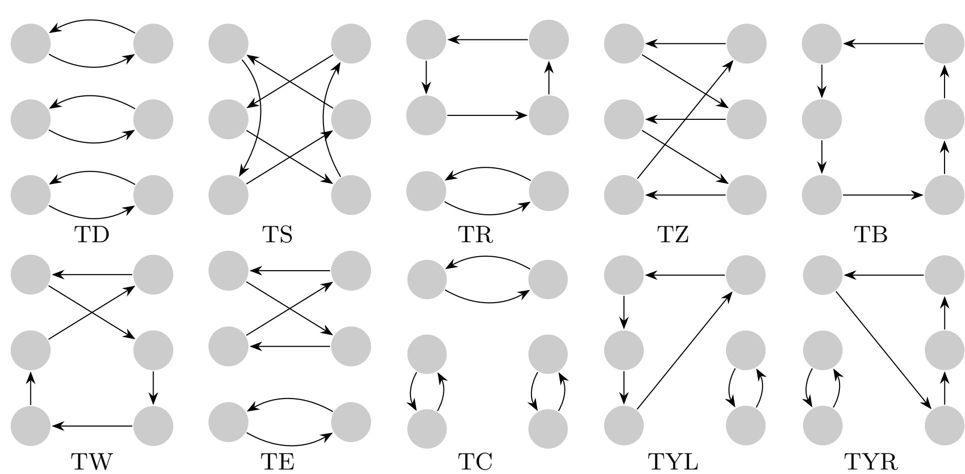
- $M(\pi(1300)) \approx 10M_\pi \gg 5M_\pi$
- QC3 will apply for heavier than physical quark masses

Simulation details

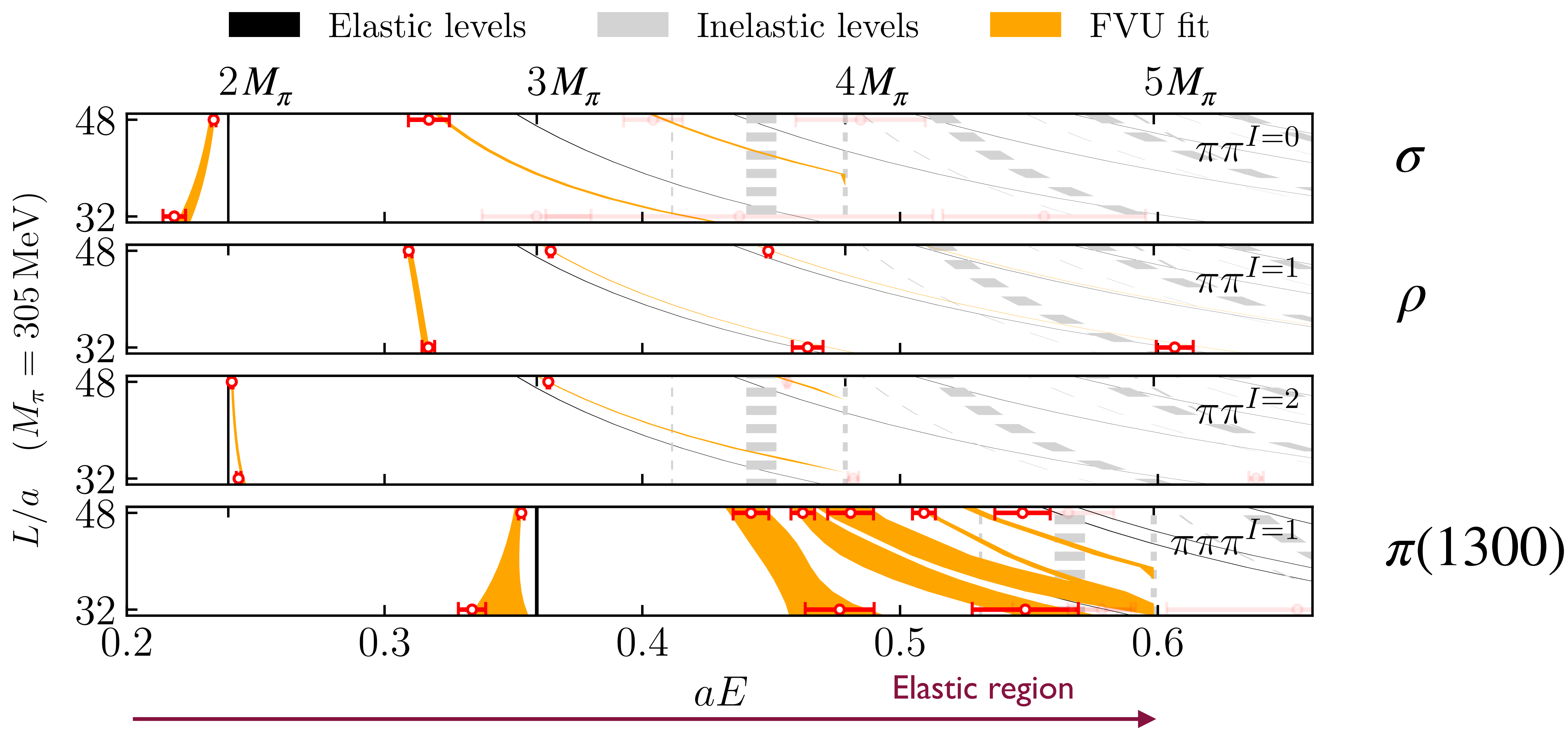
- Use CLQCD ensembles (tadpole improved Clover), fixed m_s, $a = 0.07746(18)$fm
  - Rest-frame only and single irrep

ens.	size	M_π(MeV)	M_K(MeV)	$M_\pi L$	# 2π levels	# 3π levels
F32P30	$32^3 \times 96$	303	525	3.8	5	3
F48P30	$48^3 \times 96$	303	524	5.7	7	6
F32P21	$32^3 \times 64$	211	492	2.7	5	0
F48P21	$48^3 \times 96$	207	493	3.9	7	3

- Large basis of operators: π, $\rho\pi$, $\sigma\pi$, 3π for three-particle case
  - Large set of contractions, e.g.



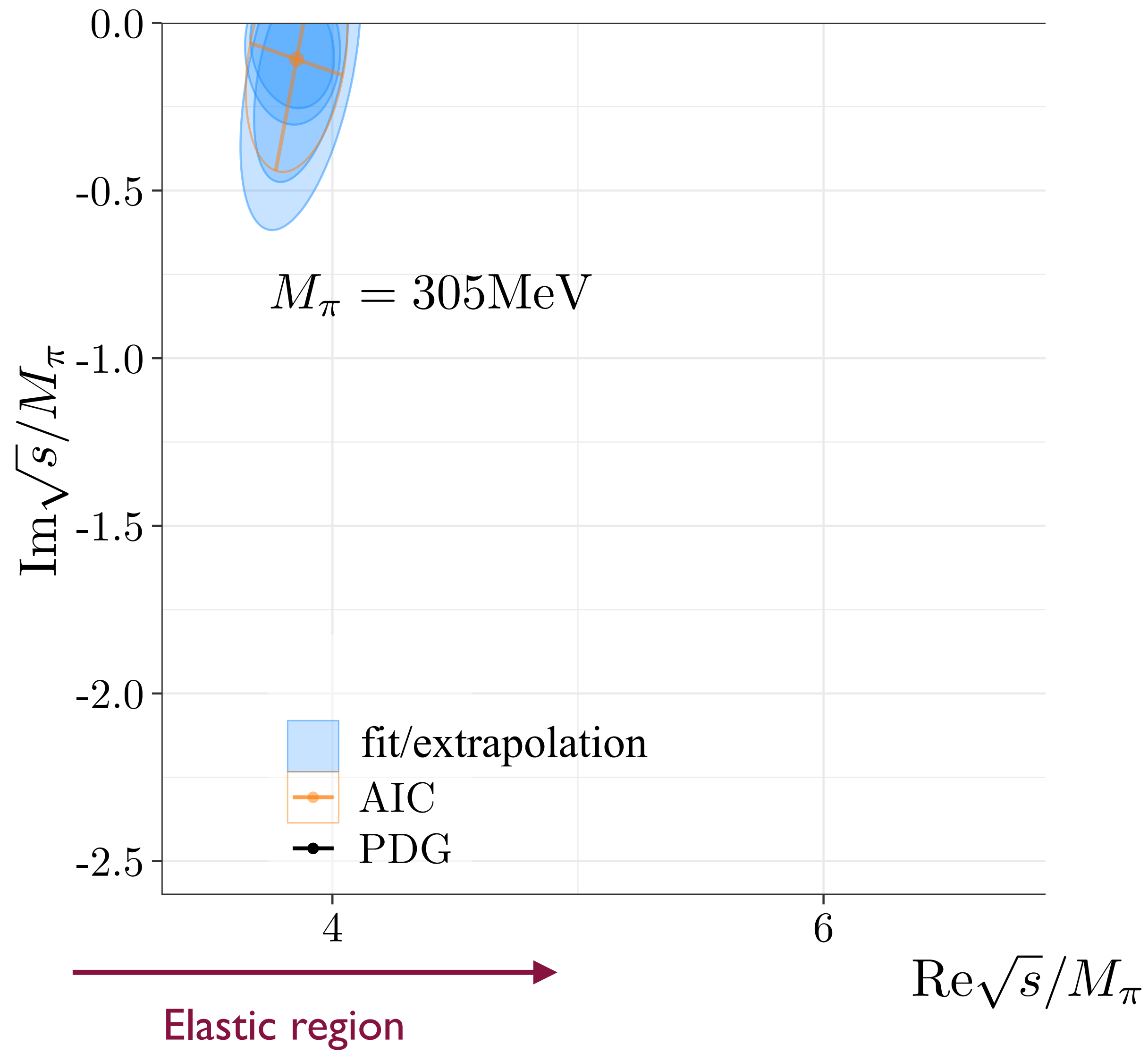
Fits



- Best global fit to both masses and volumes
  - 9 params (3 for 2π, and 6 for 3π) to 36 levels, with $\chi^2/\text{dof} = 1.08$
  - Final results from model averaging with Akaike information criterion

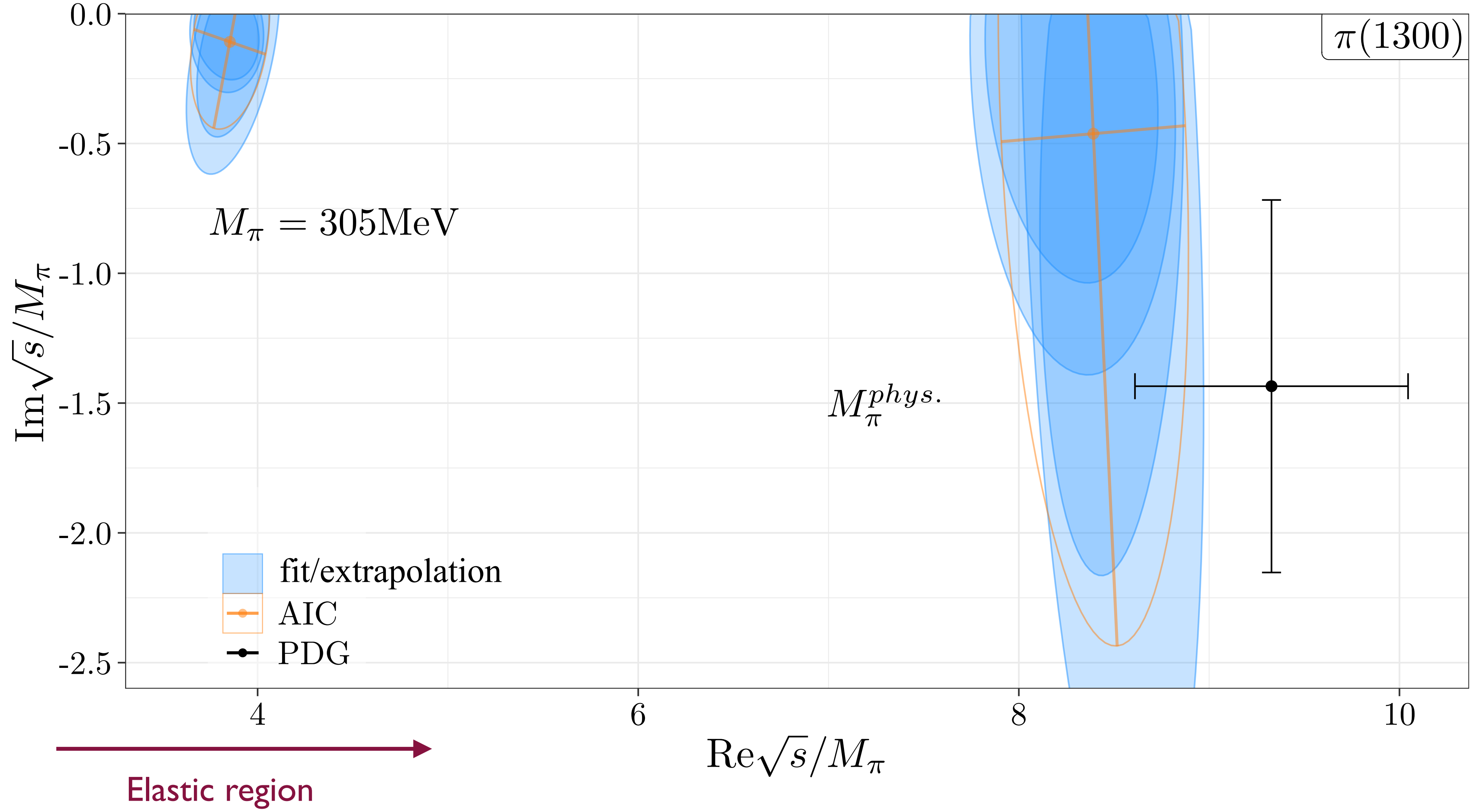
Pole positions

- Solve IVU (=infinite-volume unitarity) form of integral equations
  - Find resonance pole for $M_\pi = 305\,\mathrm{MeV}$!



Pole positions

- Solve IVU (=infinite-volume unitarity) form of integral equations
  - Find resonance pole for $M_\pi = 305\,\text{MeV}$!
- Extrapolate to physical masses using EFT-inspired form



Using three-particle methods to study the $T_{cc}^+(3875)$

Incorporating DDπ effects and left-hand cuts in lattice QCD studies of the $T_{cc}(3875)^+$

Maxwell T. Hansen,[a] Fernando Romero-López[b] and Stephen R. Sharpe[c]

[2401.06609, JHEP]

Finite- and infinite-volume study of $DD\pi$ scattering

Sebastian M. Dawid,[a] Fernando Romero-López[b,c] and Stephen R. Sharpe[a]

[2409.17059, JHEP]

Comparison of integral equations used to study T_{cc}^+ for a stable D^*

Sebastian M. Dawid,[a] Fernando Romero-López[b] and Stephen R. Sharpe[a]

[2505.05466, JHEP]

+ work in progress [talk by André Baião Raposo]

Doubly-charmed tetraquark

NATURE PHYSICS | VOL 18 | JULY 2022 | 751–754 | www.nature.com/naturephysics

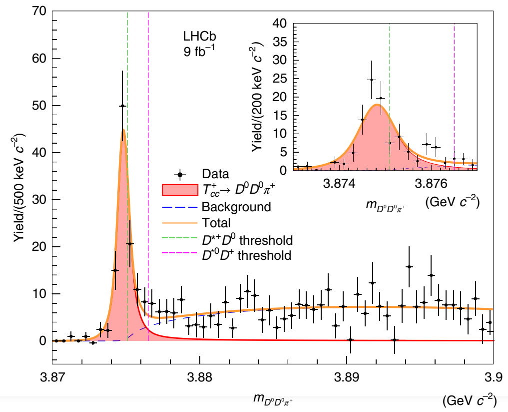
Observation of an exotic narrow doubly charmed tetraquark

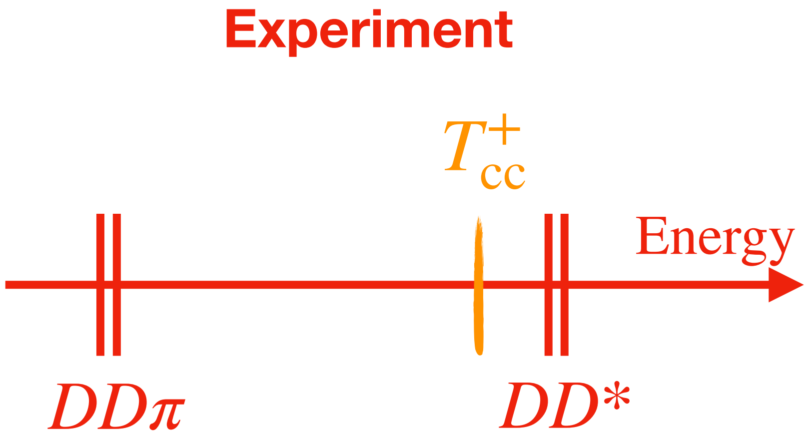
LHCb Collaboration*

$T_{cc}^{+}(3875)$

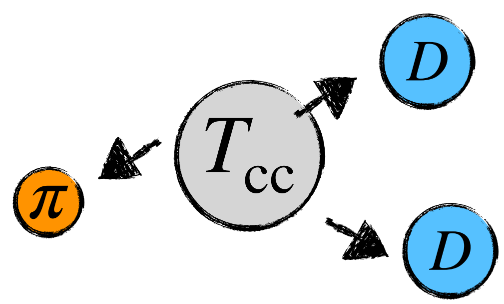
$I = 0,\ J^P = 1^+$

$cc\bar{u}\bar{d}$





▶ **For physical quark masses is a three-body resonance**



need three-body formalism!

Adapted from: Fernando Romero-López

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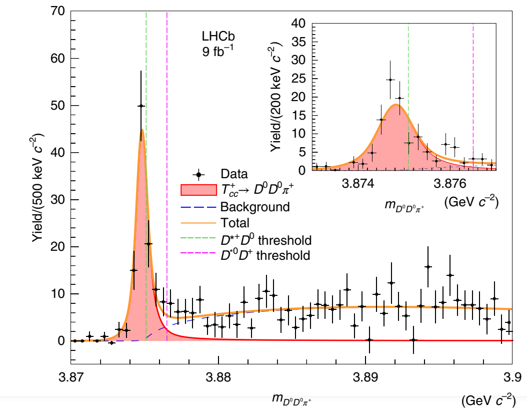
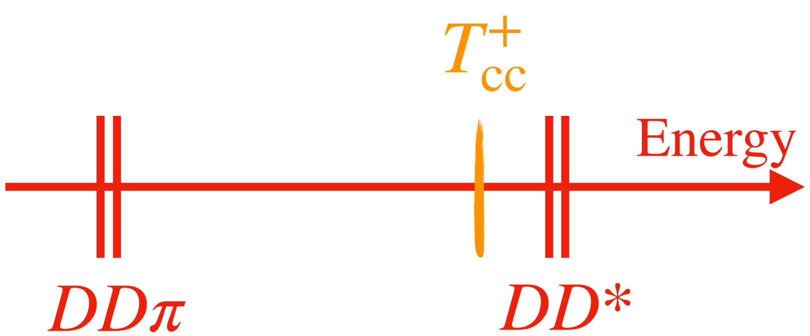


Experiment



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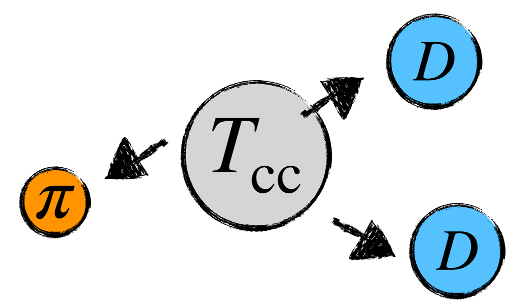
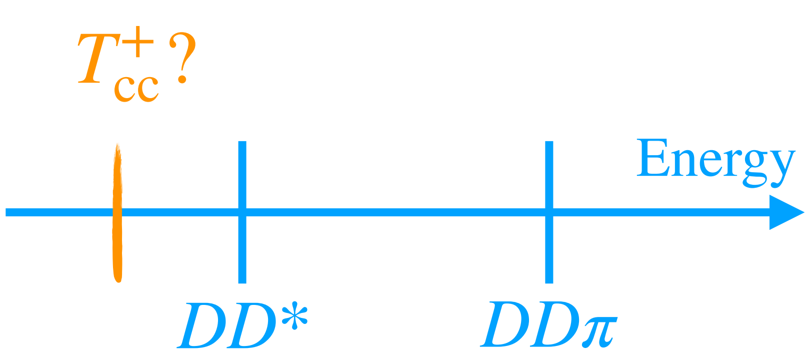
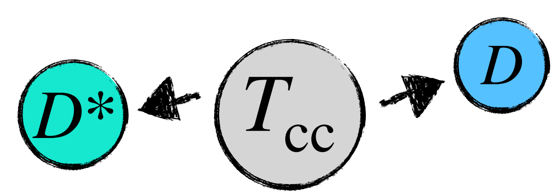


need three-body formalism!

LQCD calculations with heavier-than-physical quarks



▶ **D* is stable for slightly heavier-than-physical quarks**



suitable for the two-body finite-volume formalism?

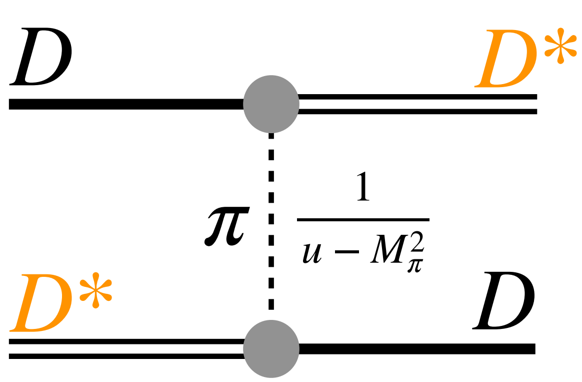
Adapted from: Fernando Romero-López

DD^* scattering

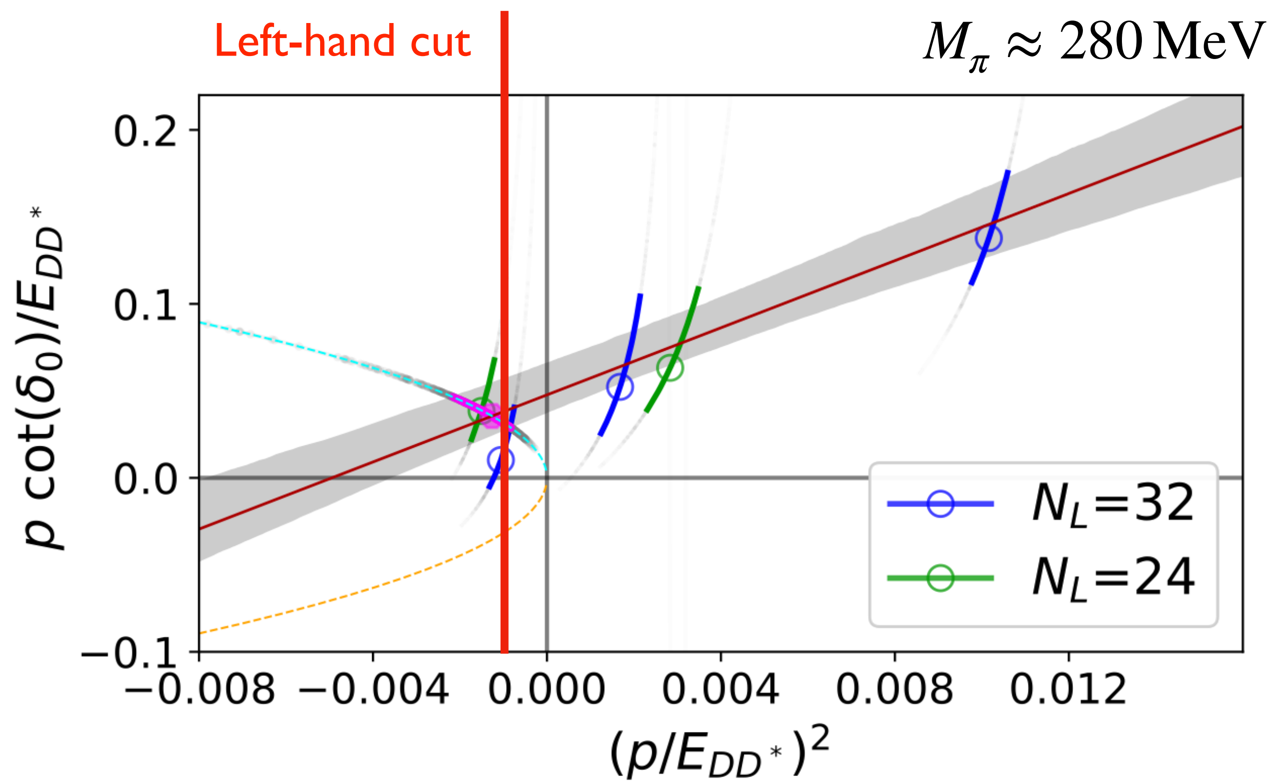
- Several LQCD studies using QC2 in T_{cc} channel

  [Chen et al., 2206.06185]
  [Lyu et al. (HALQCD), 2302.04505]
  [Padmanath & Prelovsek, 2202.10110]
  [Whyte, Thomas, Wilson, 2405.15741]
  [Several talks at Lattice 2025]

  - Use heavier-than-physical quarks
  - Find signature of virtual bound state
  - But QC2 fails at left-hand cut, below which phase shift becomes complex



[Padmanath, Prelovsek, arXiv:2202.10110]



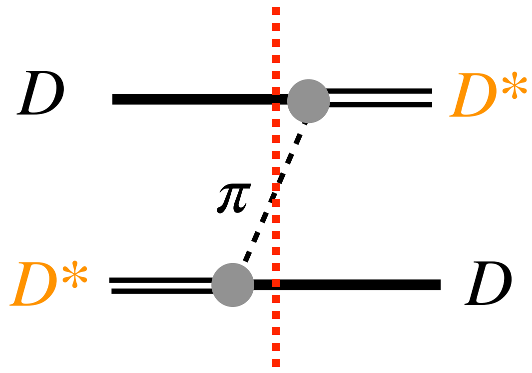
- Several solutions to failure have been proposed, by generalizing the QC2

  [Du et al (2408.09375); Abolnikov et al. (2407.04649), Bubna et al. (2402.12985); Meng et al. (2312.01930); Raposo, Hansen (2311.18793); Raposo, Hansen, Briceño, Jackura (2502.19375)]

3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

- Left-hand cut is a three-body effect



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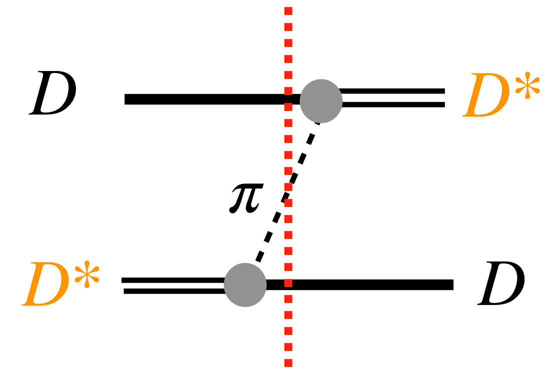
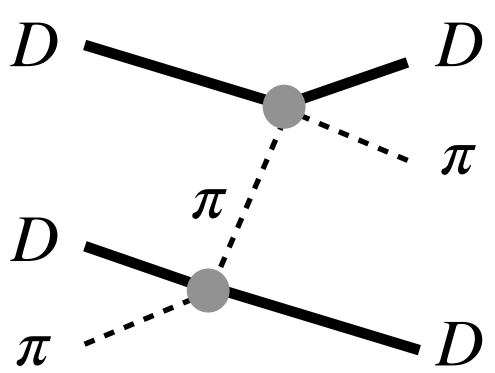


- Use $DD\pi$ QC3, but include D^* as a bound state in p-wave $D\pi$ channel



3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

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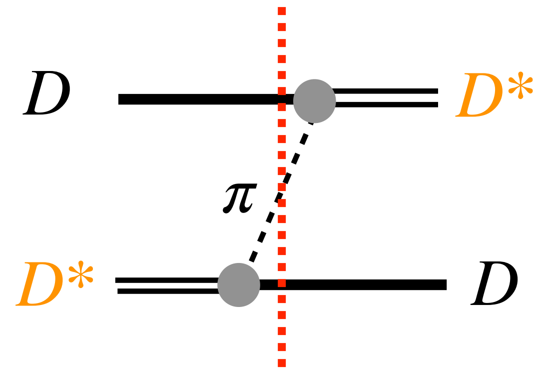

- Use $DD\pi$ QC3, but include D^* as a bound state in p-wave $D\pi$ channel

  - Finite-volume effects from u-channel pion exchange naturally incorporated

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[Hansen, Romero-López, SRS, 2401.06609, JHEP]

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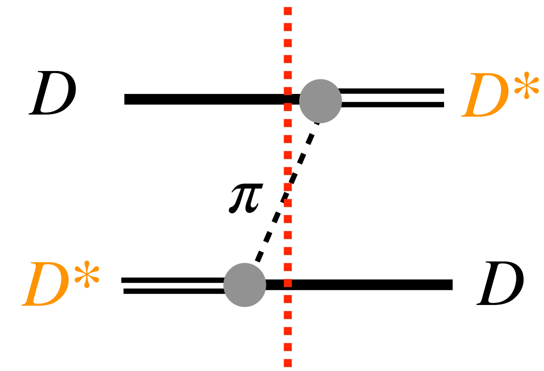

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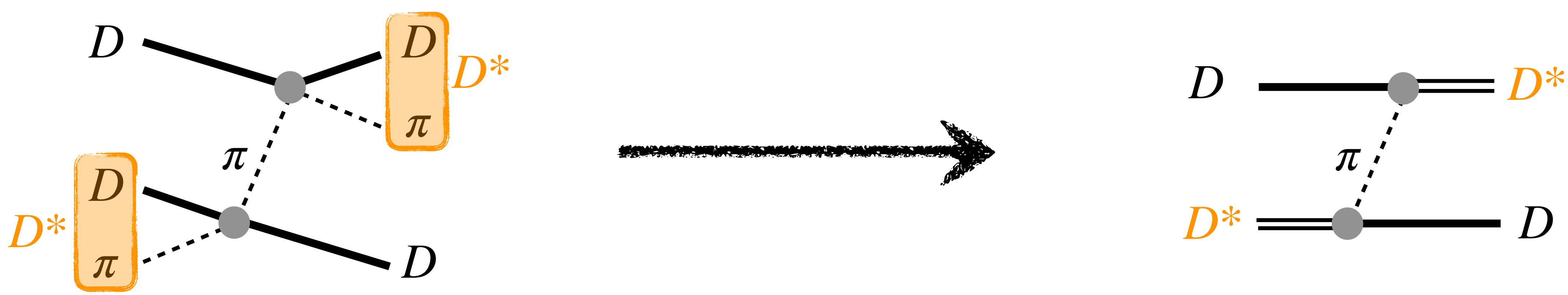


  - Finite-volume effects from u-channel pion exchange naturally incorporated

- Another advantage of QC3 approach:
  - Works for all choices of quark masses, including physical case of unbound D^*

- Successfully implemented in [Dawid, Romero-López, SRS, 2409.17059, JHEP & 2505.05466, JHEP; plus work in progress (talk by André Baião Raposo)]

Towards a formalism for the Roper

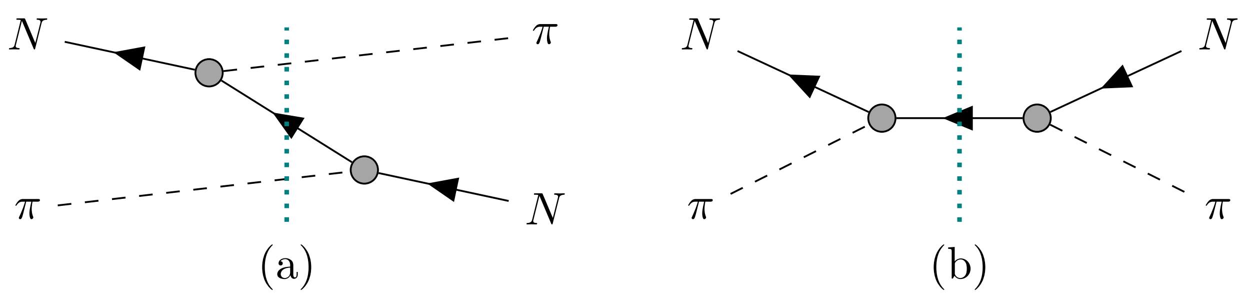
Finite-volume formalism for $N\pi\pi$ at maximal isospin

Maxwell T. Hansen[a] , Fernando Romero-López[b] , and Stephen R. Sharpe[c]

[2509.24778, talk by SRS]

A failed hope for studying N(1440)

- Need a formalism that accommodates $N\pi + N\pi\pi$
  - Can one include N as a pole in p-wave $N\pi$ scattering, and use the $N\pi\pi$ formalism?
  - No! Nucleon pole lies below the singularity due to u-channel nucleon exchange



$$\sigma_{N\pi} = M_N^2 + 2M_\pi^2 \qquad \sigma_{N\pi} = M_N^2$$

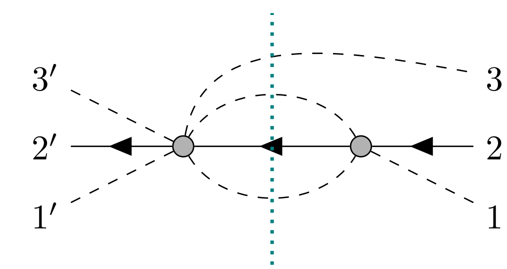
- Formalism for $N\pi\pi$ alone only valid at maximal isospin, which forbids mixing with $N\pi$
  - Interesting dynamics due to $\Delta\pi$ subchannel
  - Provides one 3-particle block of arbitrary isospin $N\pi + N\pi\pi$ formalism

Parameter space in QC3

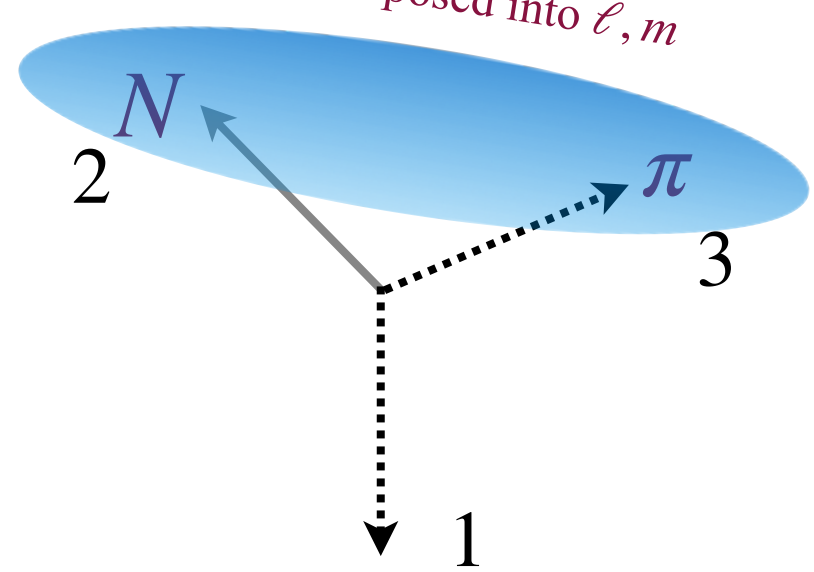
QC3 for $N\pi\pi$: $$\det_{ip\ell m m_s}\left(1+\widehat{\mathcal{K}}_{\mathrm{df},3}(E^\star)\widehat{F}_3(E,\boldsymbol{P},L)\right)=0$$

Matrix indices are:
channel, spectator momentum,
pair angular momentum,
lab frame nucleon spin

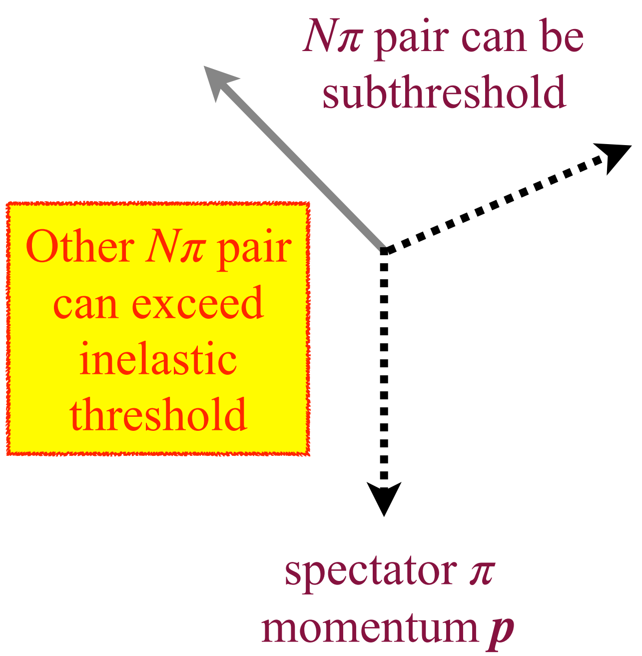
- A key quantity in derivation is B_3, $3\to 3$ Bethe-Salpeter kernel (3 Particle Irreducible)
  - Must be nonsingular in range of kinematic parameters
  - Otherwise introduce uncontrolled L^{-n} finite-volume effects



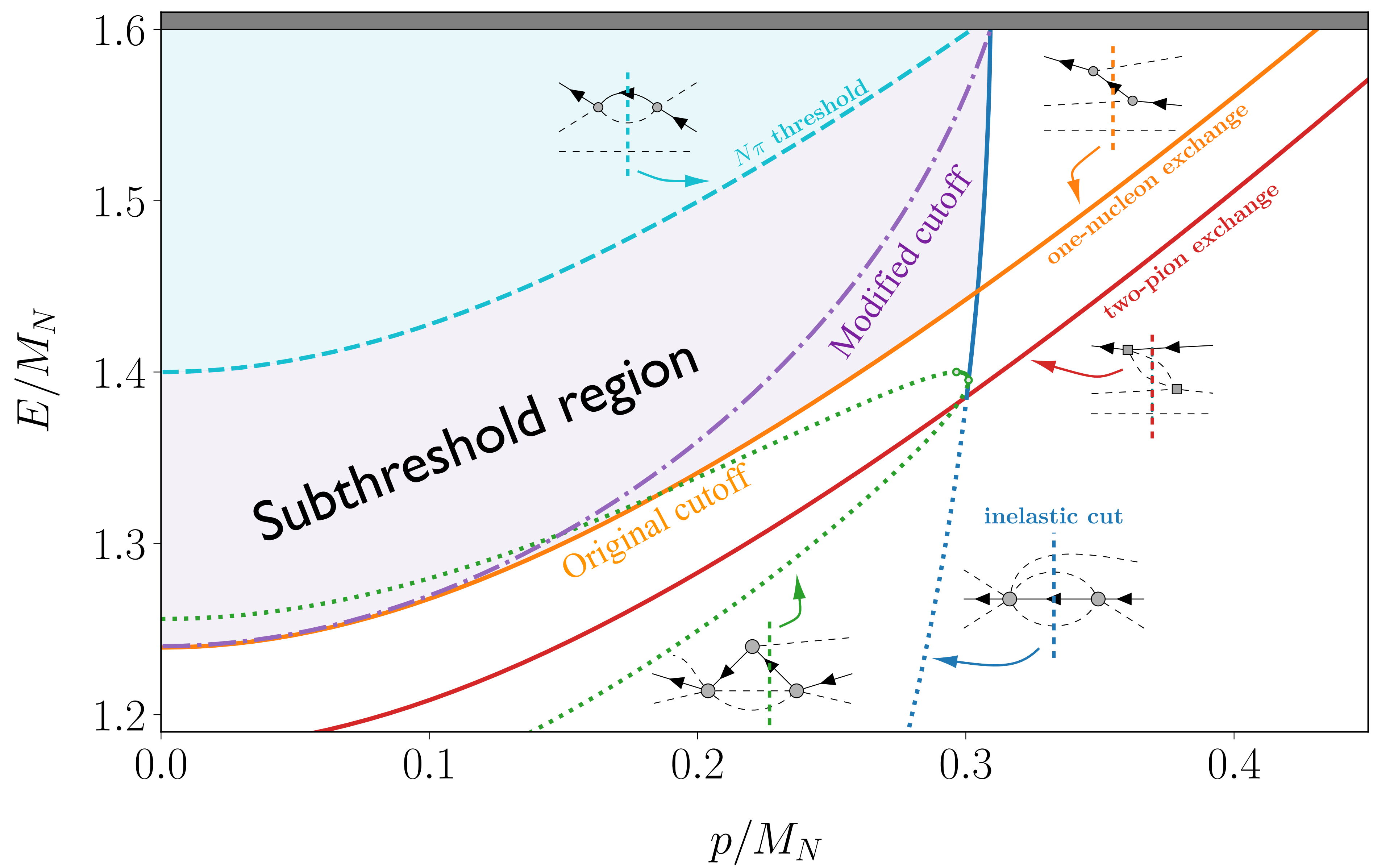
- Parameter space (for pion spectator)
  - Energy E, momentum $\boldsymbol{P}$



Subthreshold singularities



Parameter space for $\boldsymbol{P} = 0$, $M_\pi/M_N = 0.2$



Dreaming of 3 neutrons

Three relativistic neutrons in a finite volume

Zachary T. Draper,[a] Maxwell T. Hansen,[b] Fernando Romero-López[c] and Stephen R. Sharpe[a]

[2303.10219, JHEP]

Implementation of the three-neutron quantization condition

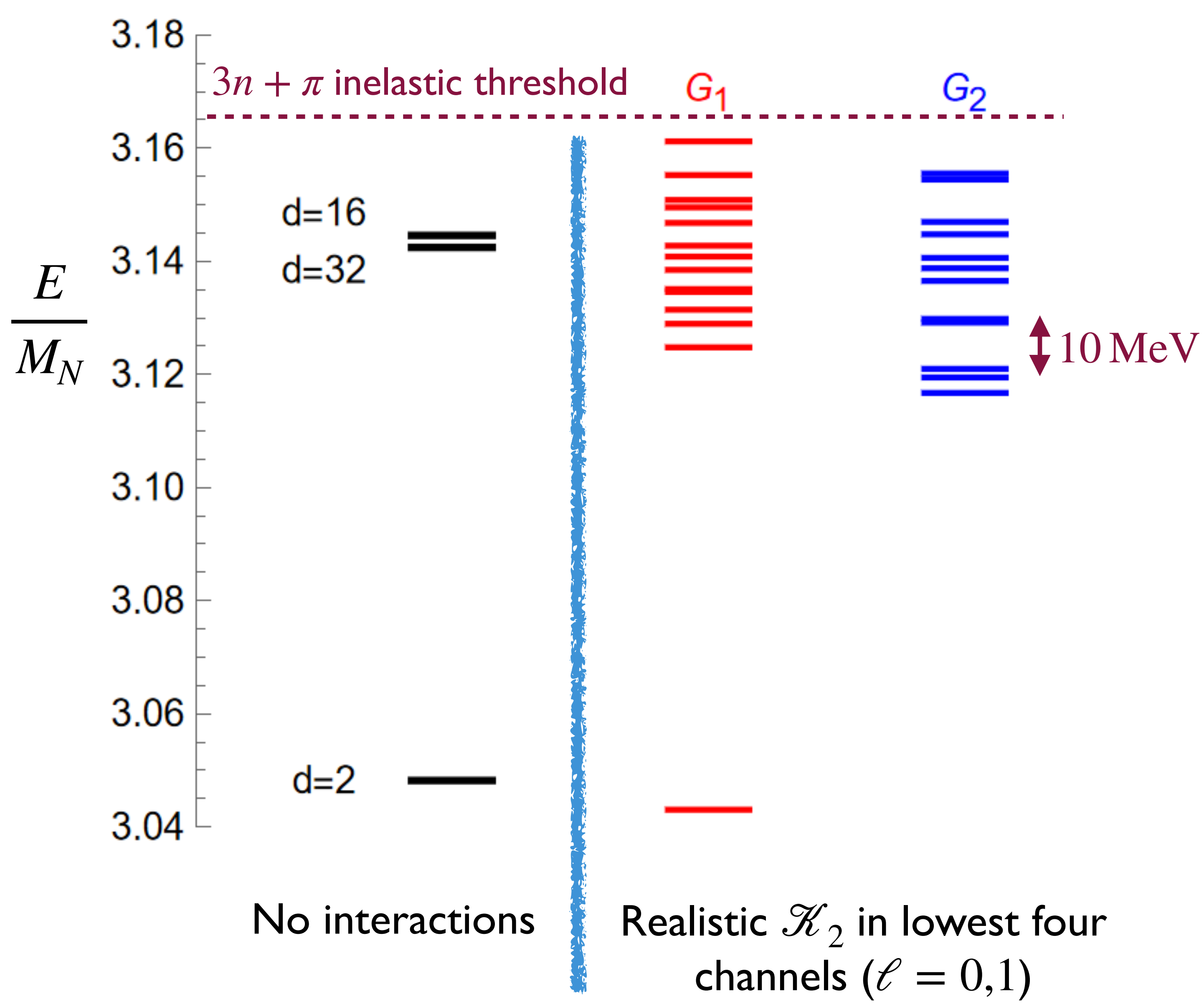
Wilder Schaaf[a] and Stephen R. Sharpe[a]

[2410.14037 (Lattice 24) & in progress]

Spectrum in realistic setup

- Physical masses, $L \approx 4.3$ fm:
  - $M_N L = 20,\ M_\pi = 0.15 M_N \Rightarrow M_\pi L = 3$

Levels in $\boldsymbol{P} = \frac{2\pi}{L}(0,0,1)$ frame



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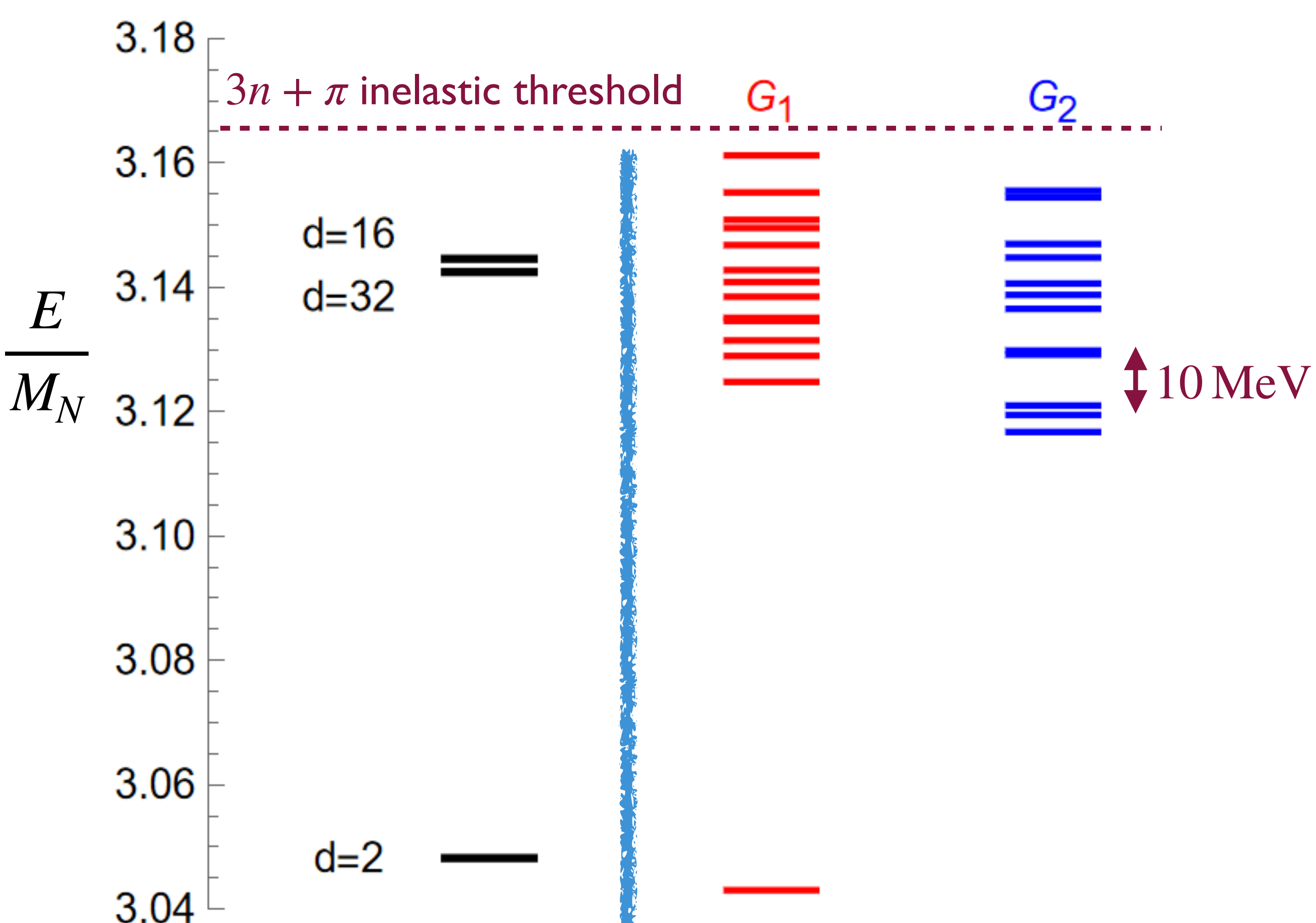
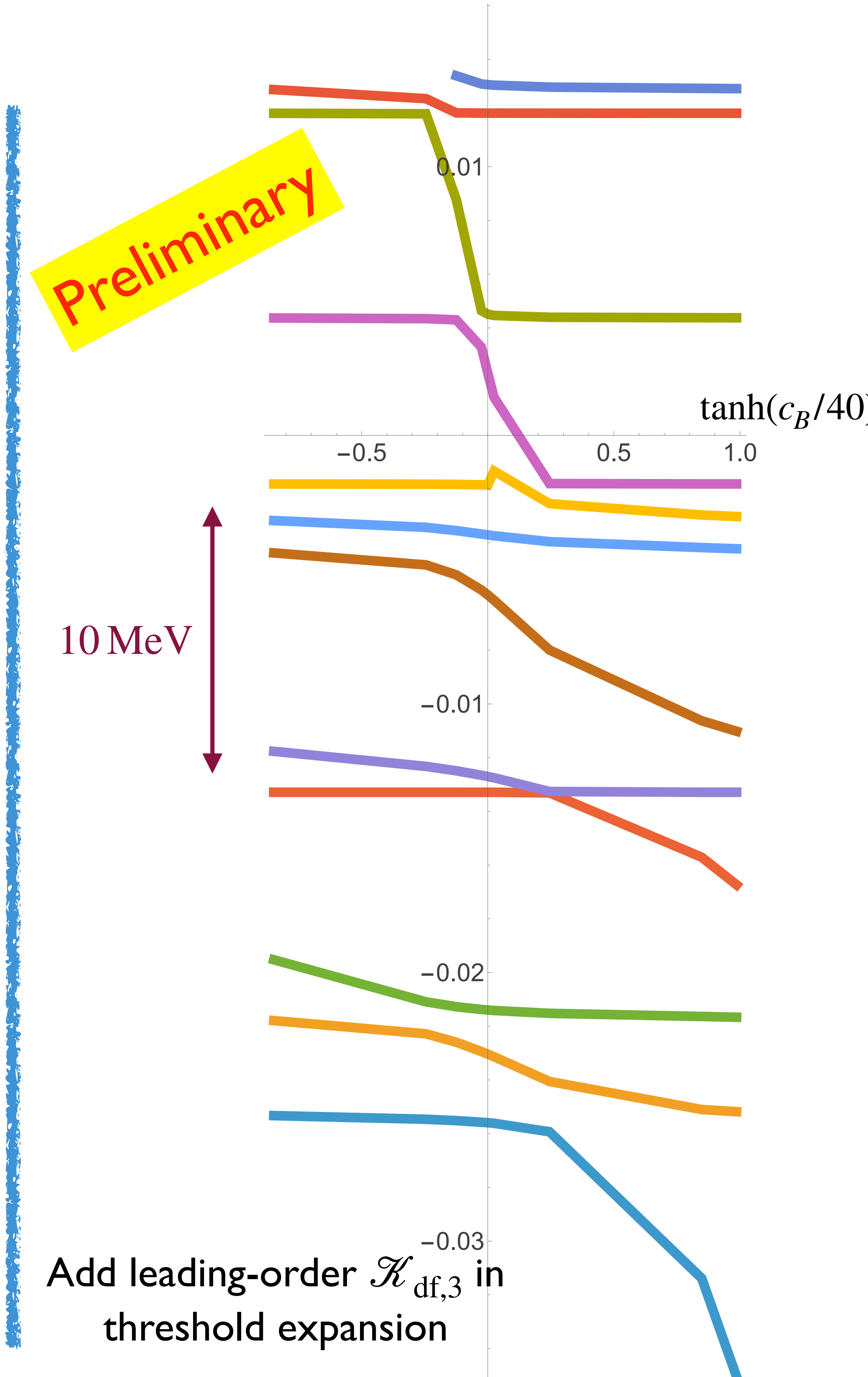


No interactions | Realistic $\mathcal{K}_2$ in lowest four channels ($\ell = 0,1$)



Add leading-order $\mathcal{K}_{\mathrm{df},3}$ in threshold expansion

Summary & Outlook

Summary & Outlook

- Major steps have been taken in the three-particle sector
  - Formalism well established & cross checked, and almost complete
  - Several pioneering end-to-end applications to three-particle spectra from LQCD
    - Another one appeared last Thursday! $3\pi(I=2) \leftrightarrow \rho\pi$ [Briceño et al. (HADSPEC) 2510.24894]
  - Path to a calculation of $K \to 3\pi$ decay amplitudes is now open
- Near-term steps in implementation
  - $T_{cc}^{+} \to D^{*}D \to DD\pi$ for lighter quark masses
  - Extended calculations for $3\pi(I=0,1,2)$
  - $N\pi\pi \leftrightarrow \Delta\pi(I=5/2)$
- Final steps in three-particle formalism
  - $3\pi(I=0) \leftrightarrow K\overline{K}$ (WZW term)
  - $NNN(I=\frac{1}{2}), N\pi\pi + N\pi$ [for Roper], $NN\pi + NN$

Challenges & Future

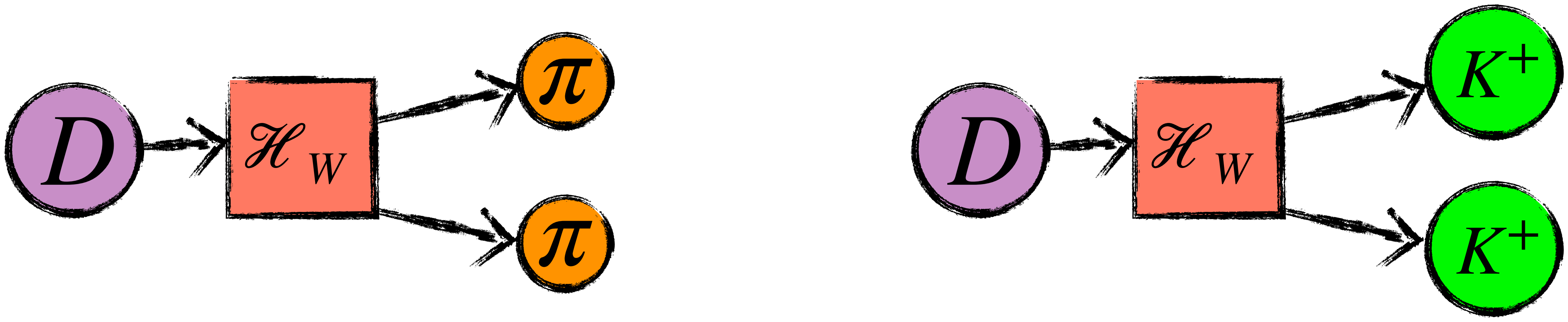
- Reducing model-dependence in fitting & analytic continuation
  - [Salg, Romero-López, Jay, 2506.16161] propose using Bayesian reconstruction + Nevanlinna interpolation
- Controlling all systematic errors (when FLAG-ready?)
  - For which processes is precision important?
  - For three-nucleon interaction even a semiquantitative result would be useful
- Comparing formalisms in detail (e.g. RFT vs FVU)
  - Equivalent up to technical details (e.g. smooth vs hard cutoff functions)
  - Share data for cross-checking analysis? Bootcamp?
- Moving to more particles (needed, e.g., for $D \to \pi\pi, K\overline{K}$ & physical $\pi(1300)$)
  - QC4 (which is not yet known except in threshold expansion) may be practical [Talk by Raj Mukerjee]
  - QCN seems impractical—combine QC2-4 with inclusive methods in some yet-to-developed manner?
    - "Scattering amplitudes from finite-volume spectral functions" [Bulava & Hansen, 1903.11735]
    - Squeeze as much information as we can out of the lattice simulations!
- Investigate alternative approaches
  - "Scattering amplitudes from Euclidean correlators: Haag-Ruelle theory and approximation formulae", [Patella & Tantalo, 2407.02069]
  - HALQCD approach? [Doi et al., 1106.2276: 3 nucleon potentials in NR regime]
- Combine LQCD results with EFT and amplitude-analysis methods (Exohad collab.)

Thank you!
Questions?

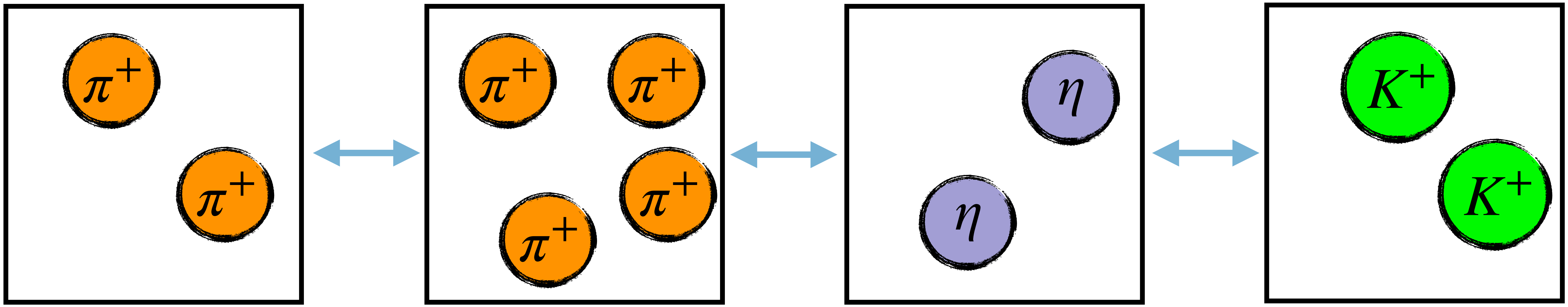
Backup slides

Long-term dream

CP violation in D decays

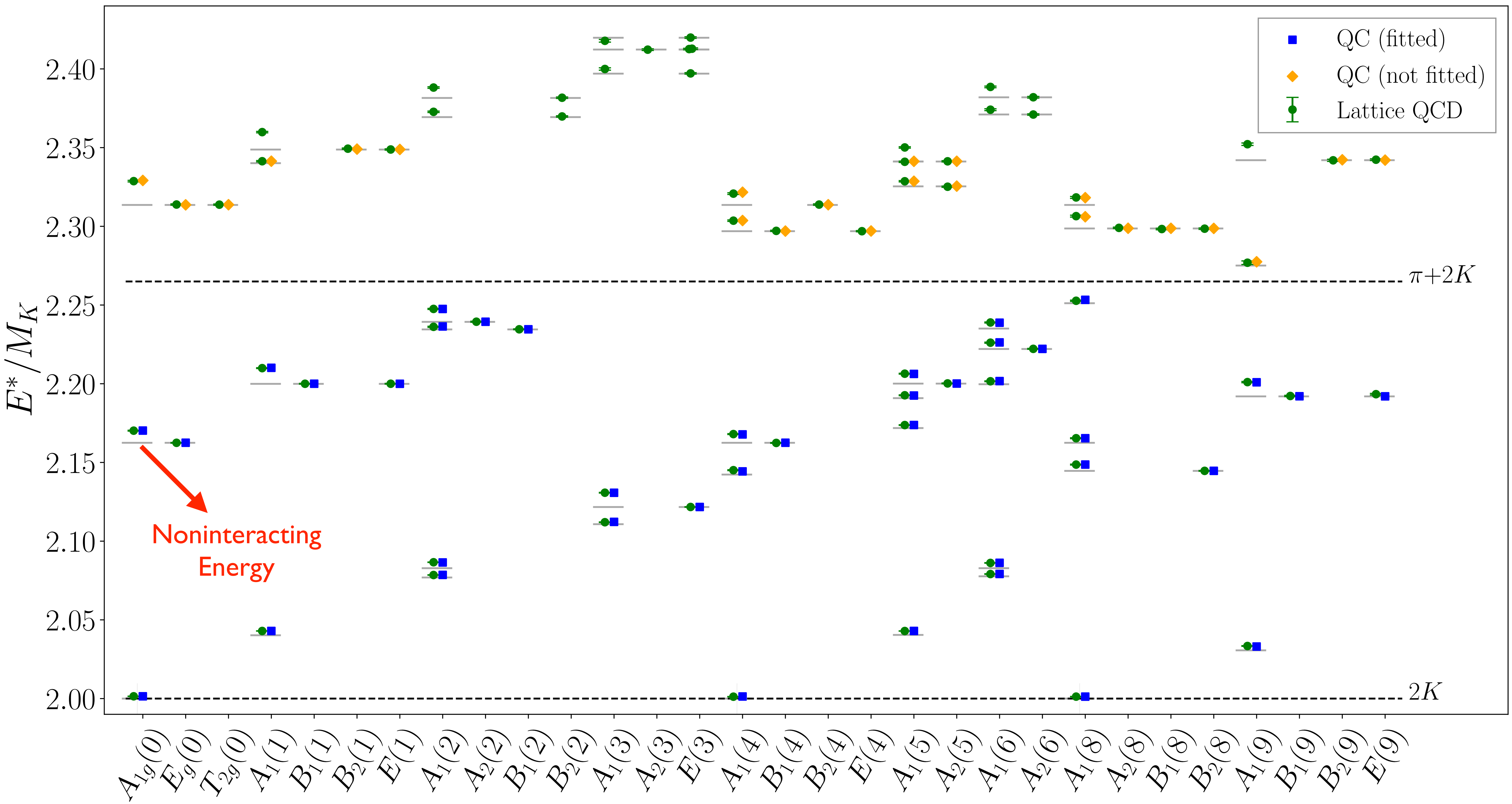


Challenge: finite-volume mixing with $4\pi, 6\pi, \eta\eta, \ldots$



4+-particle formalism not yet developed

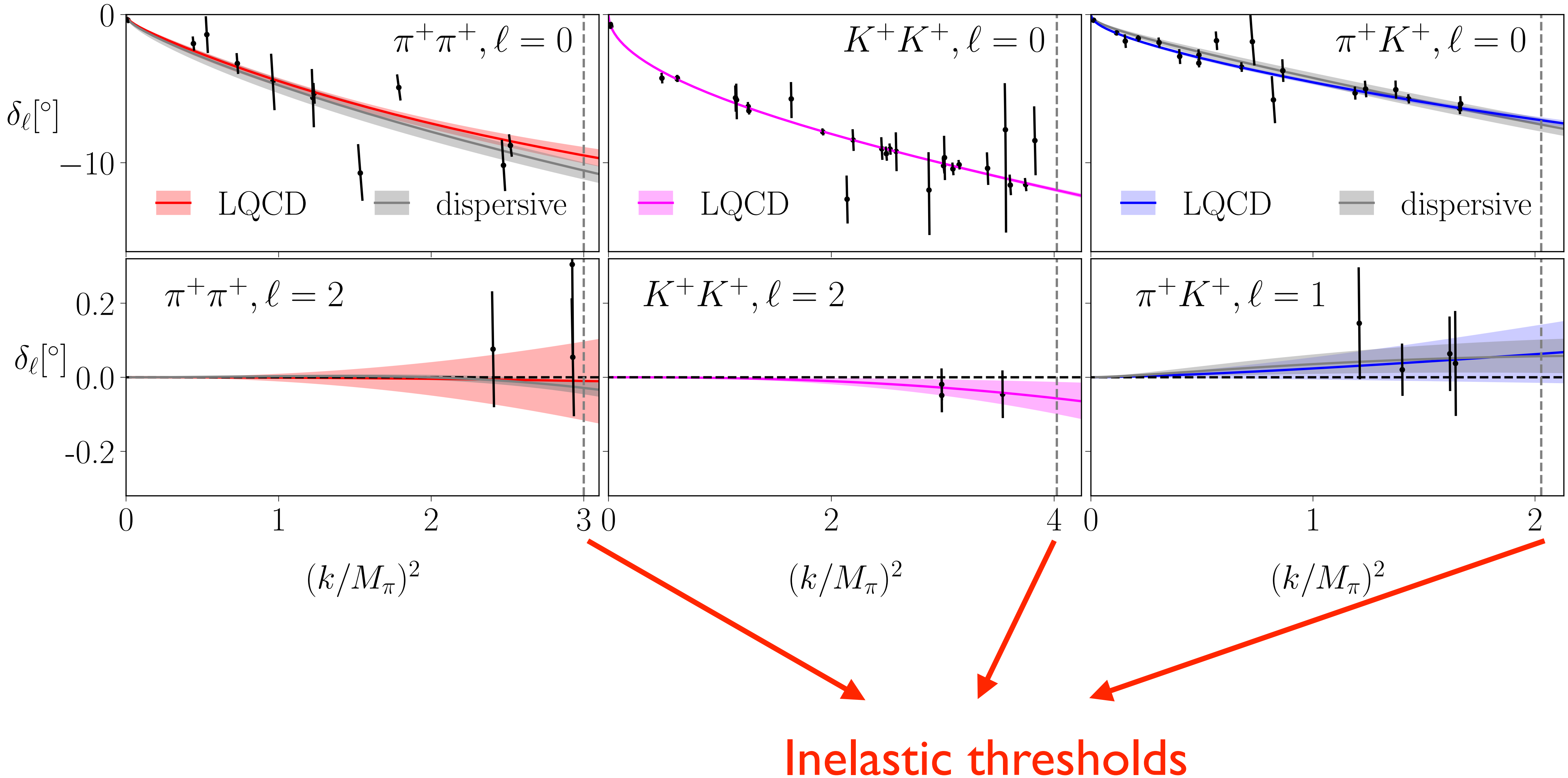
Example of spectrum



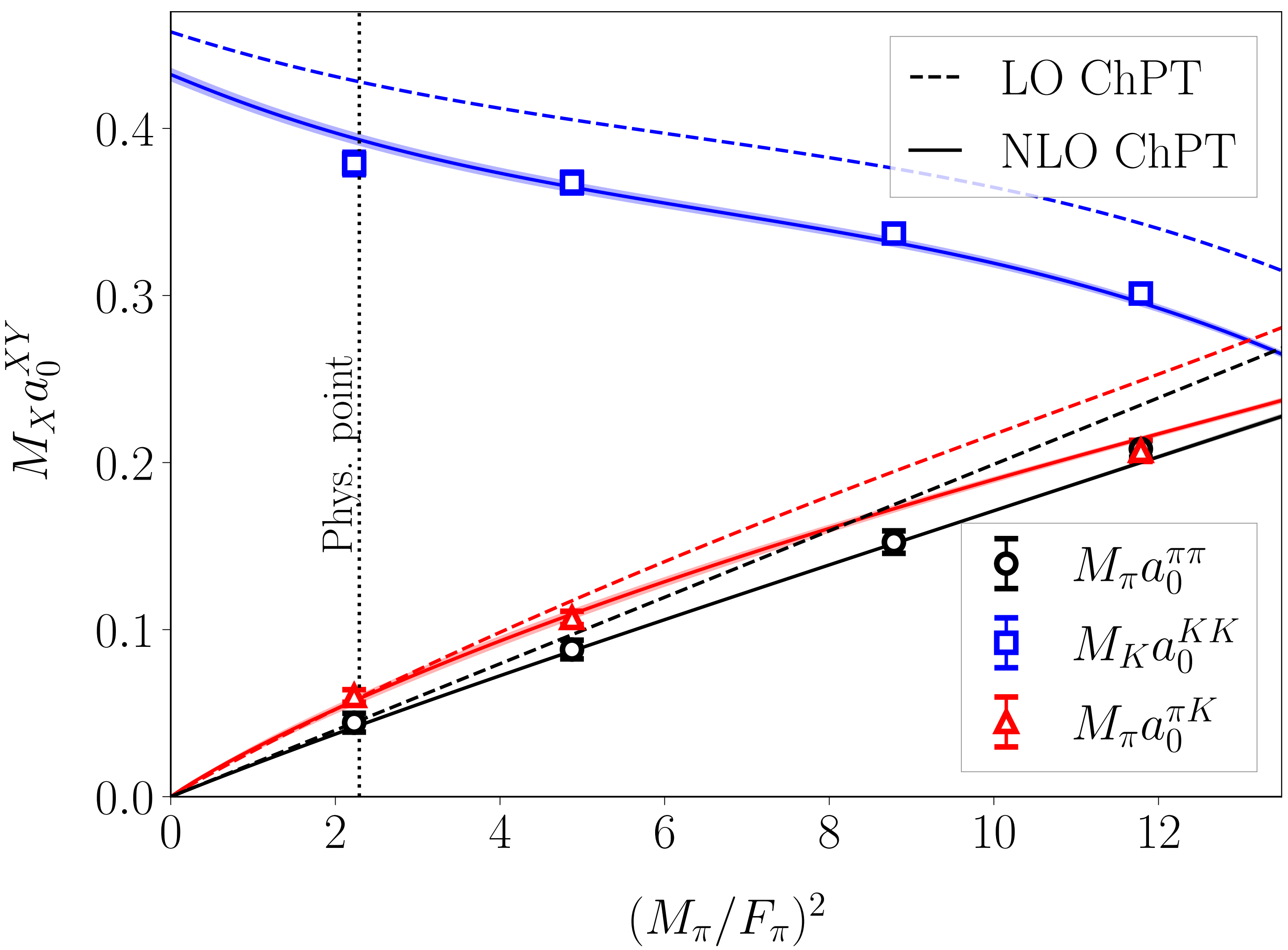
E250: K^+K^+

2-particle interactions

E250: ~physical quark masses



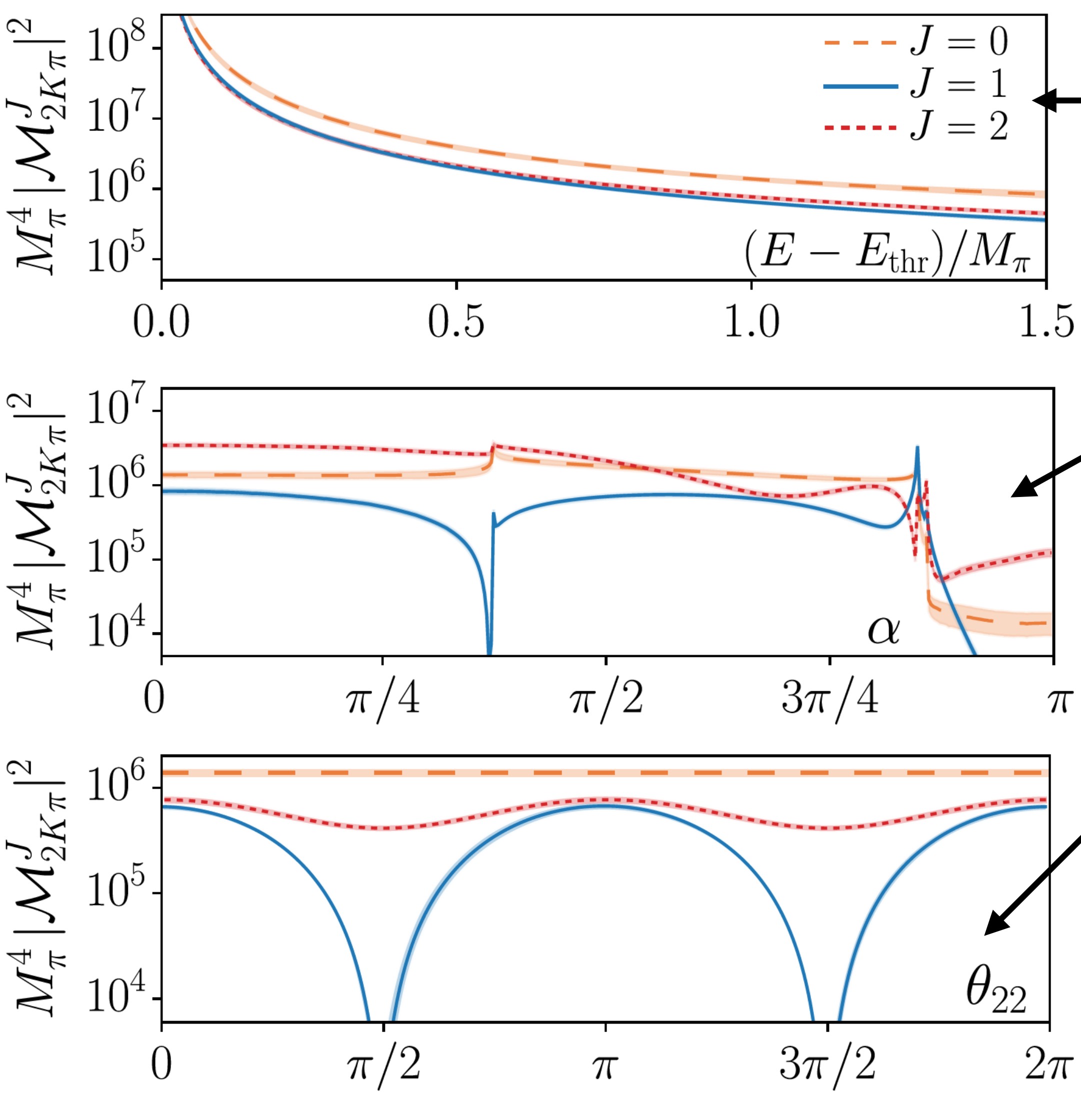
Scattering lengths



$\chi^2/\mathrm{dof} = 12.4/(11-2)$

- Simultaneous chiral fit to $\pi\pi$, πK, KK scattering lengths
  - Fit involves two LECs, one of which is determined with 2% stat. Errors

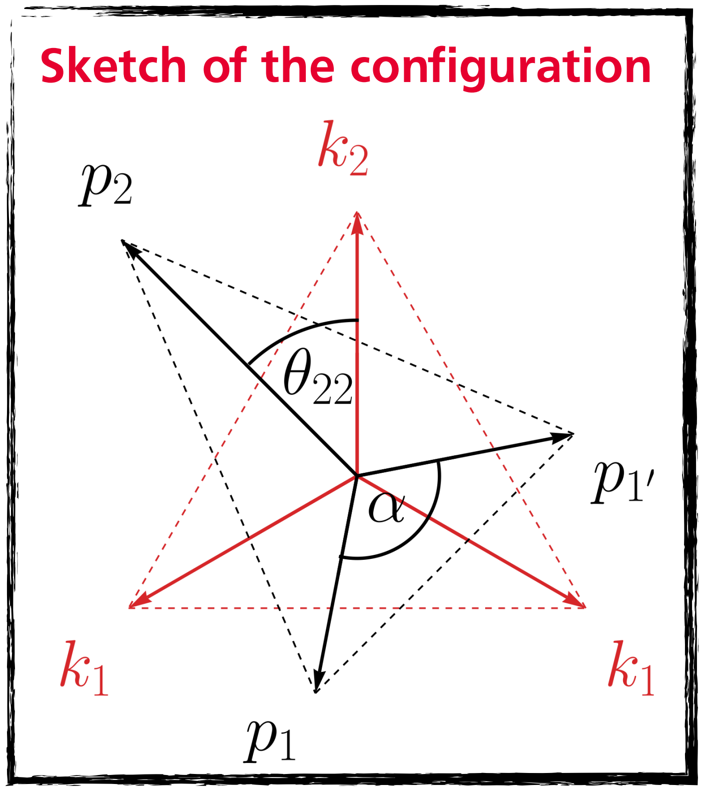
Some angular dependence



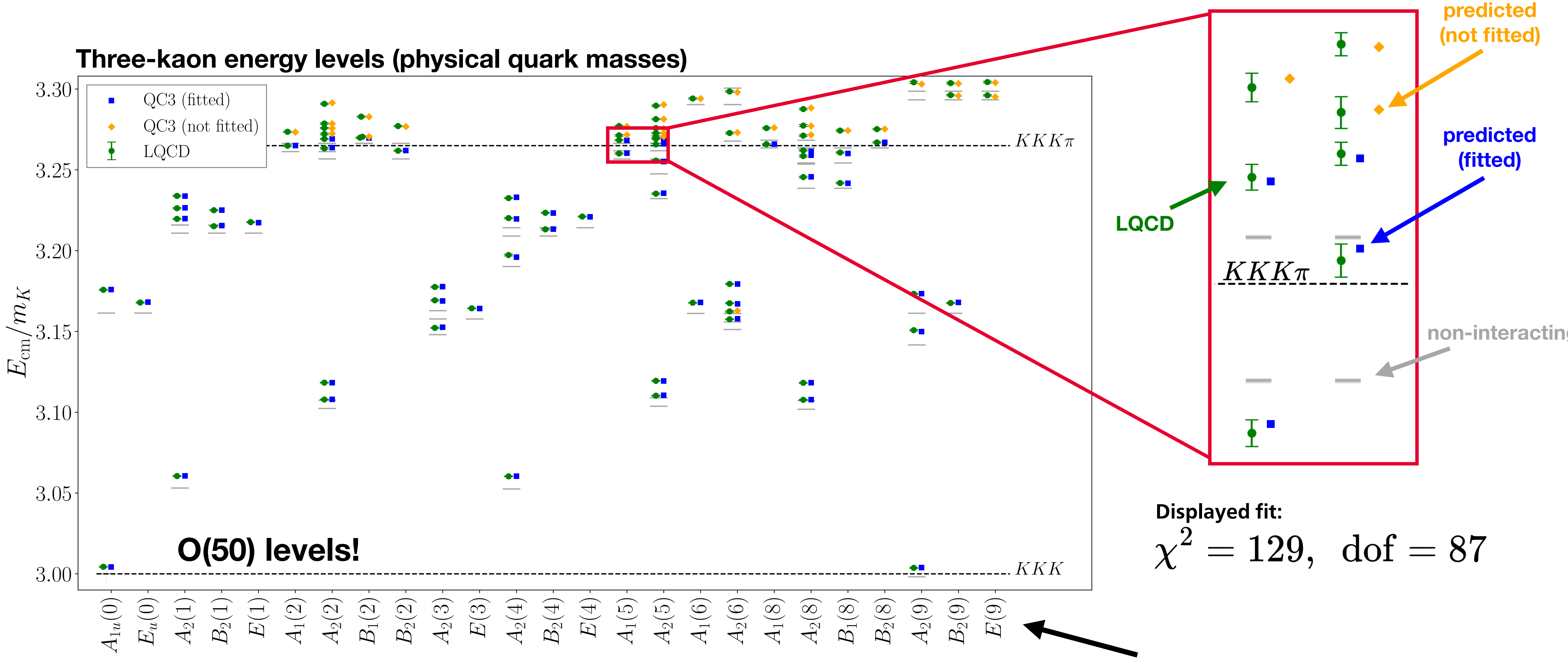
Different partial waves (largest is $J=0$)

Angle in the triangle: equilateral ($\alpha=\pi/3$) vs isosceles

Angle between in/out momenta



Three-meson spectrum



Displayed fit:

$$\chi^2 = 129, \quad \text{dof} = 87$$

Fernando Romero-López, Uni Bern

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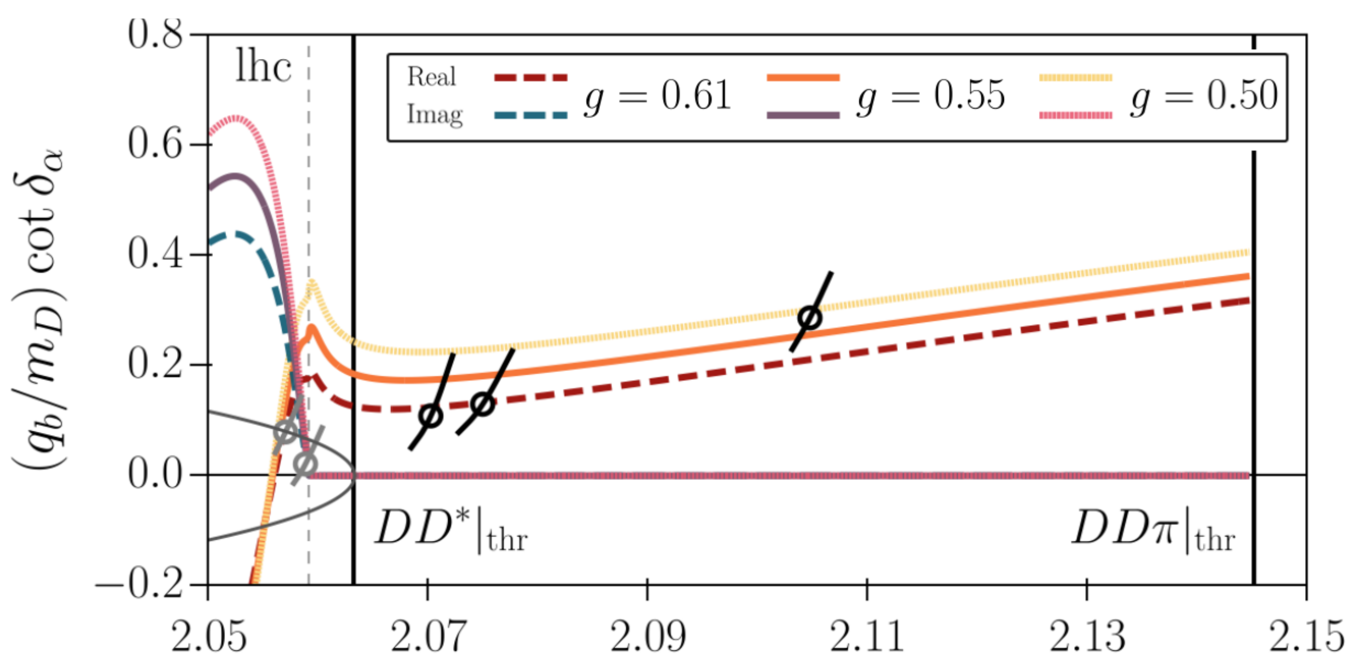
Application to LQCD data

[Dawid, Romero-López, SRS, 2409.17059, JHEP]

- Proof of concept study to (limited) existing LQCD data with bound D^*
  - Choose p-wave $D\pi$ scattering amplitude to obtain D^*, for given choice of $g_{D^*D\pi}$
  - Choose reasonable s-wave $D\pi$ and DD amplitudes
  - Eyeball fit to results of [Padmanath, Prelovsek, arXiv:2202.10110] requires inclusion of $\mathscr{K}_{\text{df},3}$
  - Solve integral equations, analytically continue to D^* pole, use LSZ to obtain $\mathscr{M}(DD^*)$
  - Result shows appropriate behavior at and below left-hand cut



- Virtual bound-state pole moves into complex plane!
- Looking forward: complete analysis requires more extensive LQCD spectra
  - DD, $D\pi$, $DD\pi$ & DD^* levels