Three-particle scattering amplitudes from lattice QCD

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Collaborators

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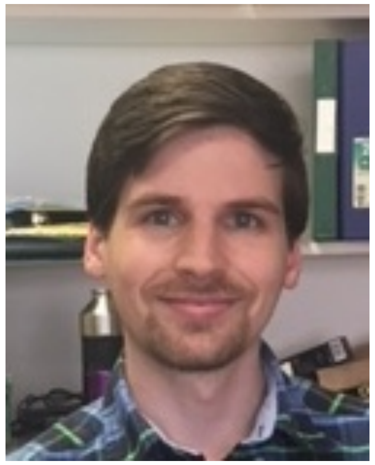
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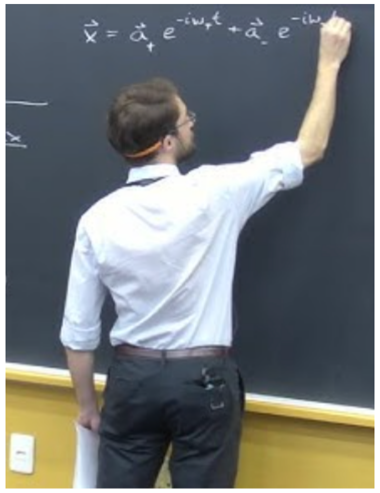
Zach Draper (ex-UW)


Sebastian Dawid (Indiana)

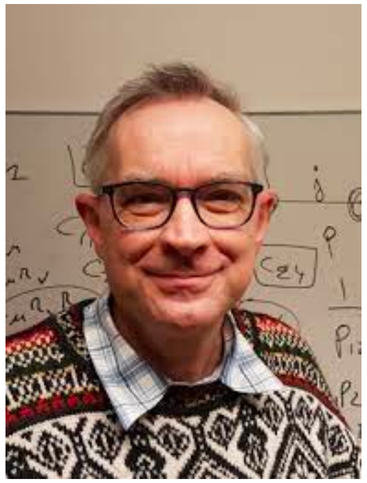
Drew Hanlon (CMU)

Ben Hörz (Intel)

Sarah Skinner (ex-CMU)

Colin Morningstar (CMU)

Jorge Baeza Ballesteros (Berlin)


Hans Bijnens (Lund)

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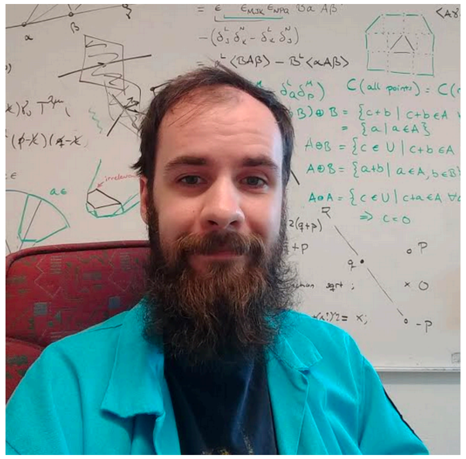
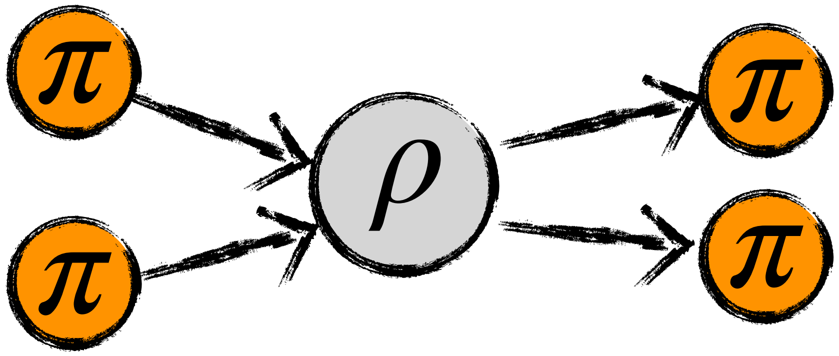

Mattias Sjö (Marseille)

Wilder Schaaf (UW)

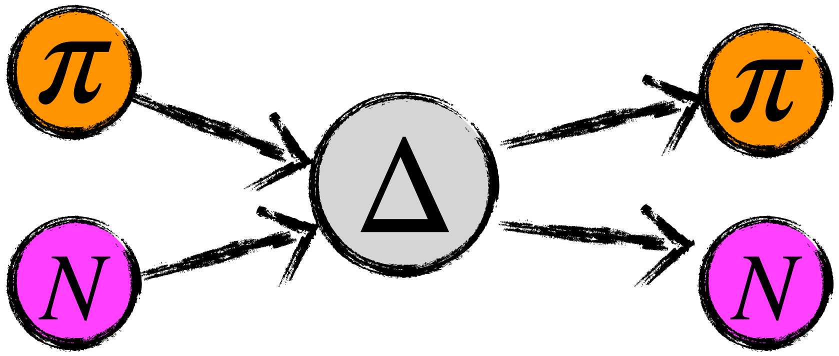
Overview: present status

Overview: present status



Physical quark masses

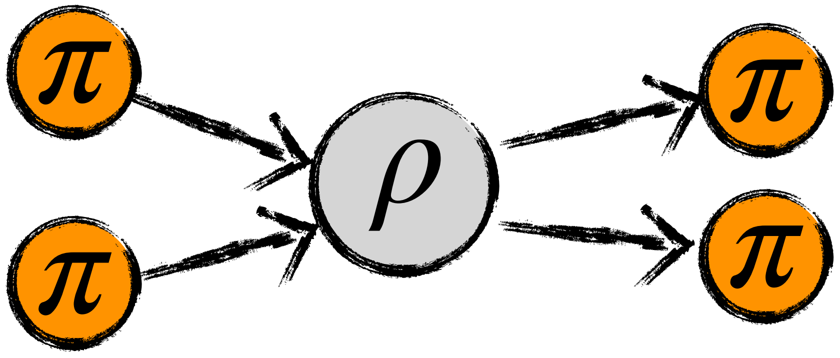
E.g.: Wang, Leinweber, Liu, Liu, Sun, Thomas, Wu, Xing, Yu, 2502.03700



$M_\pi = 200 \text{ MeV}$

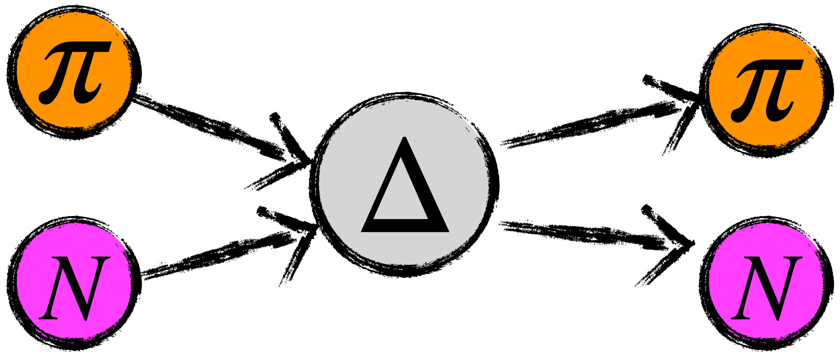
Bulava, Hanlon, Hörz, Morningstar, Nicholson, Romero-López
Skinner, Varnas, Walker-Loud, 2208.03867

Overview: present status


π
π
ρ
π
π

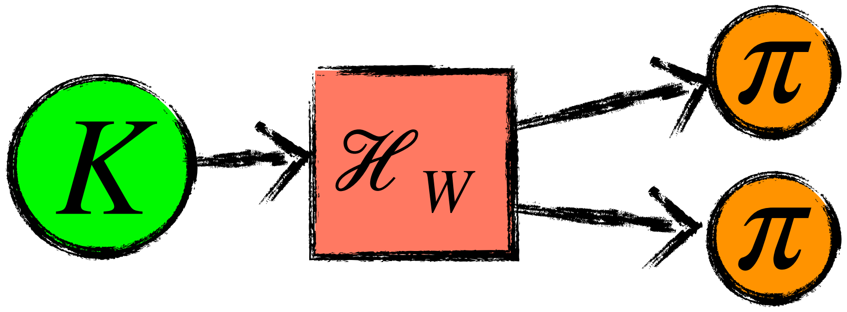
Physical quark masses

E.g.: Wang, Leinweber, Liu, Liu, Sun, Thomas, Wu, Xing, Yu, 2502.03700


π
N
Δ
π
N

$M_\pi = 200$ MeV

Bulava, Hanlon, Hörz, Morningstar, Nicholson, Romero-López Skinner, Varnas, Walker-Loud, 2208.03867

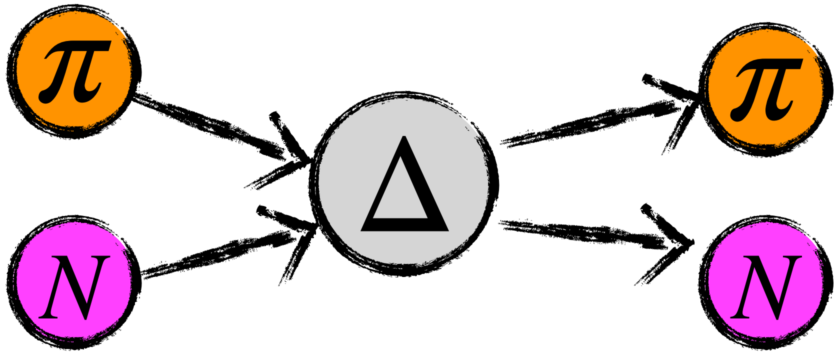

K
$\mathscr{H}_W$
π
π

Almost "FLAG-ready", i.e. fully controlled

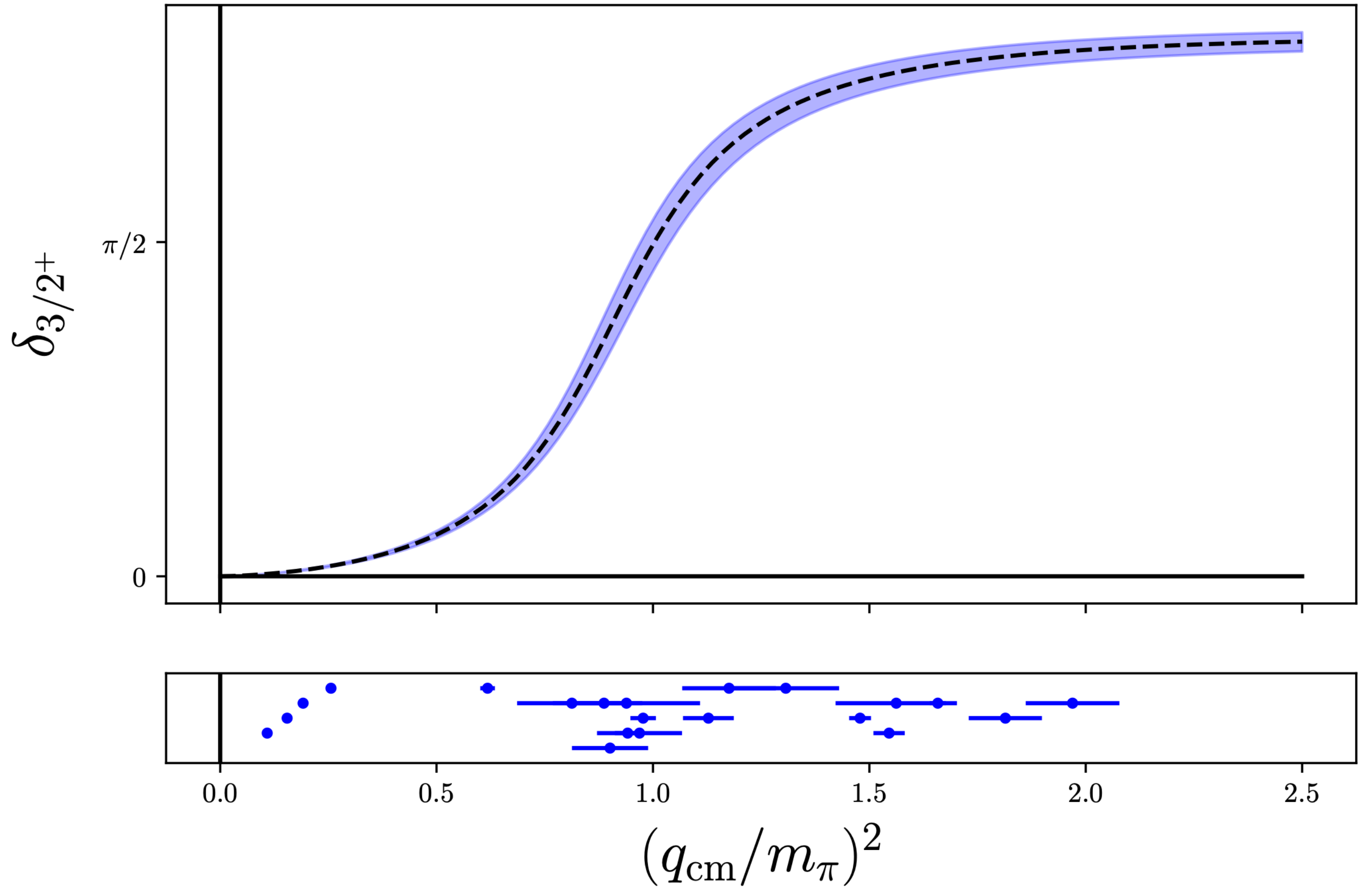
Abbott et al. [RBC-UKQCD collaboration], 2004.09440

Δ resonance

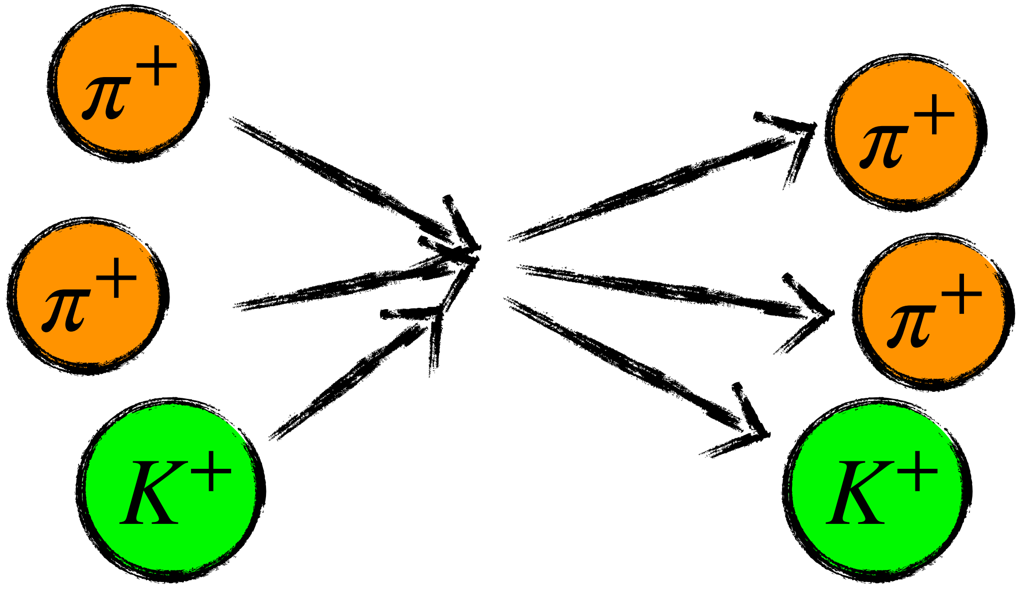
Bulava, Hanlon, Hörz, Morningstar, Nicholson, Romero-López, Skinner, Varnas, Walker-Loud, 2208.03867



$M_\pi \approx 200$ MeV
$M_N \approx 950$ MeV
$a = 0.063$ fm
$L^3 \times T = 64^3 \times 128$
Clover fermions (CLS)

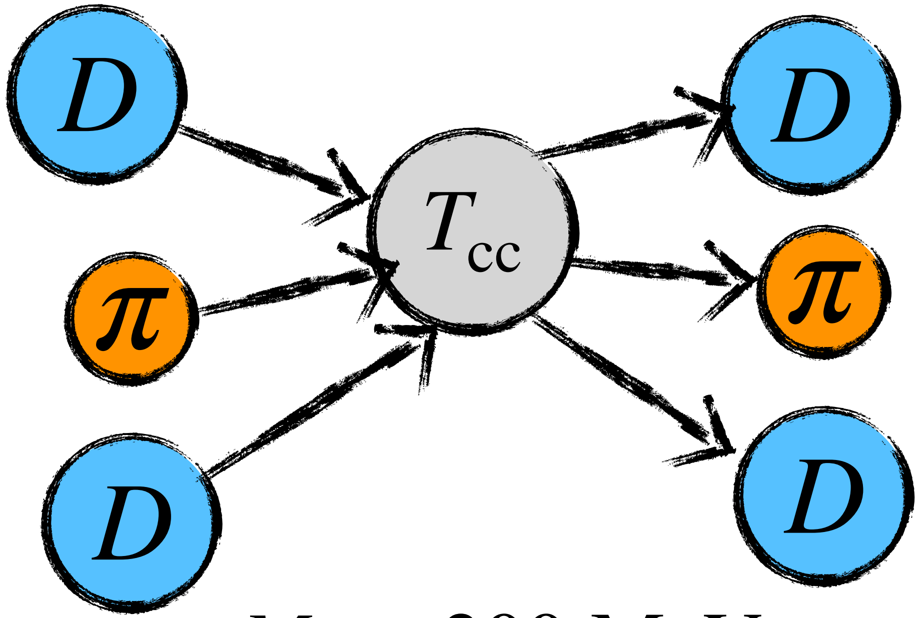


Overview: present frontier


π^+
π^+
K^+
π^+
π^+
K^+

Physical quark masses

Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López, SRS, Skinner, 2502.14348 & 2502.17976

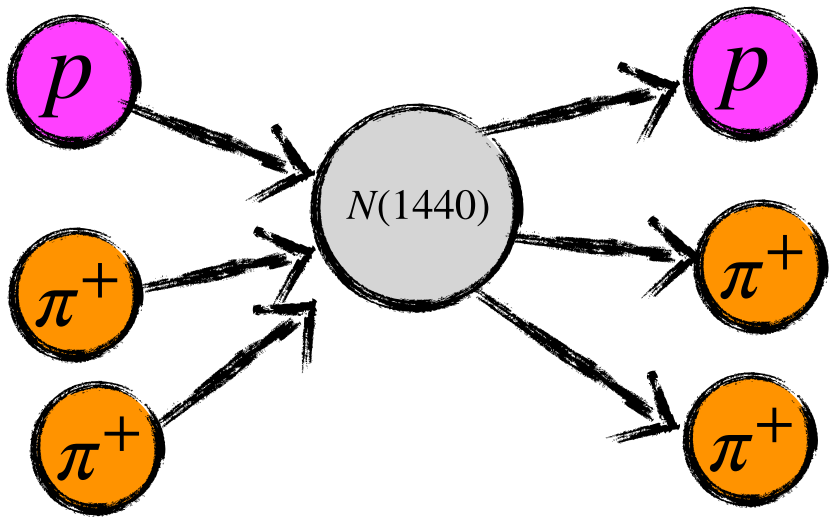

D
π
D
T_{cc}
D
π
D

$M_\pi = 200$ MeV

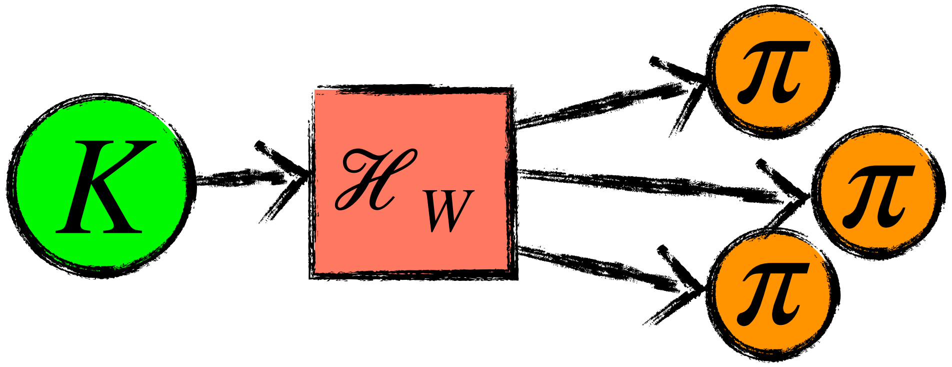
Padmanath, Prelovsek, 2202.10110

…

Dawid, Romero-López, SRS, 2409.17059

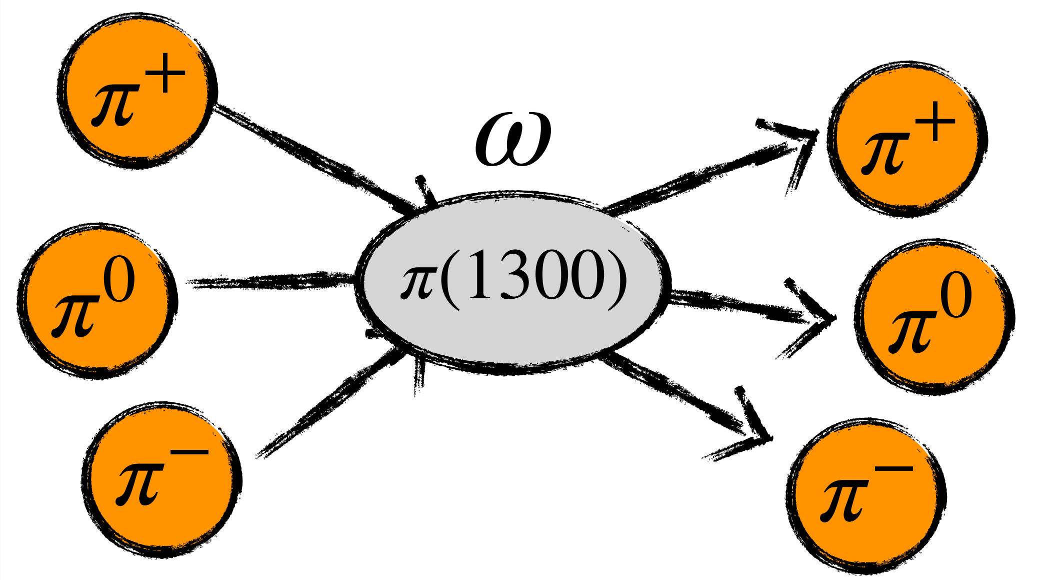

p
π^+
π^+
$N(1440)$
p
π^+
π^+

Formalism: Hansen, Romero-López, SRS, 2509.24778

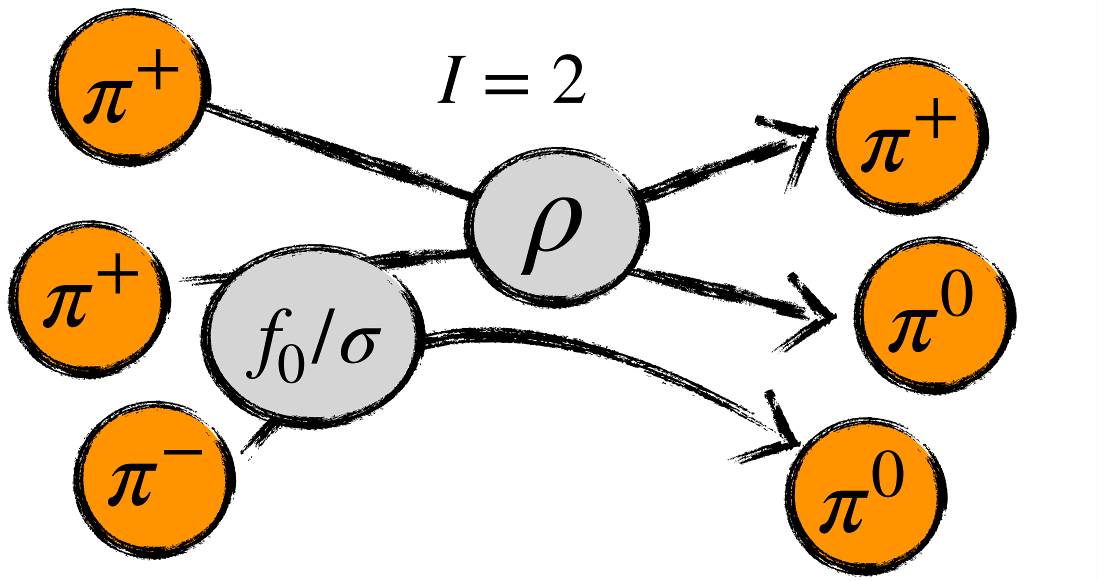

K
$\mathscr{H}_W$
π
π
π

Formalism: Müller, Rusetsky, 2012.13957
Hansen, Romero-López, SRS, 2101.10246
Müller, Pang, Rusetsky, Wu, 2211.10126
Pang, Bubna, Müller, Rusetsky, Wu, 2312.04391

Overview: present frontier

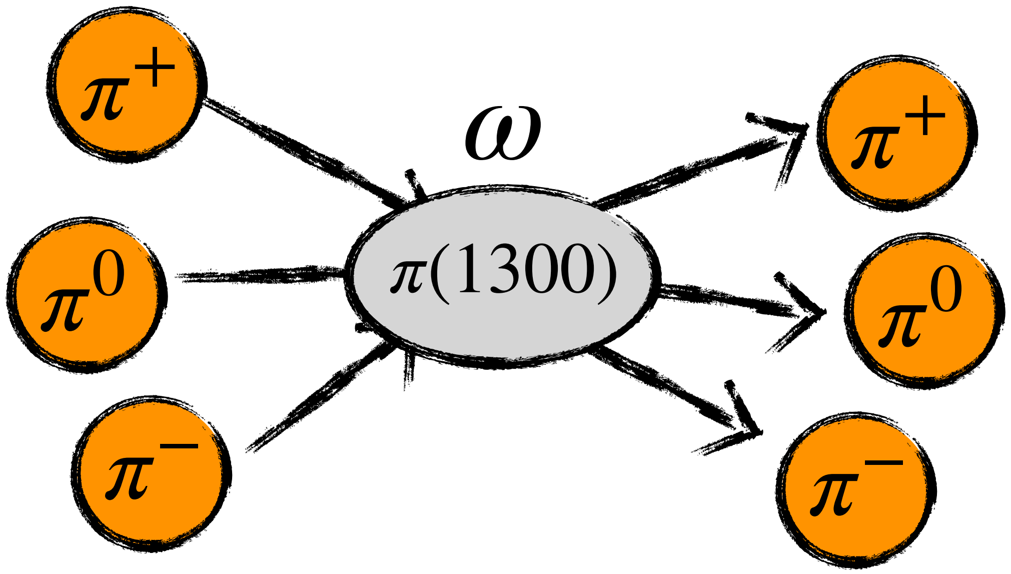

π^+
π^0
π^-
ω
$\pi(1300)$
π^+
π^0
π^-

Formalism: Hansen, Romero-López, SRS, 2003.10974
Alternative formalism & first LQCD results:
Yan, Mai, Garofalo, Meißner, Liu, Liu, Urbach, 2407.16659; Yan, Mai, Garofalo, Feng, Döring, Liu, Liu, Meißner, Urbach, 2510.0947


$I = 2$
π^+
π^+
π^-
f_0/σ
ρ
π^+
π^0
π^0

Formalism: Hansen, Romero-López, SRS, 2003.10974; Jackura, 2208.10587
First LQCD results:
Briceño, Hansen, Jackura, Edwards, Thomas, 2510.24894

Overview: present frontier


π^+
ω
π^+
π^0
$\pi(1300)$
π^0
π^-
π^-

Formalism: Hansen, Romero-López, SRS, 2003.10974
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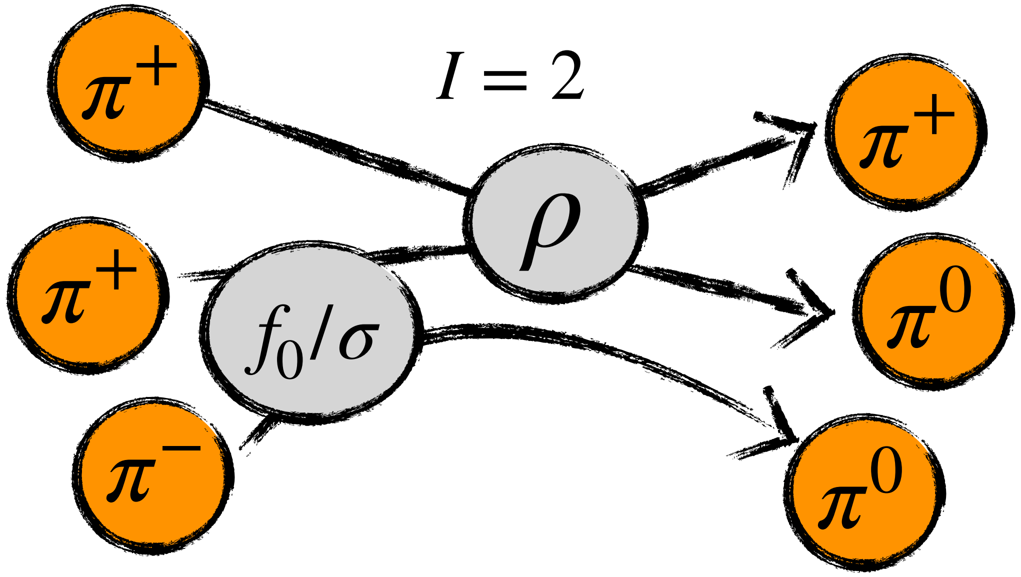
π^+
$I = 2$
π^+
ρ
π^+
π^0
f_0/σ
π^-
π^0

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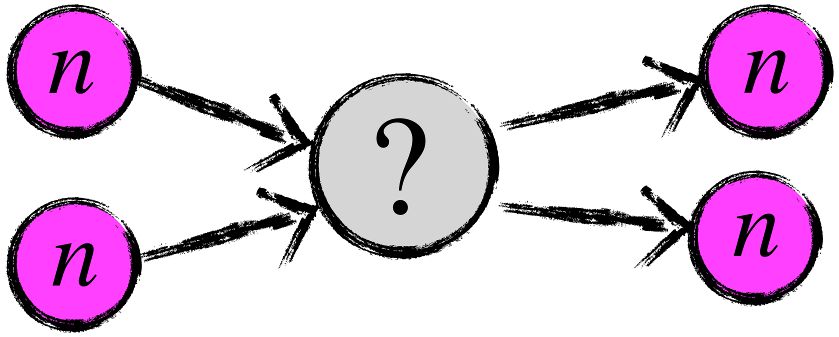
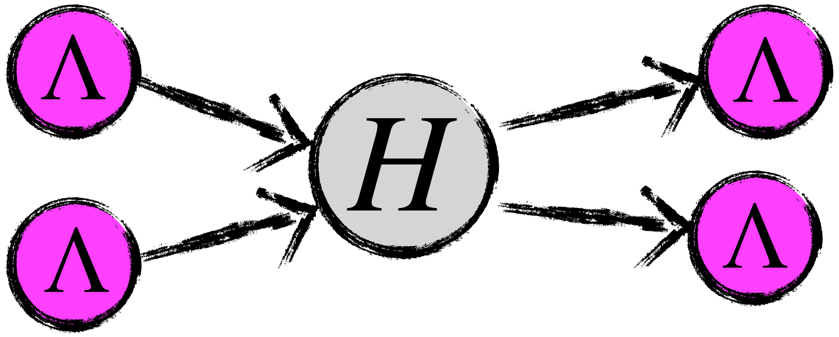


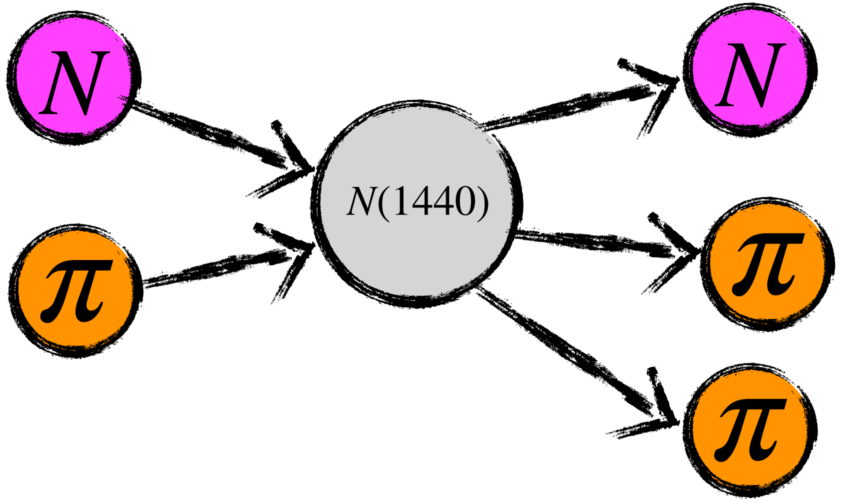
Di-neutron bound state?


Λ
Λ
H
Λ
Λ

Is H dibaryon bound?

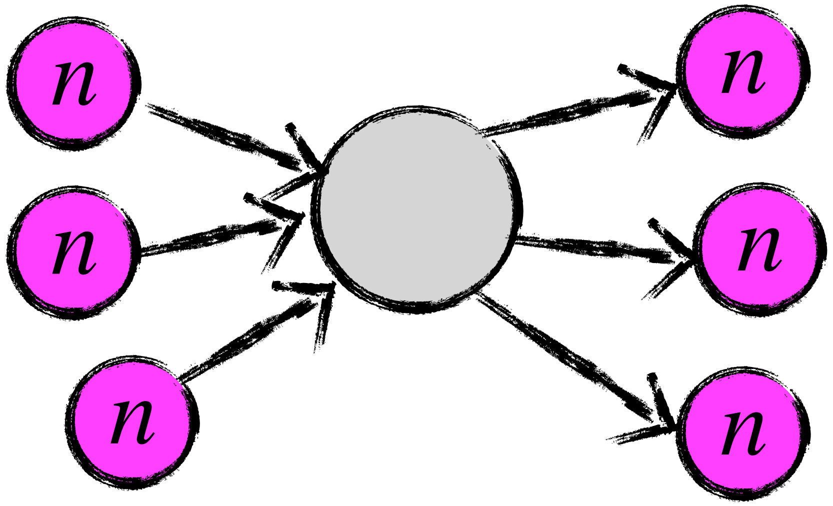
Overview: long-term dreams

Roper resonance


N
π
N(1440)
N
π
π

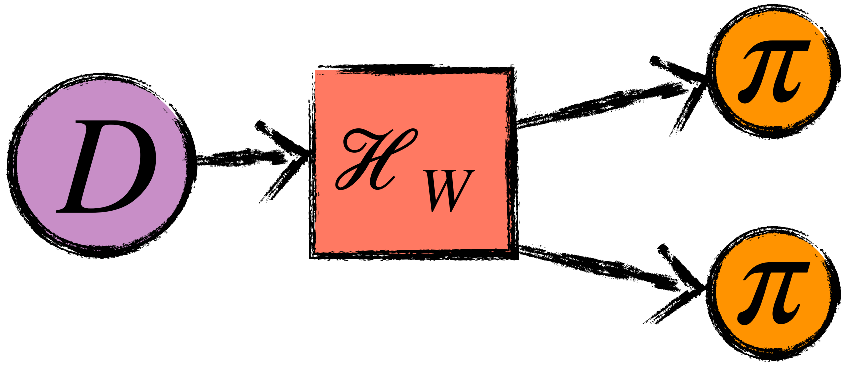
Formalism: Hansen, Romero-López, SRS, … in prep.

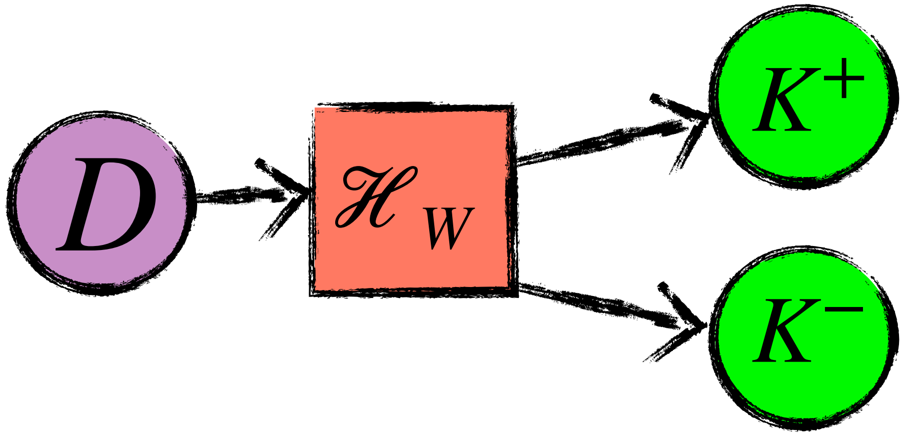
3 neutron interactions



Formalism: Draper, Hansen, Romero-López, SRS, 2303.10219
Toy application: Schaaf, SRS, 2512.24508

CP violation in D decays


D
$\mathcal{H}_W$
π
π


D
$\mathcal{H}_W$
K^+
K^-

Outline

- Motivations
- The fundamental issue & its resolution
- Recent applications of three-particle formalism
    - Three-particle amplitudes involving pions & kaons at physical quark masses
    - $DD\pi$ scattering, relevant for the T_{cc}^+
    - Progress on $N\pi\pi$ — towards the Roper resonance
    - Dreaming of three neutrons
- Summary & Outlook

Motivations

Underlying motivations

- Determine properties of strong interaction resonances from QCD
  - E.g. exotics such as $T_{cc}(3875)^+ \to DD^* \to DD\pi$
  - Long-standing puzzles, e.g. Roper $N(1440) \to N\pi + N\pi\pi$

Underlying motivations

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  - E.g. exotics such as $T_{cc}(3875)^+ \to DD^* \to DD\pi$
  - Long-standing puzzles, e.g. Roper $N(1440) \to N\pi + N\pi\pi$
- Determine three particle "forces" for $3n,\ 3\pi,\ 3K, \ldots$
  - Needed to understand neutron star EoS, properties of large nuclei, …

Underlying motivations

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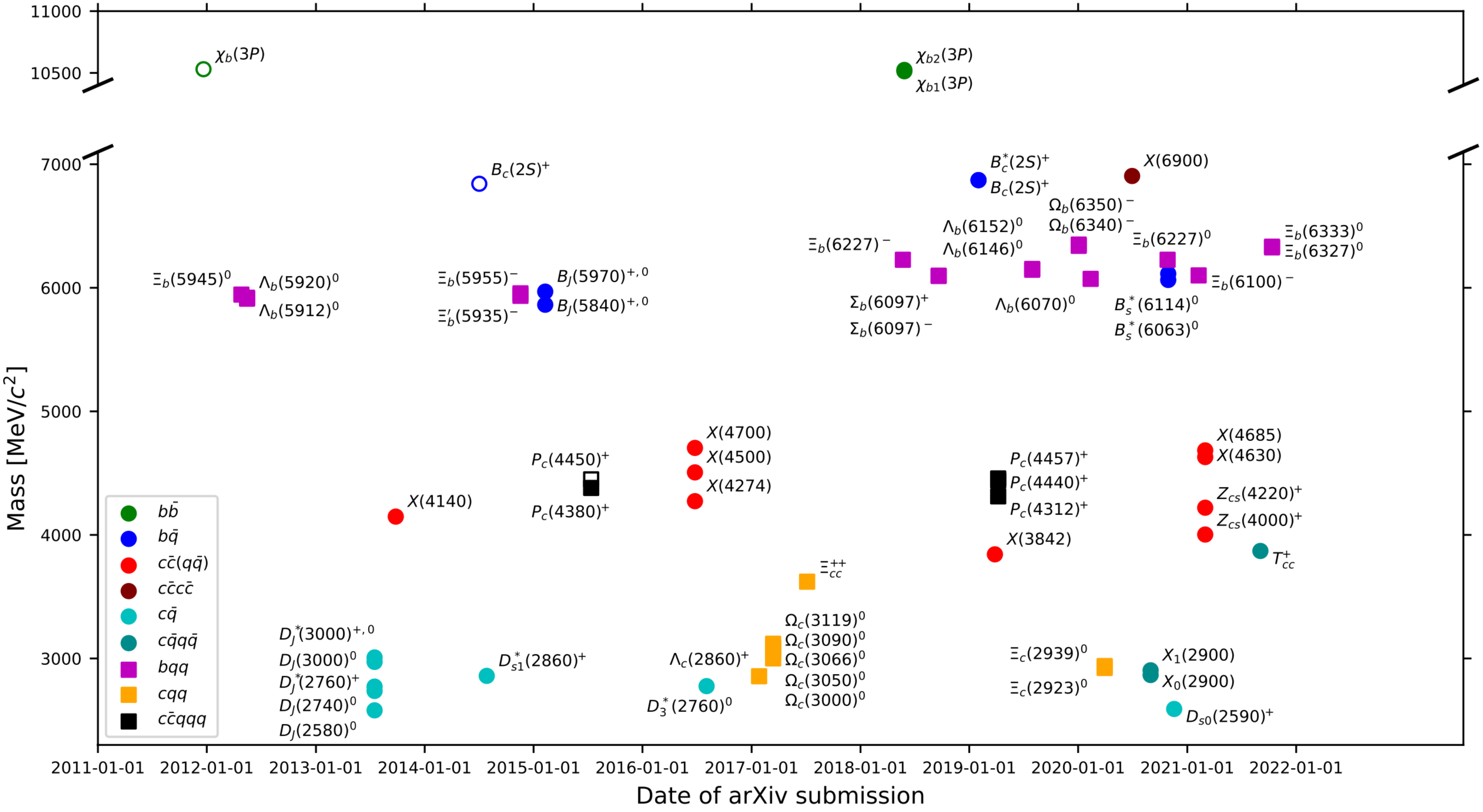
  - E.g. exotics such as $T_{cc}(3875)^+ \to DD^* \to DD\pi$
  - Long-standing puzzles, e.g. Roper $N(1440) \to N\pi + N\pi\pi$
- Determine three particle "forces" for $3n,\ 3\pi,\ 3K, \ldots$
  - Needed to understand neutron star EoS, properties of large nuclei, …
- Calculate weak decay amplitudes within the Standard Model, in order to search for new physics
  - E.g. $K \to 2\pi$ (essentially done), $K \to 3\pi$ (method known), & $D \to \pi^+\pi^-,\ K^+K^-$ (open question)

Cornucopia of exotics



[I. Danilkin, talk at INT workshop, March 23]

Data from Babar, Belle, BESIII, COMPASS, LHCb

The fundamental issue & its resolution

On the one hand…

- LQCD determines energies and properties of finite-volume eigenstates
  - Obtained by fits to (numerically-evaluated) Euclidean correlation functions:

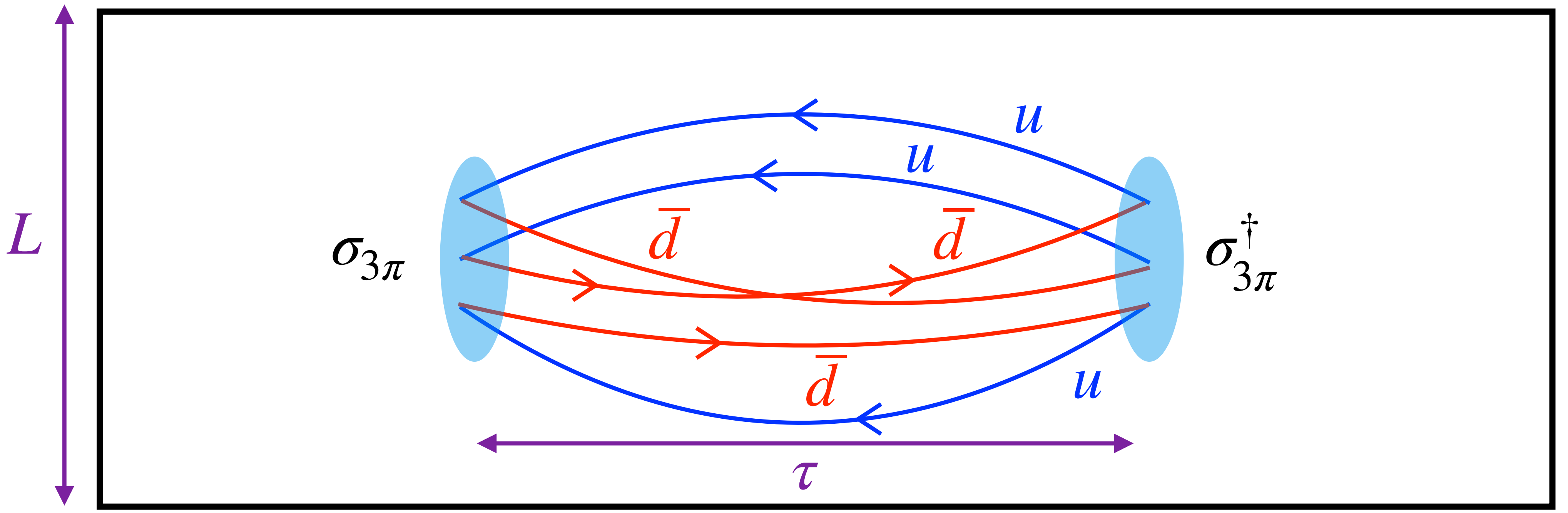
$$\int_L d^3x\, e^{-i\vec{P}\cdot\vec{x}}\, {}_L\langle\Omega|\,\sigma_{3\pi}(\tau,\vec{x})\sigma^\dagger_{3\pi}(0)\,|\Omega\rangle_L \propto \sum_n \left|{}_L\langle 0|\,\sigma^\dagger_{3\pi}(0)\,|3\pi,\vec{P},n\rangle_L\right|^2 e^{-E_n\tau}; \qquad (\tau>0)$$

$\vec{P} = 2\pi\vec{n}/L$
Assuming L^3 box with PBC

$\sigma_{3\pi} \sim 3\pi^+$
Lives on timeslice

Tower of finite-volume states
with quantum numbers of $3\pi^+$,
with momentum $\vec{P}$, and living
in irreps of cubic group

Energies of said states



On the one hand…

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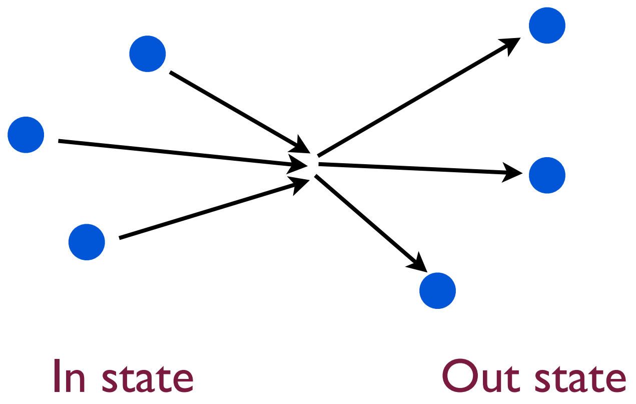
- E_n are physical quantities!
  - Can determine 5-10 levels for each choice of quantum numbers ($\vec{P}$, irrep, …)
  - Can now begin to calculate with physical quark masses
  - Results come with statistical & systematic errors (e.g. need $a \to 0$)
  - Today, I assume that the physical E_n are provided by LQCD simulations

…while on the other

- We want infinite-volume scattering amplitudes, e.g.

$$\mathcal{M}_3 \sim$$


In state
Out state

…while on the other

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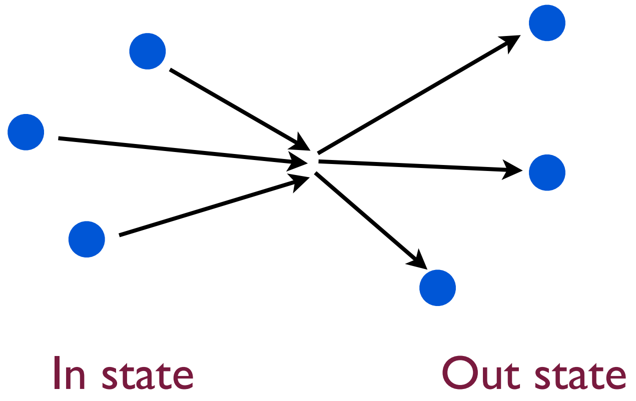
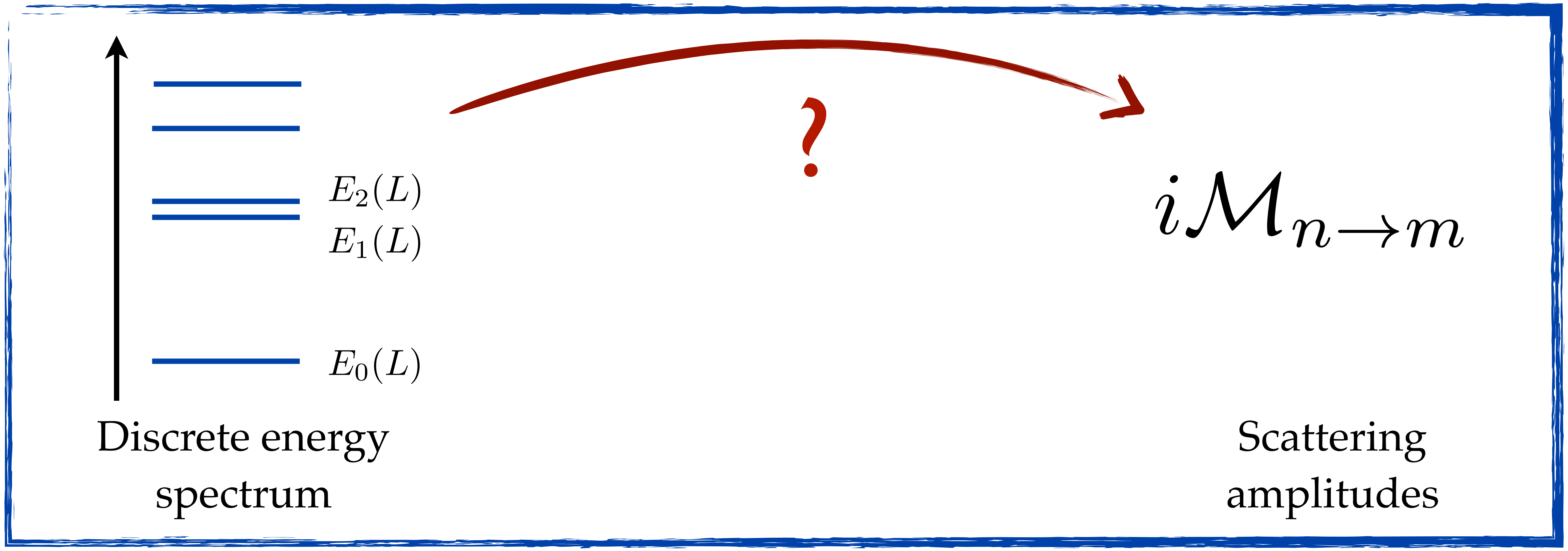
$$\mathcal{M}_3 \sim$$


In state
Out state

- How do we relate these? A finite-volume QFT problem.


$E_2(L)$
$E_1(L)$
$E_0(L)$
?
$i\mathcal{M}_{n\to m}$

Discrete energy spectrum

Scattering amplitudes

A related question:

- LQCD can also calculate matrix elements between finite-volume states

$$ {}_L\langle \Omega | \sigma_{3\pi}(\tau_f, \overrightarrow{P}) [\int_L d^3x \mathscr{H}_W(0, \vec{x})] K^\dagger(\tau_i, \overrightarrow{P}) | \Omega \rangle_L \propto \sum_{n',n} c_{n',n} e^{-E_n \tau_f} \underline{{}_L\langle 3\pi, \overrightarrow{P}, n' | \mathscr{H}_W(0) | K, n \rangle_L} e^{E_{K_n'}\tau_i} $$

$\tau_f > 0$

$\tau_i < 0$

A physical quantity if $E_{n'} = E_n$

A related question:

- LQCD can also calculate matrix elements between finite-volume states

$$ {}_L\langle\Omega|\sigma_{3\pi}(\tau_f,\vec{P})\Big[\int_L d^3x\,\mathscr{H}_W(0,\vec{x})\Big]K^\dagger(\tau_i,\vec{P})|\Omega\rangle_L \propto \sum_{n',n} c_{n',n}\, e^{-E_n\tau_f}\, \underbrace{{}_L\langle 3\pi,\vec{P},n'|\mathscr{H}_W(0)|K,n\rangle_L}_{\text{A physical quantity if } E_{n'}=E_n}\, e^{E_{K_n'}\tau_i} $$

$\tau_f > 0$

$\tau_i < 0$

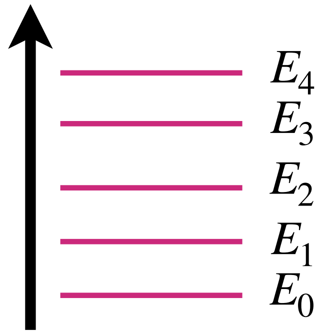
- How are these related to decay amplitudes?

$$ \mathscr{A}(K\to 3\pi) = {}_{\text{out}}\langle 3\pi|\mathscr{H}_W(0)|K\rangle $$

2-particle formalism

[Lüscher, 1986-91 + many subsequent works]

Two-particle spectrum



↔

Quantization condition: QC2

$$\det_{\ell m}\left[\mathcal{K}_2 + F_2^{-1}\right] = 0$$

Known kinematical "Lüscher zeta" function

Fit →

K-matrix

$\mathcal{K}_2$

Unitarity relation →

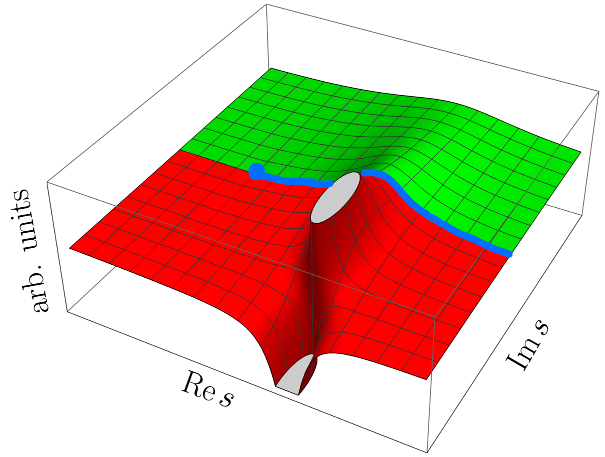
Scattering amplitude

$\mathcal{M}_2$

$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho\mathcal{K}_2}$$

Phase space

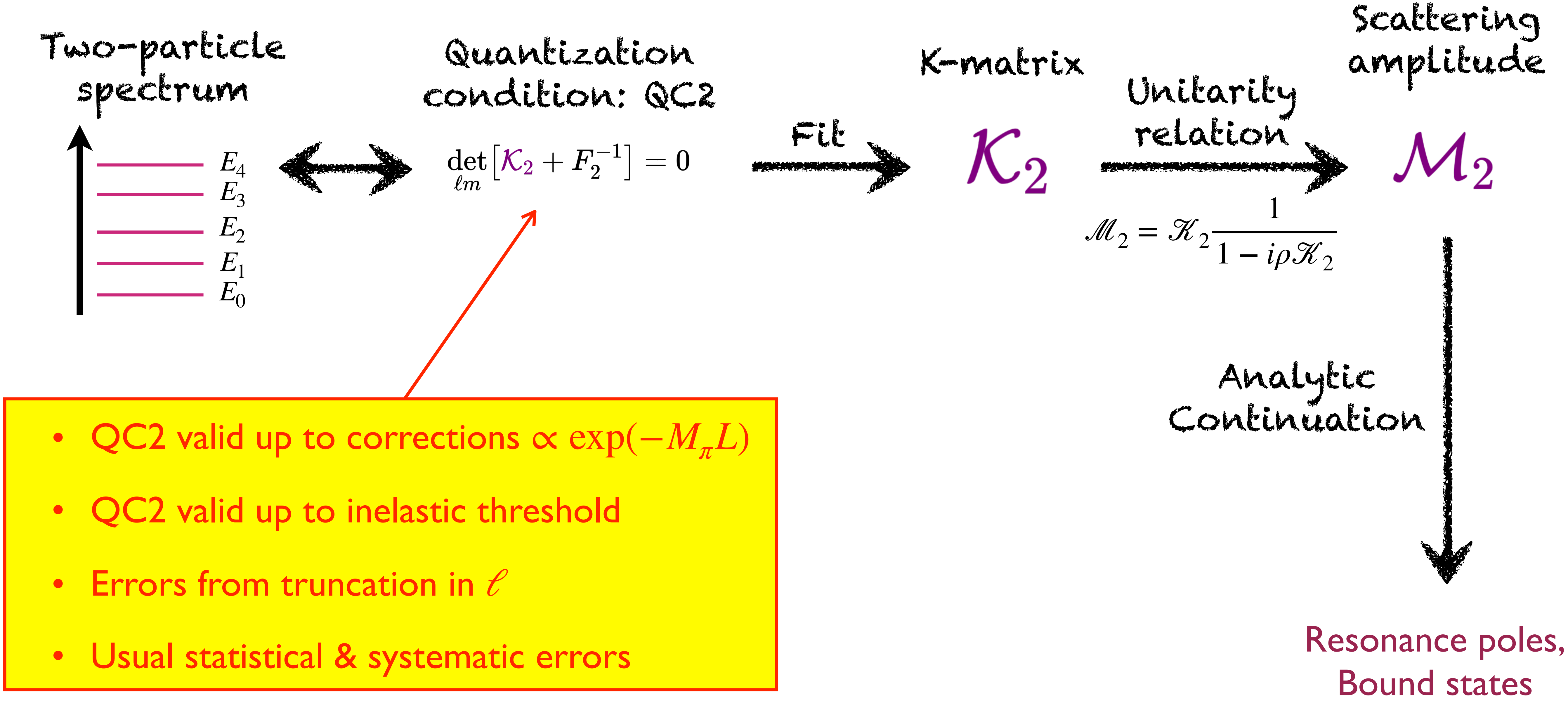
Analytic Continuation ↓



Resonance poles, Bound states

2-particle formalism

[Lüscher, 1986-91 + many subsequent works]



Two-particle spectrum

E_4
E_3
E_2
E_1
E_0

Quantization condition: QC2

$$\det_{\ell m}\left[\mathcal{K}_2 + F_2^{-1}\right] = 0$$

Fit

K-matrix

$$\mathcal{K}_2$$

Unitarity relation

$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho\mathcal{K}_2}$$

Scattering amplitude

$$\mathcal{M}_2$$

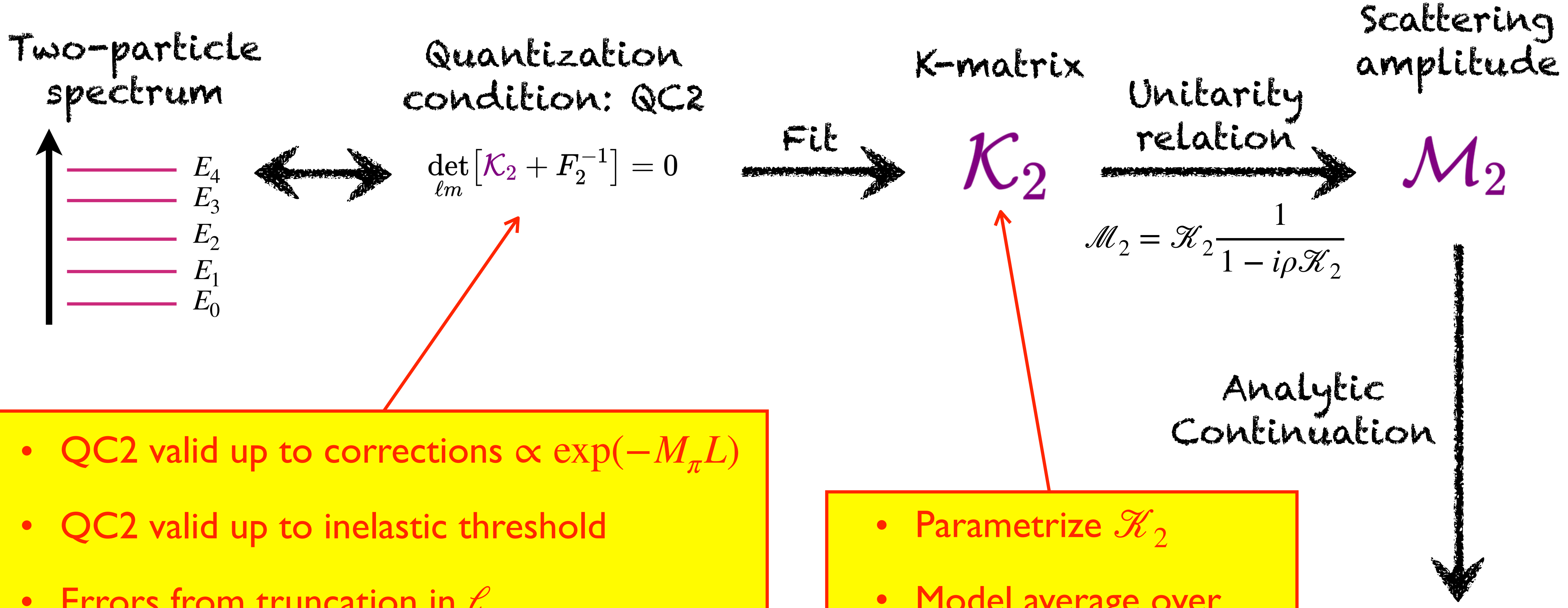
Analytic Continuation

Resonance poles,
Bound states

- QC2 valid up to corrections ∝ $\exp(-M_\pi L)$
- QC2 valid up to inelastic threshold
- Errors from truncation in ℓ
- Usual statistical & systematic errors

2-particle formalism

[Lüscher, 1986-91 + many subsequent works]



Two-particle spectrum

E_4, E_3, E_2, E_1, E_0

Quantization condition: QC2

$$\det_{\ell m}\left[\mathcal{K}_2 + F_2^{-1}\right] = 0$$

Fit

K-matrix

$$\mathcal{K}_2$$

Unitarity relation

$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho\mathcal{K}_2}$$

Scattering amplitude

$$\mathcal{M}_2$$

Analytic Continuation

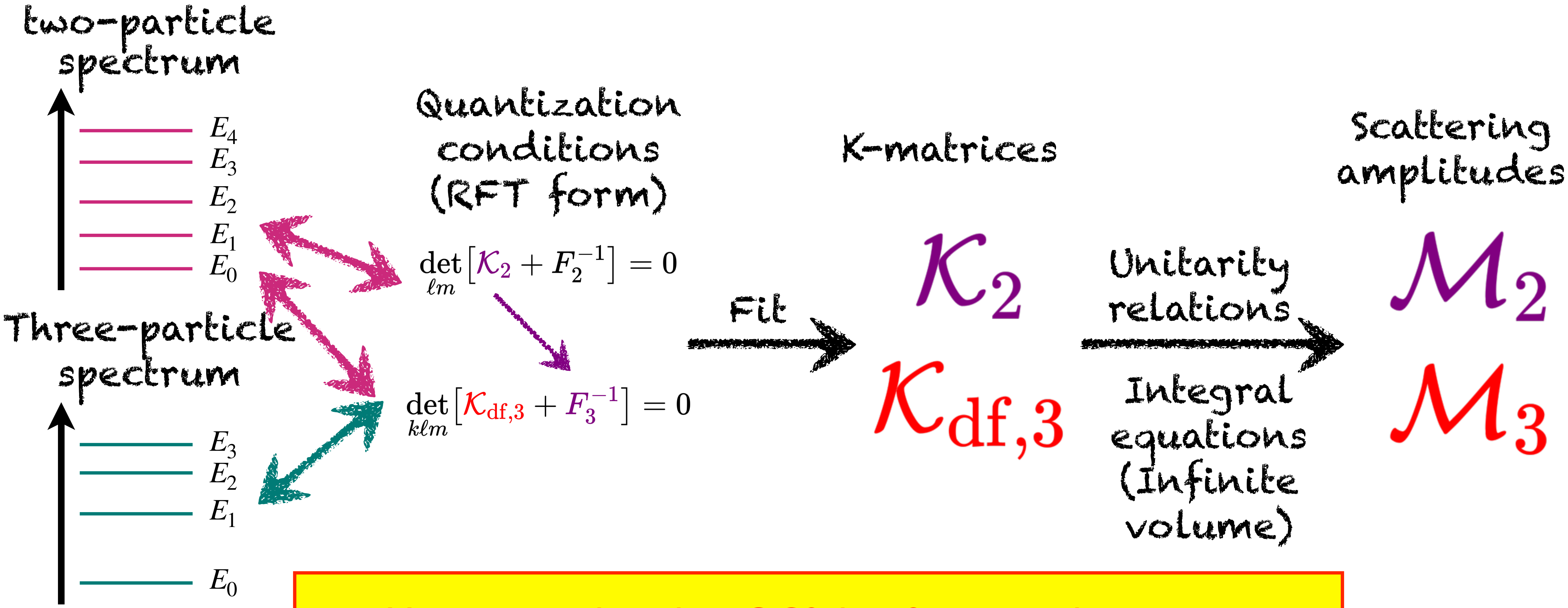
Resonance poles,
Bound states

- QC2 valid up to corrections ∝ $\exp(-M_\pi L)$
- QC2 valid up to inelastic threshold
- Errors from truncation in ℓ
- Usual statistical & systematic errors

- Parametrize $\mathcal{K}_2$
- Model average over "reasonable" forms
- Uncertainties increase as move into complex plane

3-particle formalism

[Hansen & SRS, 2014 & 2015 + many subsequent works]


two-particle spectrum
E_4
E_3
E_2
E_1
E_0
Three-particle spectrum
E_3
E_2
E_1
E_0
Quantization conditions (RFT form)
Fit
K-matrices
Unitarity relations
Integral equations (Infinite volume)
Scattering amplitudes

$$\det_{\ell m}\left[\mathcal{K}_2 + F_2^{-1}\right] = 0$$

$$\det_{k\ell m}\left[\mathcal{K}_{\mathrm{df},3} + F_3^{-1}\right] = 0$$

$$\mathcal{K}_2 \qquad \mathcal{K}_{\mathrm{df},3}$$

$$\mathcal{M}_2 \qquad \mathcal{M}_3$$

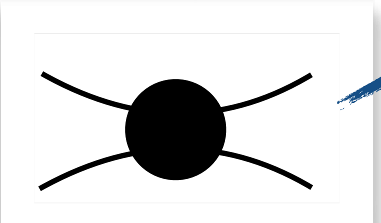
- Using more data than QC2, but fitting much more challenging
- More quantities to parametrize, with limited phenomenological knowledge for $\mathscr{K}_{\mathrm{df},3}$
- Solving integral equations involves new challenges

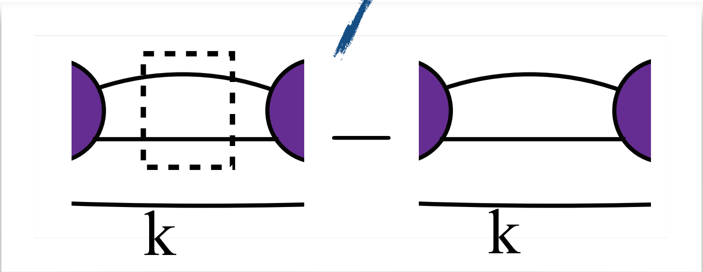
QC3

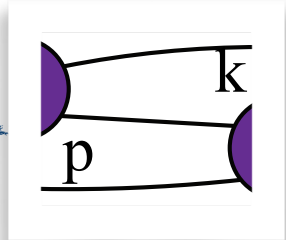
$$\det_{k\ell m}\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$$

- QC3 valid up to corrections $\propto \exp(-M_\pi L)$
- QC3 valid up to first inelastic threshold, e.g. $E^* = 5M_\pi$ for 3 pion system
- Matrix indices are spectator finite-volume momentum $\boldsymbol{k}$, and pair CM-frame ℓ, m
- RFT formalism includes smooth cutoff in $\boldsymbol{k}$; must truncate "by hand" in ℓ

$$F_3 = \frac{1}{2\omega L^3}\left[\frac{F}{3} - F\frac{1}{1/\mathscr{K}_{2,L} + F + G}F\right]$$



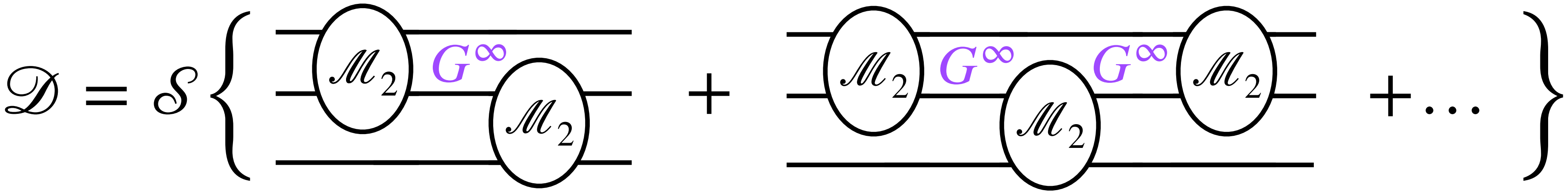




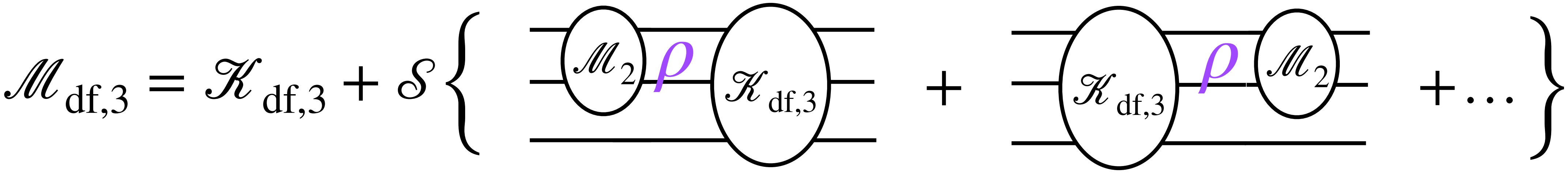
Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

$$\mathcal{M}_3 = \lim_{L\to\infty} \mathcal{S}\left\{ \mathcal{D}_L^{(u,u)} + \mathcal{M}_{\mathrm{df,3,L}}^{(u,u)} \right\} = \mathcal{D} + \mathcal{M}_{\mathrm{df},3}$$

- $\mathcal{D}$ contains all divergent contributions to $\mathcal{M}_3$, but depends on cutoff function $H(\vec{k})$

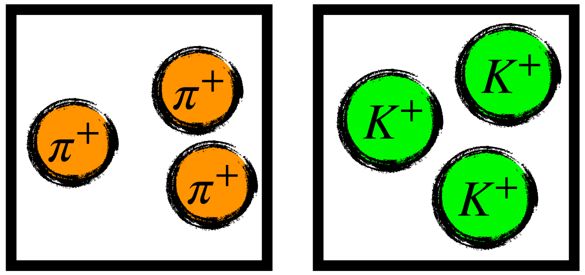

$\mathcal{D} = \mathcal{S}\{$
$\mathcal{M}_2$
G^∞
$\mathcal{M}_2$
$+$
$\mathcal{M}_2$
G^∞
$\mathcal{M}_2$
G^∞
$\mathcal{M}_2$
$+\ldots\}$

- $\mathcal{M}_{\mathrm{df},3}$ is divergence-free, equals $\mathcal{K}_{\mathrm{df},3}$ at leading order, and is also cutoff-dependent


$\mathcal{M}_{\mathrm{df},3} = \mathcal{K}_{\mathrm{df},3} + \mathcal{S}\{$
$\mathcal{M}_2$
ρ
$\mathcal{K}_{\mathrm{df},3}$
$+$
$\mathcal{K}_{\mathrm{df},3}$
ρ
$\mathcal{M}_2$
$+\ldots\}$

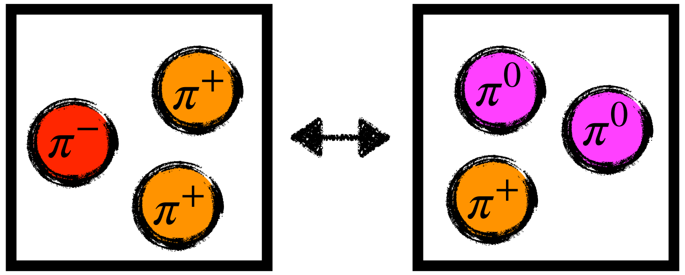
- "Decorations" ensure that $\mathcal{M}_3$ is unitary

Status: formalism

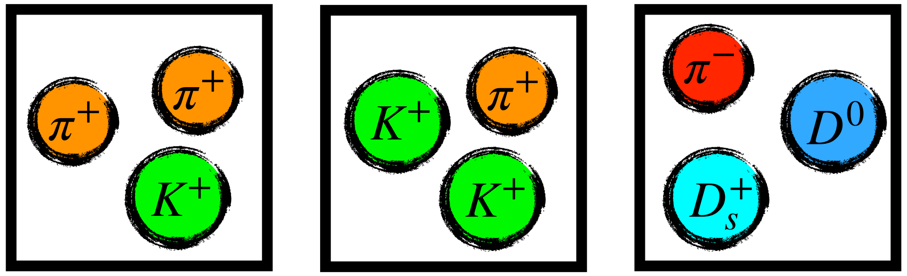


[Hansen & SRS 14,15 (RFT); Hammer, Pang, Rusetsky 17 (NREFT); Mai, Döring 17 (FVU)]

[Many subsequent refinements & tests]

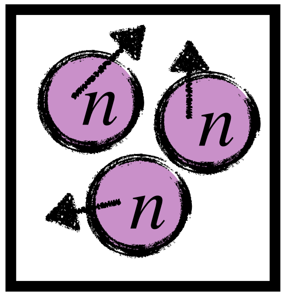

π^- π^+ π^+
π^0 π^0 π^+

[Hansen, Romero-López, SRS (RFT) 20;
Mai, Alexandru, Brett, Culver, Döring, Lee, Sadasivan (FVU) 21; Feng, Gil, Döring, Molina, Mai, Shastry, Szczepaniak (FVU) 24; Yan, Mai, Garofalo, Meißner, Liu, Liu, Urbach (FVU) 24; Yan, Mai, Garofalo, Feng, Döring, Liu, Liu, Meißner, Urbach (FVU) 25]

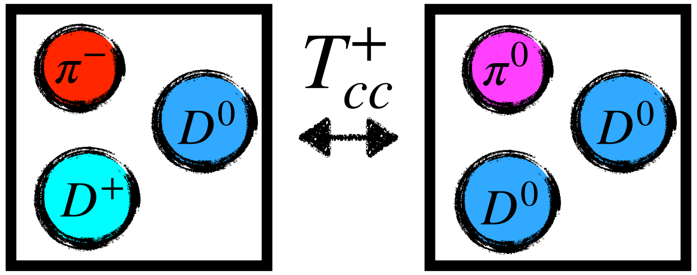

π^+ π^+ K^+
K^+ π^+ K^+
π^- D^0 D_s^+

[Blanton, Sharpe (RFT) 20, 21;

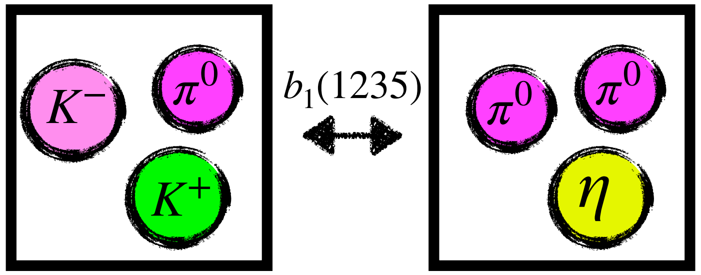
Feng et al. (FVU) 24]



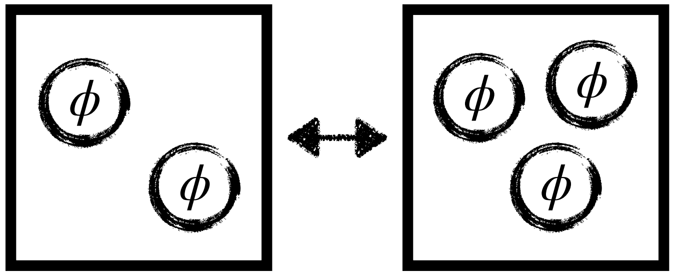
[Draper, Hansen, Romero-López, SRS 23]


π^- D^0 D^+
T_{cc}^+
π^0 D^0 D^0

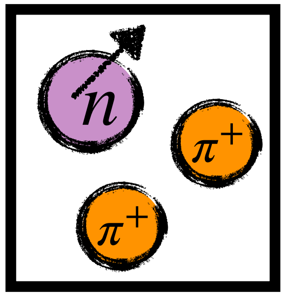
[Hansen, Romero-López, SRS 24]


K^- π^0 K^+
$b_1(1235)$
π^0 π^0 η

[Draper, SRS 24]

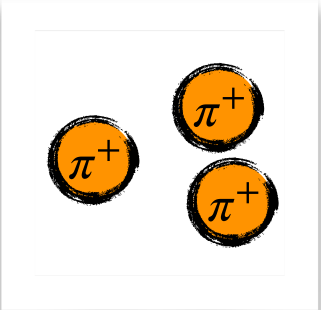

ϕ ϕ
ϕ ϕ ϕ

[Briceño, Hansen, SRS 17]

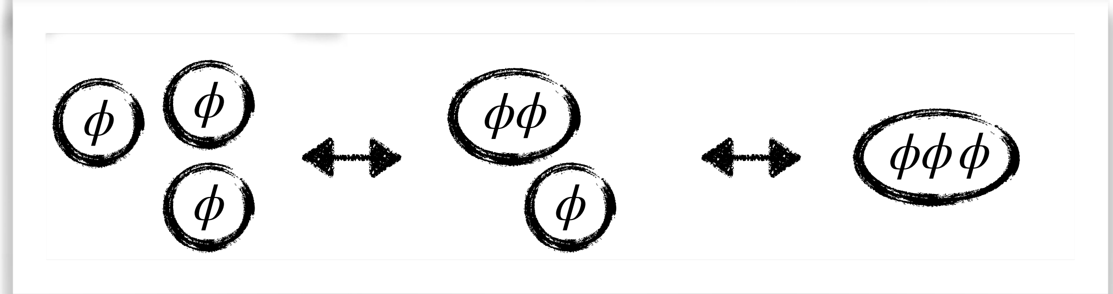


[Hansen, Romero-López, SRS 25]

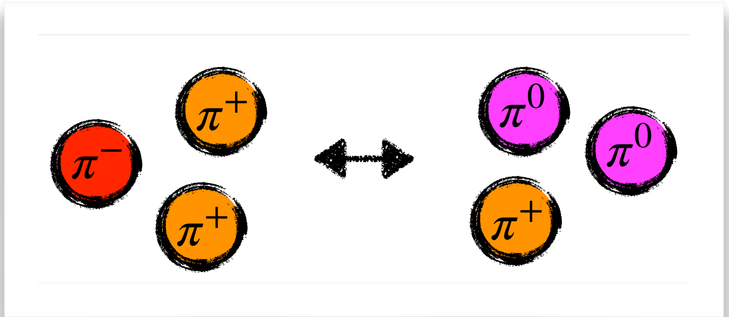
Status: solving integral equations



[Hansen, Briceño, Edwards, Thomas, Wilson (RFT) 20]


ϕ
ϕ
ϕ
$\phi\phi$
ϕ
$\phi\phi\phi$

[Jackura, Briceño, Dawid, Islam, McCarty, 20;
Dawid, Islam, Briceño (RFT) 23;
Dawid, Islam, Briceño, Jackura (RFT) 23]

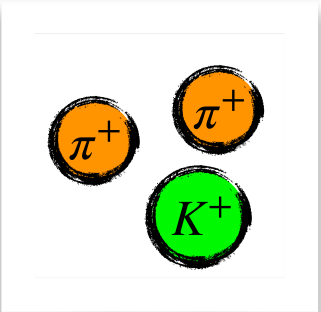

π^-
π^+
π^+
π^0
π^0
π^+

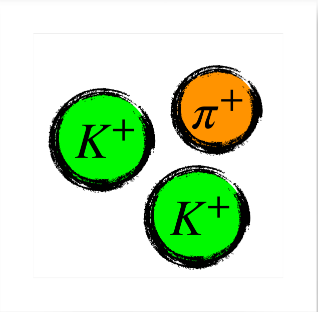
[Mai, Alexandru, Brett, Culver, Döring, Lee, Sadasivan (IVU) 21;

Jackura, Briceño, 23; Briceño, Costa, Jackura (RFT) 24; Jackura, Chambers, Briceño (RFT) 25; Briceño, Hansen, Jackura, Edwards, Thomas (RFT) 25;

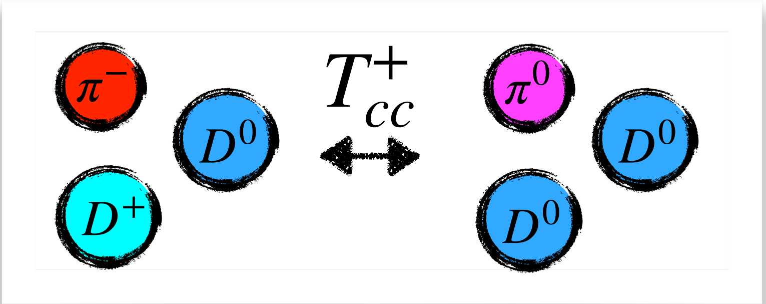
Yan, Mai, Garofalo, Feng, Meißner, Liu, Liu, Urbach (IVU) 24;

Yan, Mai, Garofalo, Feng, Döring, Liu, Liu, Meißner, Urbach (IVU) 25]



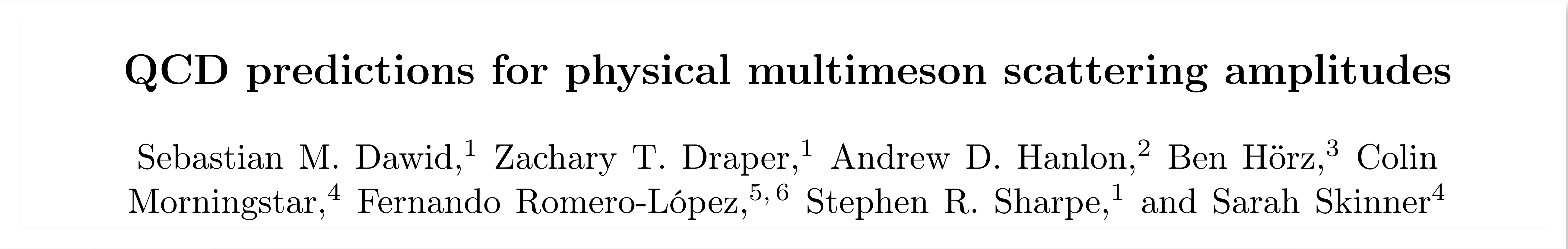


[Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López, SRS, Skinner, 25]


π^-
D^0
D^+
T_{cc}^+
π^0
D^0
D^0

[Dawid, Romero-López, SRS 24, 25]

Recent applications of three-particle formalism

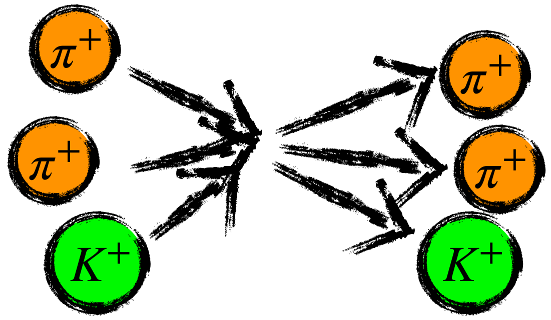


QCD predictions for physical multimeson scattering amplitudes

Sebastian M. Dawid,[1] Zachary T. Draper,[1] Andrew D. Hanlon,[2] Ben Hörz,[3] Colin Morningstar,[4] Fernando Romero-López,[5, 6] Stephen R. Sharpe,[1] and Sarah Skinner[4]

[2502.14348 (PRL) & 2502.17976 (PRD)]

Three PGBs at maximal isospin



$$\boxed{3\pi^+, \quad 3K^+, \quad \pi^+\pi^+K^+, \quad K^+K^+\pi^+}$$

- Benchmark system, with simple repulsive dynamics
  - First calculation at physical quark masses
  - Study $J^P = 0^-, 1^+, 2^-$, including subchannels with $\ell = 0,1,2$
  - Compare to expectations from ChPT
- Use CLS ensembles ($\mathcal{O}(a)$ improved Wilson) + GEVP

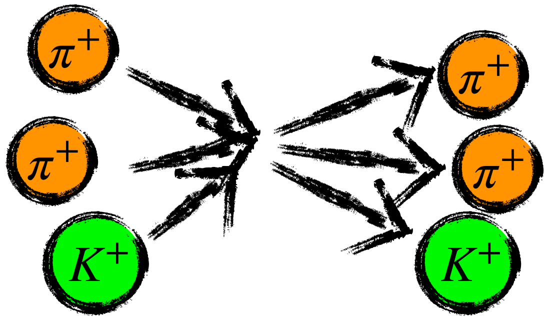
	$(L/a)^3 \times (T/a)$	M_π[MeV]	M_K[MeV]	$N_{\rm cfg}$	$M_\pi L$
N203	$48^3 \times 128$	340	440	771	5.41
N200	$48^3 \times 128$	280	460	1712	4.42
D200	$64^3 \times 128$	200	480	2000	4.20
E250	$96^3 \times 192$	130	500	505	4.05

[Blanton, SRS, et al., PRL 2020 & JHEP 2021]
[Draper, SRS, et al., JHEP 2023]

New work

$$a \simeq 0.063 \text{ fm}$$

Three PGBs at maximal isospin



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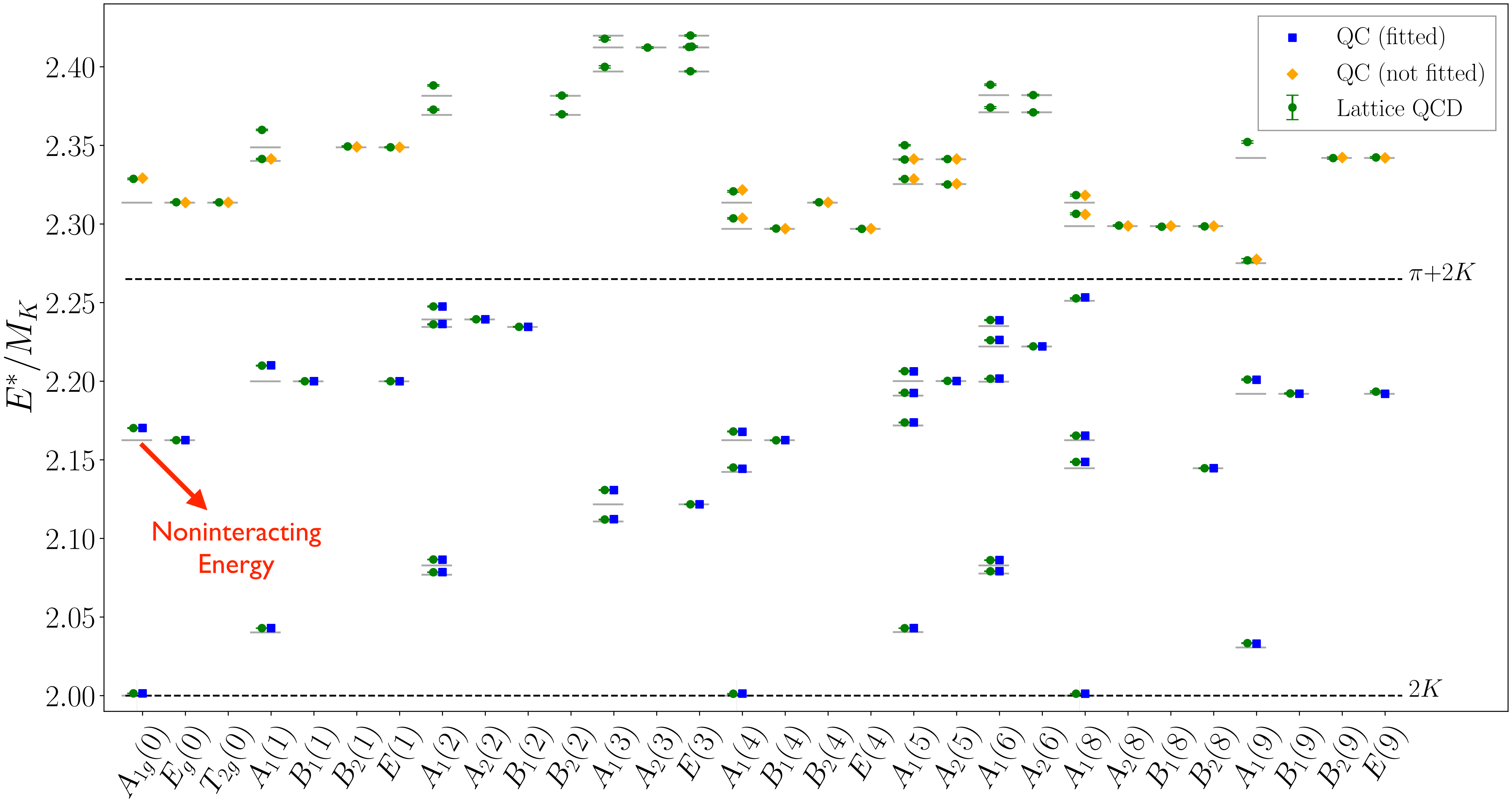
[Draper, SRS, et al., JHEP 2023]

New work

$$a \simeq 0.063 \text{ fm}$$

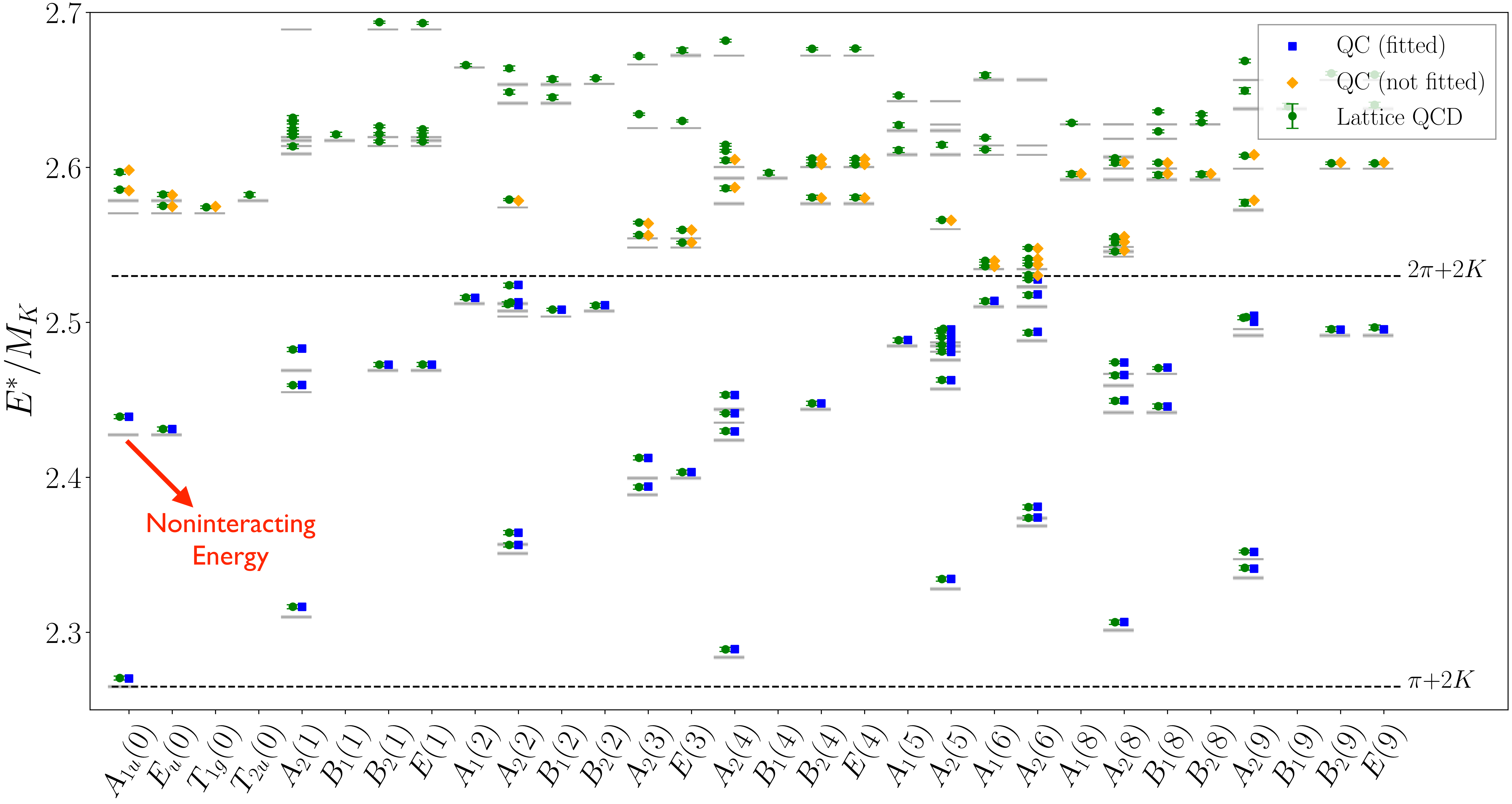
$$\text{tr } m_q = 2m_{ud} + m_s \simeq \text{const}$$

Example of spectrum



E250: K^+K^+

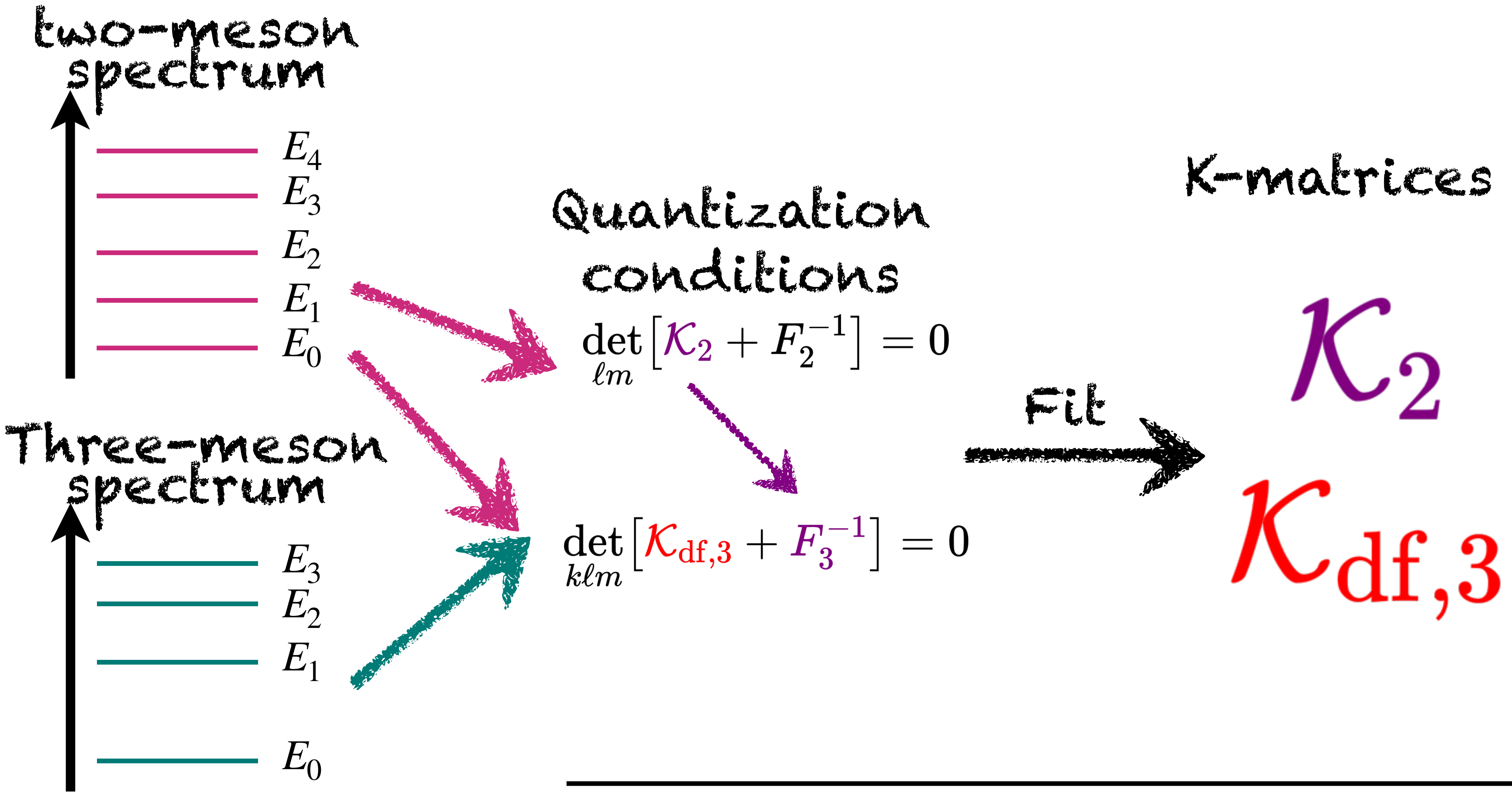
Example of spectrum



Noninteracting Energy

E250: $K^+K^+\pi^+$

Fits to spectra


two-meson spectrum
E_4
E_3
E_2
E_1
E_0
Three-meson spectrum
E_3
E_2
E_1
E_0
Fit

Quantization conditions

$$\det_{\ell m}\left[\mathcal{K}_2 + F_2^{-1}\right] = 0$$

$$\det_{k\ell m}\left[\mathcal{K}_{\mathrm{df},3} + F_3^{-1}\right] = 0$$

K-matrices

$$\mathcal{K}_2$$

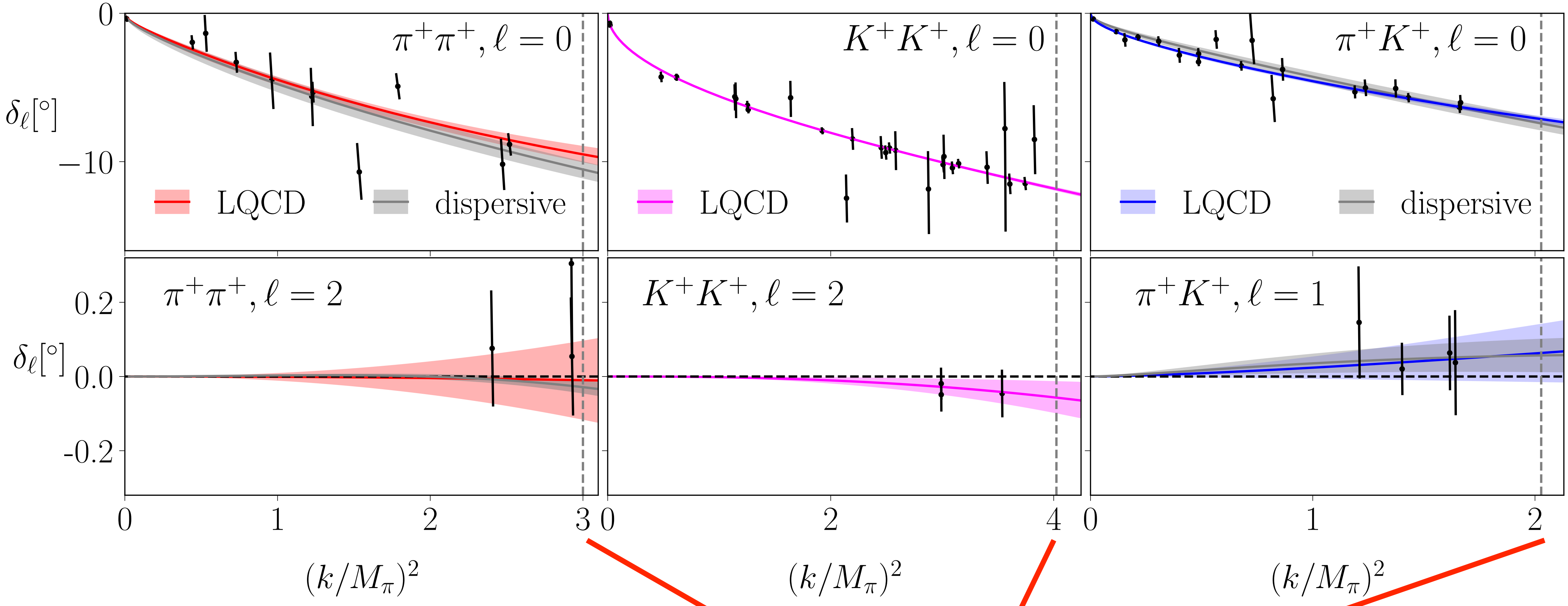
$$\mathcal{K}_{\mathrm{df},3}$$

channels	# levels	# parameters	χ^2/DOF
$2\pi/3\pi$	$34+32$	6	1.14
$2\pi/\pi K/2\pi K$	$24+25+23$	9	1.23
$2K/\pi K/2K\pi$	$25+40+50$	10	1.95
$2K/3K$	$40+53$	6	1.49

Parameters in $\mathscr{K}_2$ and $\mathscr{K}_3$: use threshold expansions

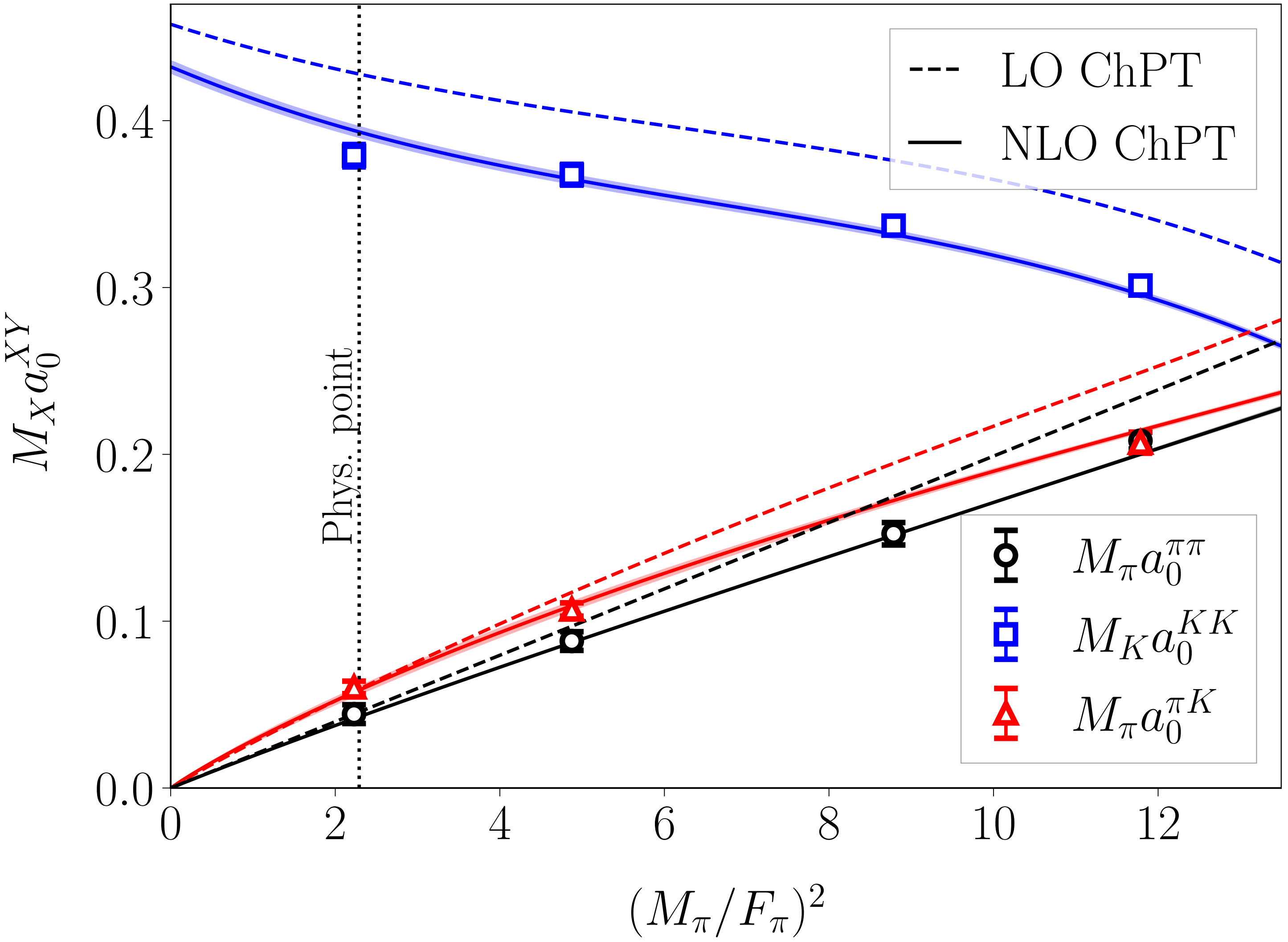
2-particle interactions

E250: ~physical quark masses



Inelastic thresholds

Scattering lengths



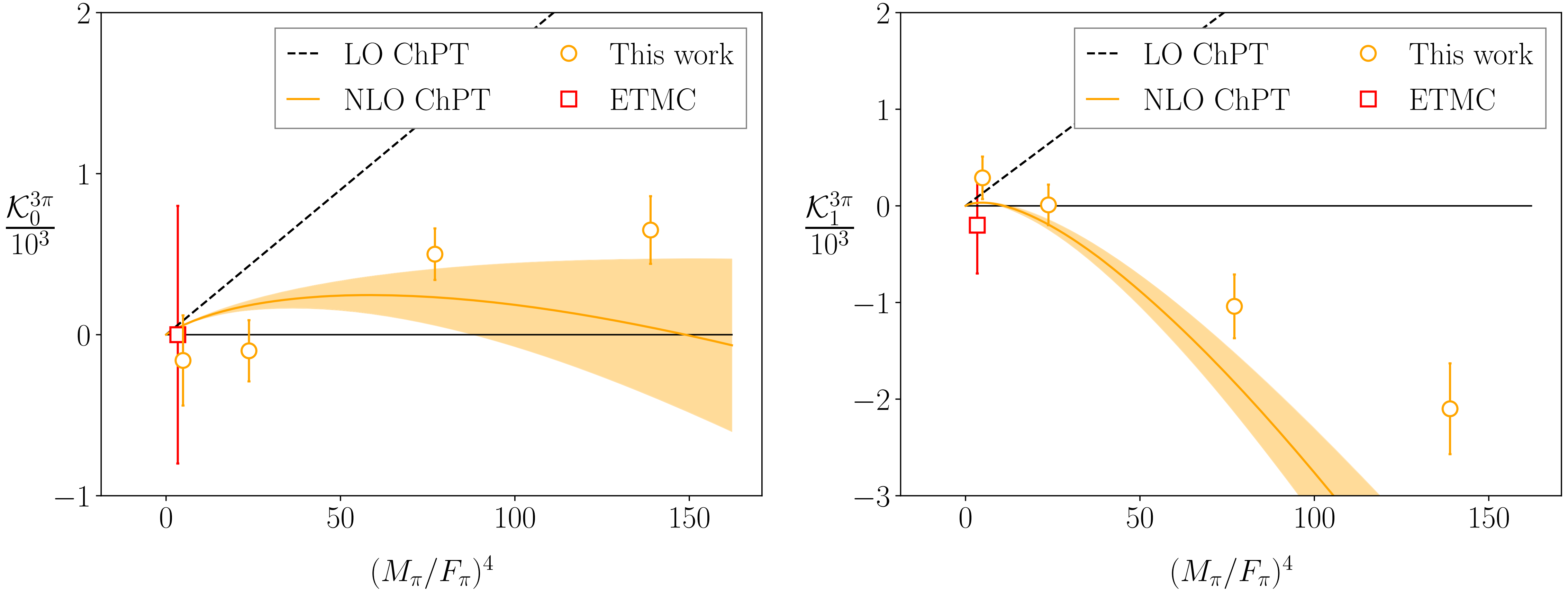
$\chi^2/\mathrm{dof} = 12.4/(11-2)$

- Simultaneous chiral fit to $\pi\pi$, πK, KK scattering lengths
  - Fit involves two LECs, one of which is determined with 2% stat. errors

Comparing $\mathcal{K}_{\text{df},3}$ to ChPT

- Parametrize $\mathcal{K}_{\text{df},3}$ in a threshold expansion, keeping first two terms
- From fits to spectra, we find non vanishing results for these terms at heavier quark masses
- Compare to NLO Chiral Perturbation theory calculation of $\mathcal{K}_{\text{df},3}$ for $3\pi \to 3\pi$

[Baeza-Ballesteros, Bijnens, Husek, Romero-López, SRS, Sjö, 2303.13206 (JHEP) & 2401.14293 (JHEP)]

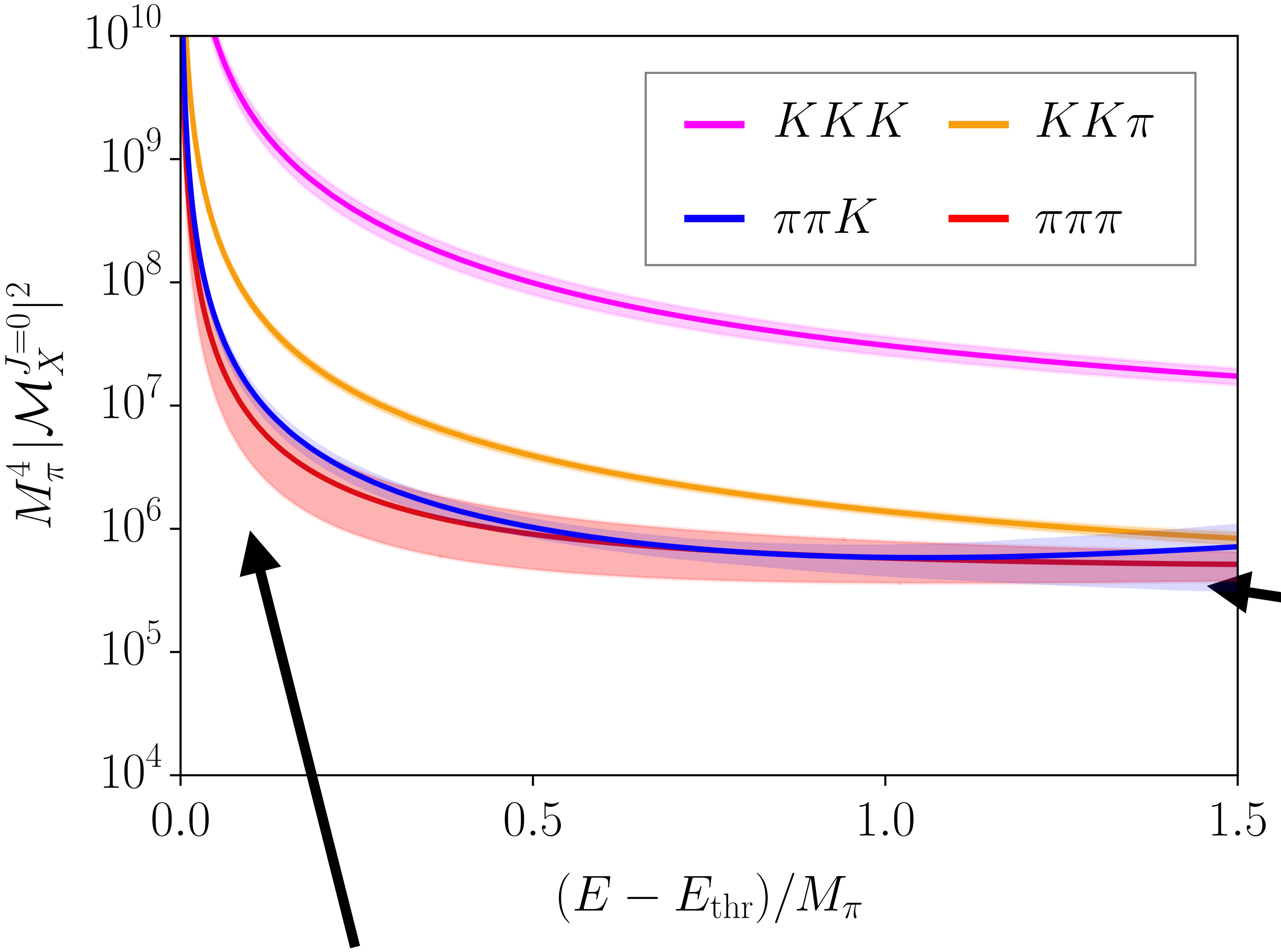


Large NLO corrections in ChPT resolve LO inconsistency

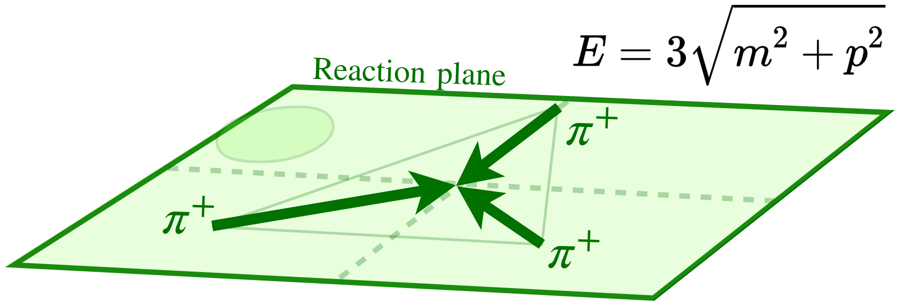
Physical 3-particle amplitudes

$M_\pi^4|\text{Amplitude}|^2$

$J^P = 0^-$

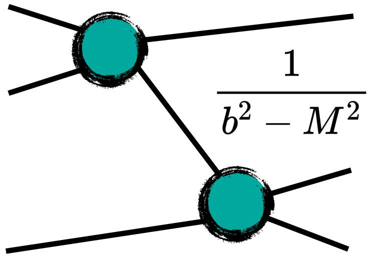


Equilateral kinematic configuration

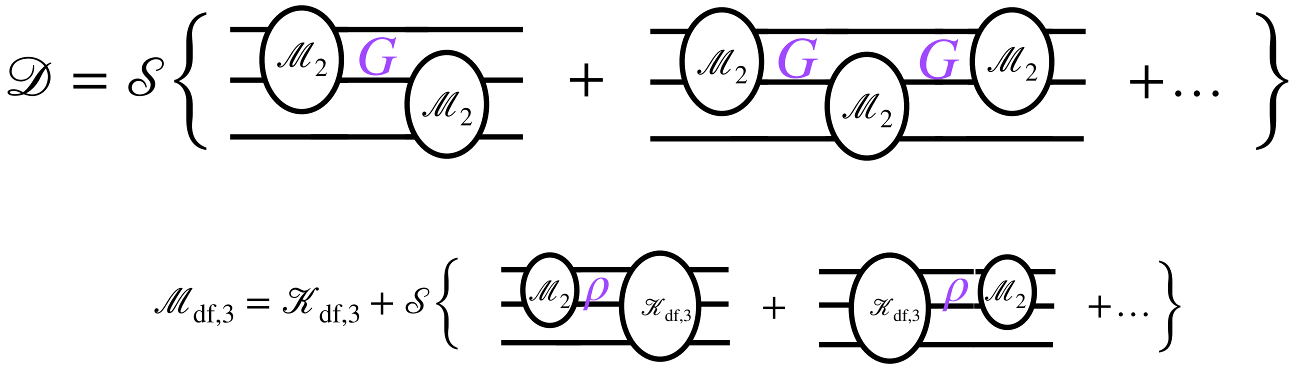

$E = 3\sqrt{m^2 + p^2}$
Reaction plane
π^+
π^+
π^+

Pion interactions are chirally suppressed

Divergent at threshold



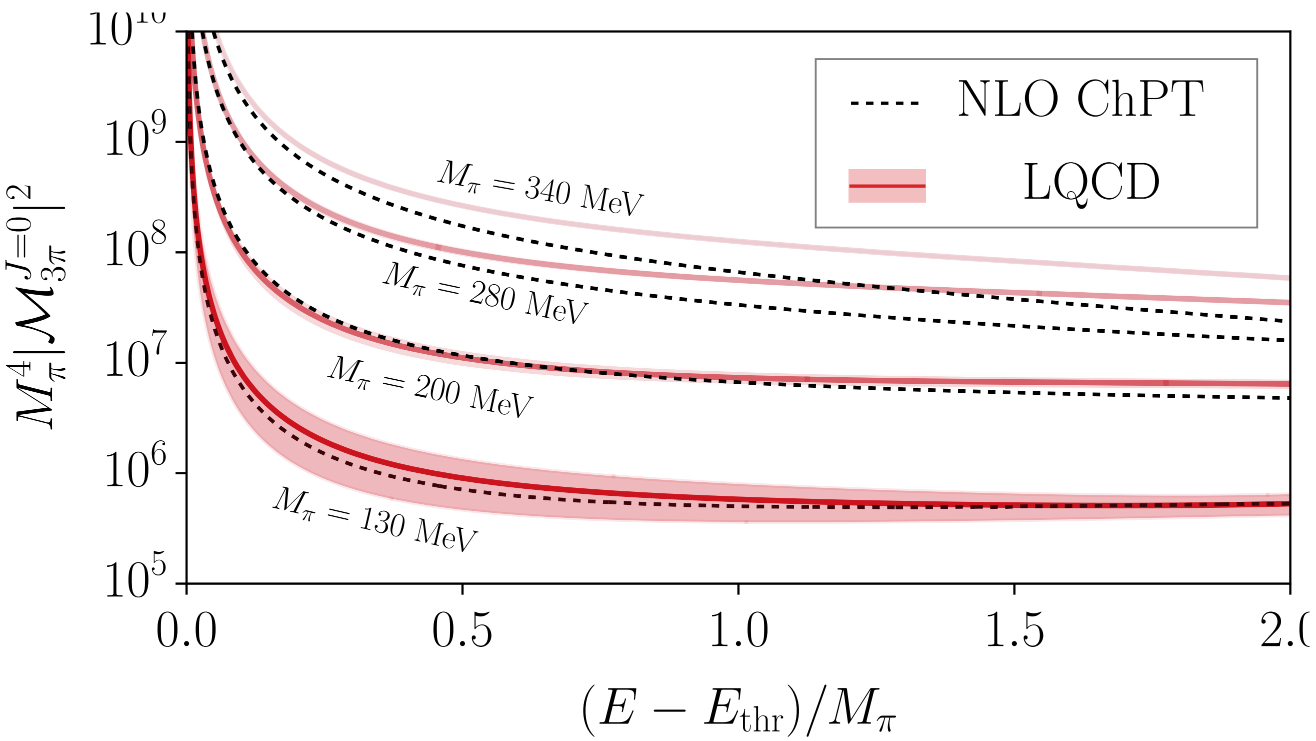
$$\mathcal{M}_3 = \mathcal{D} + \mathcal{M}_{\text{df},3}$$

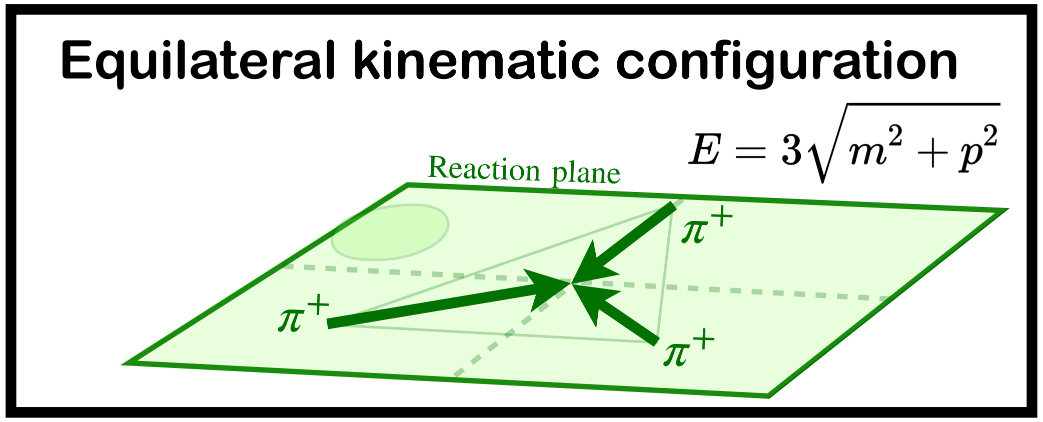

$\mathcal{D} = \mathcal{S}\{$
$\mathcal{M}_2$
G
$\mathcal{M}_2$
$+$
$\mathcal{M}_2$
G
$\mathcal{M}_2$
G
$\mathcal{M}_2$
$+ \ldots \}$
$\mathcal{M}_{\text{df},3} = \mathcal{K}_{\text{df},3} + \mathcal{S}\{$
$\mathcal{M}_2$
ρ
$\mathcal{K}_{\text{df},3}$
$+$
$\mathcal{K}_{\text{df},3}$
ρ
$\mathcal{M}_2$
$+ \ldots \}$

Comparison with ChPT

$M_\pi^4 |\text{Amplitude}|^2$

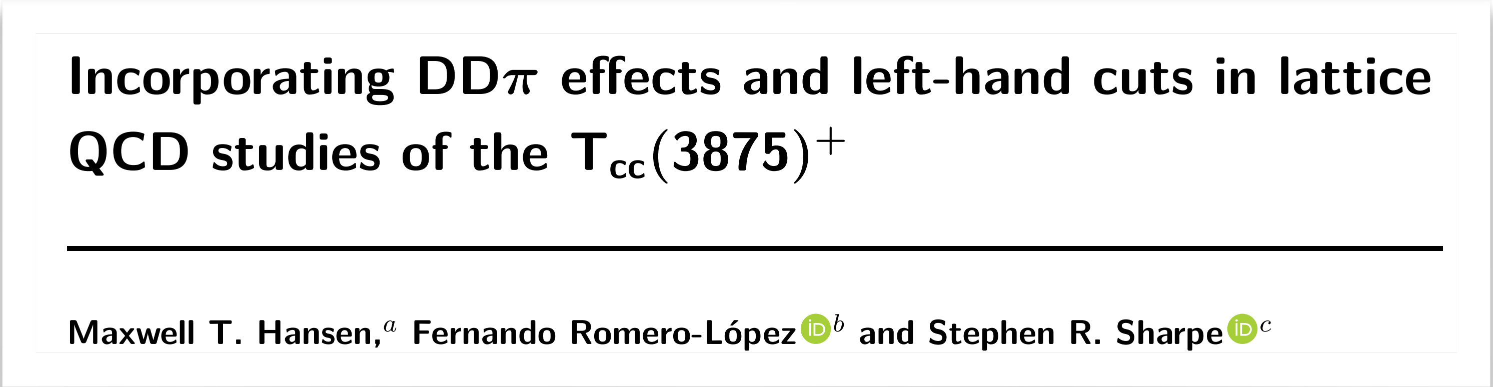
$J^P = 0^-$




Equilateral kinematic configuration
$E = 3\sqrt{m^2+p^2}$
Reaction plane
π^+
π^+
π^+

Observe expected breakdown of ChPT as M_π or p_π increase

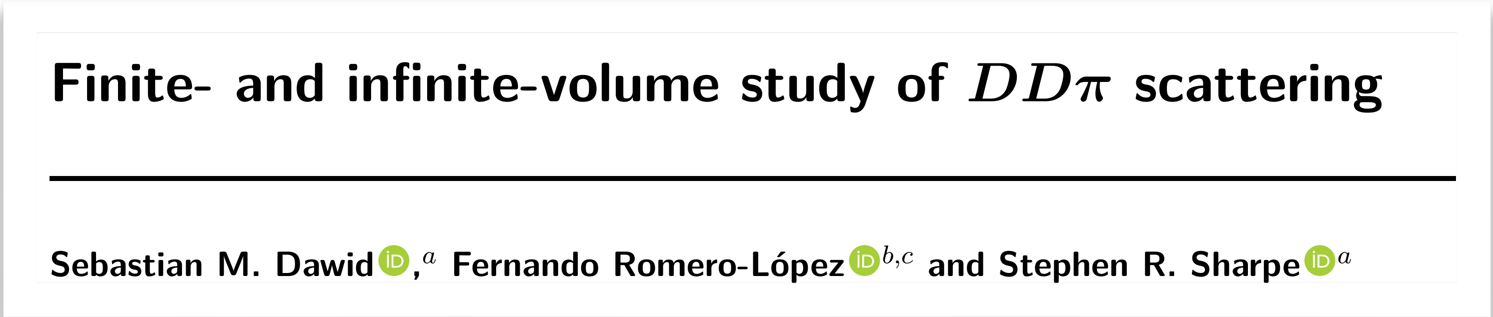
Using three-particle methods to study the $T_{cc}^{+}(3875)$



Incorporating DDπ effects and left-hand cuts in lattice QCD studies of the T$_{\rm cc}$(3875)$^+$

Maxwell T. Hansen,[a] Fernando Romero-López[b] and Stephen R. Sharpe[c]

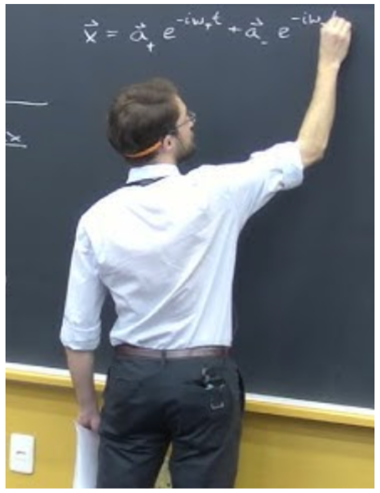
[2401.06609, JHEP]

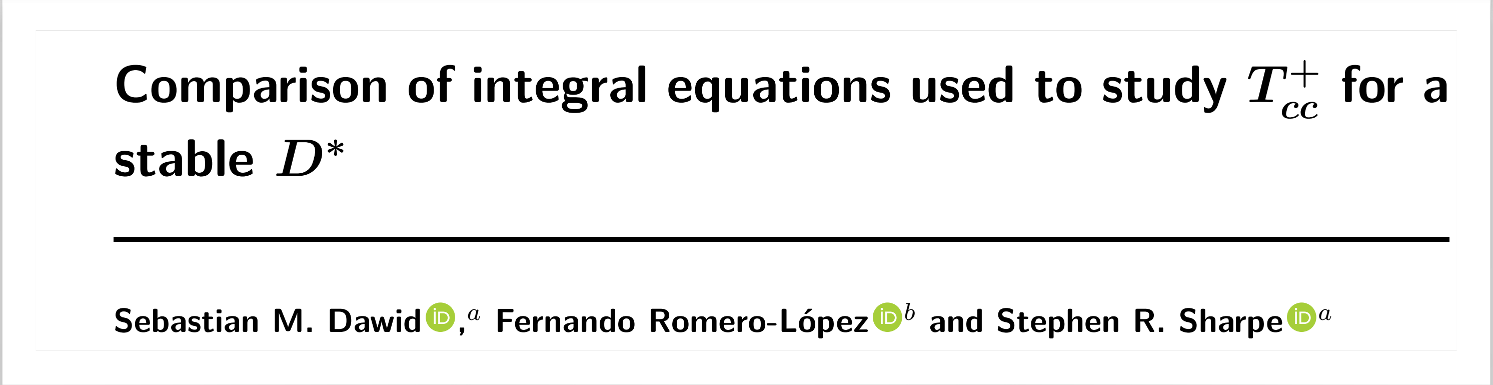


Finite- and infinite-volume study of $DD\pi$ scattering

Sebastian M. Dawid,[a] Fernando Romero-López[b,c] and Stephen R. Sharpe[a]

[2409.17059, JHEP]





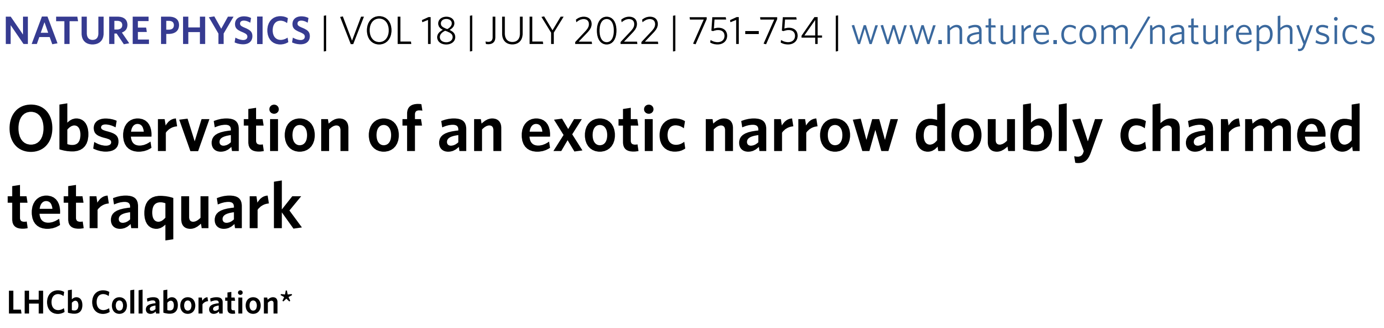
Comparison of integral equations used to study T_{cc}^{+} for a stable D^*

Sebastian M. Dawid,[a] Fernando Romero-López[b] and Stephen R. Sharpe[a]

[2505.05466, JHEP]

+ work in progress

Doubly-charmed tetraquark



NATURE PHYSICS | VOL 18 | JULY 2022 | 751–754 | www.nature.com/naturephysics

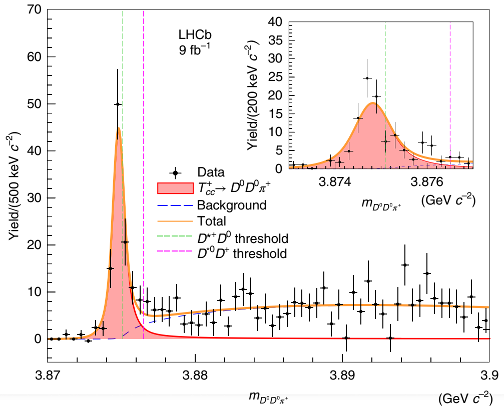
Observation of an exotic narrow doubly charmed tetraquark

LHCb Collaboration*

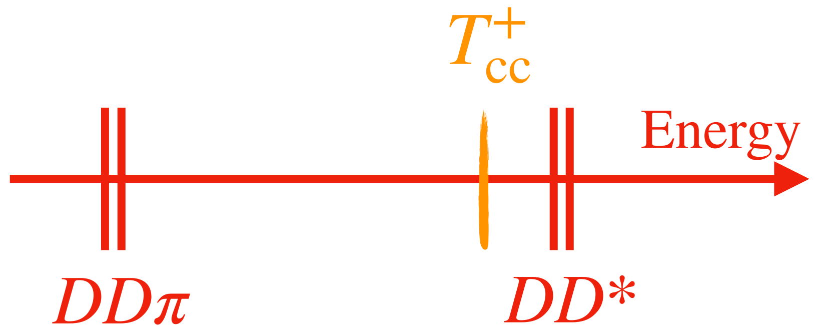
$$T_{cc}^{+}(3875)$$

$$I = 0,\ J^P = 1^+$$

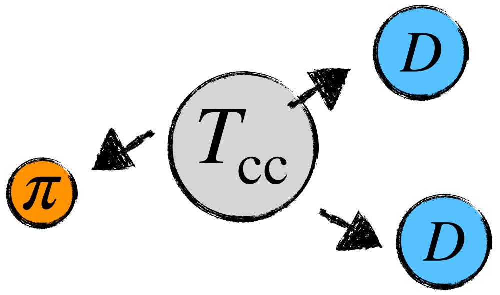
$$cc\bar{u}\bar{d}$$



Experiment



- **For physical quark masses is a three-body resonance**



need three-body formalism!

Adapted from: Fernando Romero-López

Doubly-charmed tetraquark

NATURE PHYSICS | VOL 18 | JULY 2022 | 751–754 | www.nature.com/naturephysics

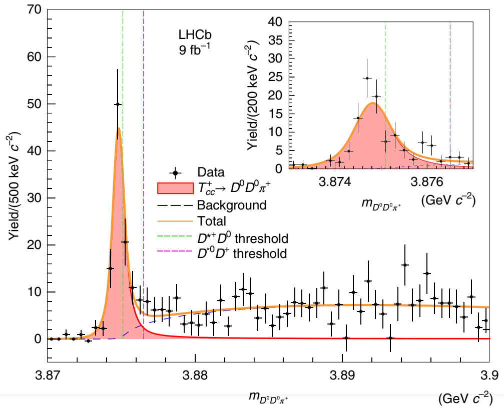
Observation of an exotic narrow doubly charmed tetraquark

LHCb Collaboration*

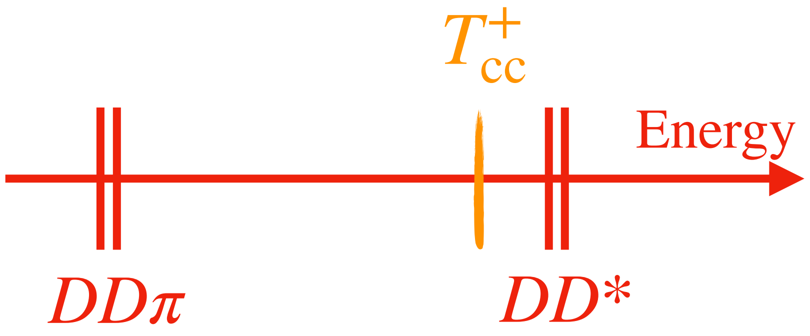
$T_{cc}^+(3875)$

$I = 0,\ J^P = 1^+$

$cc\bar{u}\bar{d}$



Experiment



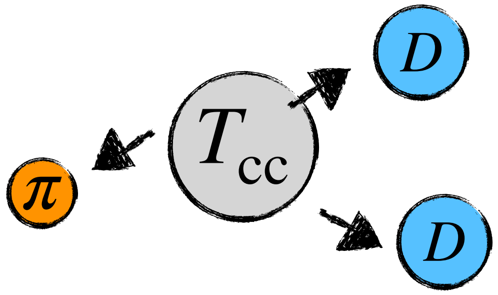
T_{cc}^+

Energy

$DD\pi$

DD^*

- **For physical quark masses is a three-body resonance**



D

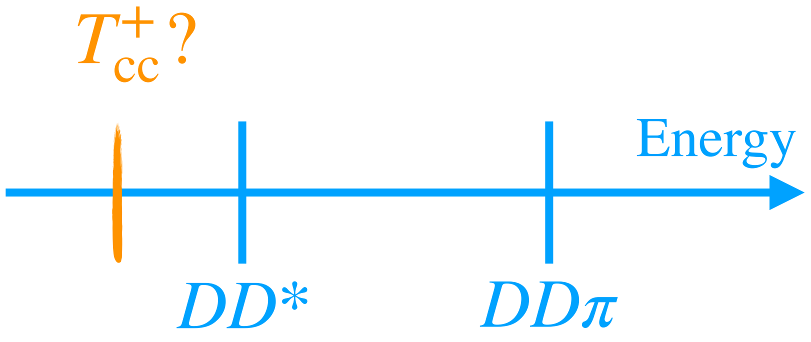
T_{cc}

π

D

need three-body formalism!

LQCD calculations with heavier-than-physical quarks



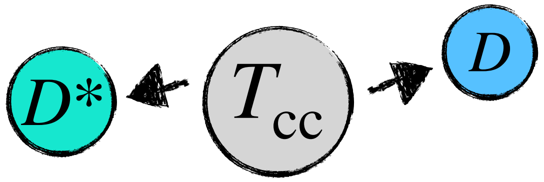
T_{cc}^+?

Energy

DD^*

$DD\pi$

- **D* is stable for slightly heavier-than-physical quarks**



D^*

T_{cc}

D

suitable for the two-body finite-volume formalism?

Adapted from: Fernando Romero-López

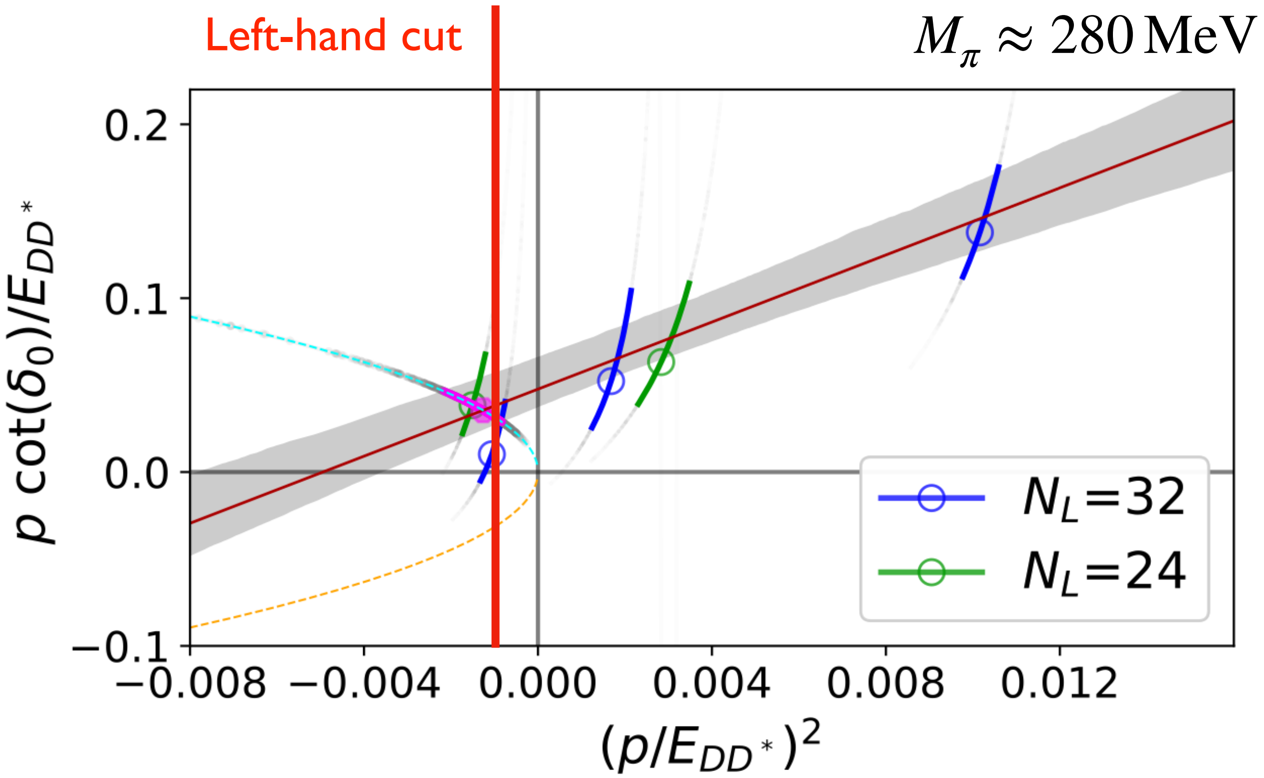
DD^* scattering

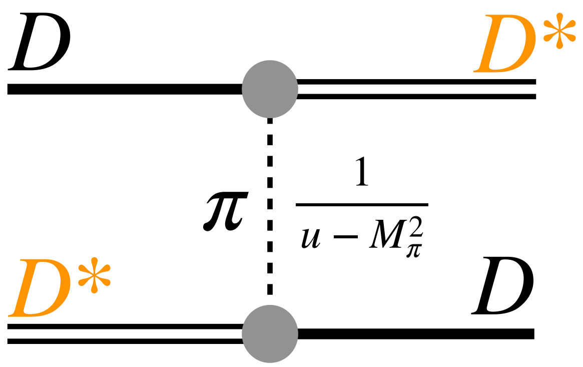
- Several LQCD studies using QC2 in T_{cc} channel

  [Chen et al., 2206.06185]
  [Lyu et al. (HALQCD), 2302.04505]
  [Padmanath & Prelovsek, 2202.10110]
  [Whyte, Thomas, Wilson, 2405.15741]
  [Several talks at Lattice 2025]

  - Use heavier-than-physical quarks
  - Find signature of virtual bound state
  - But QC2 fails at left-hand cut, below which phase shift becomes complex

[Padmanath, Prelovsek, arXiv:2202.10110]





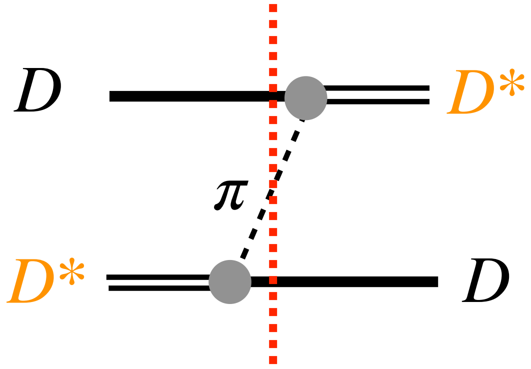
- Several solutions to failure have been proposed, by generalizing the QC2

  [Du et al (2408.09375); Abolnikov et al. (2407.04649), Bubna et al. (2402.12985); Meng et al. (2312.01930); Raposo, Hansen (2311.18793); Raposo, Hansen, Briceño, Jackura (2502.19375)]

3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

- Left-hand cut is a three-body effect



3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

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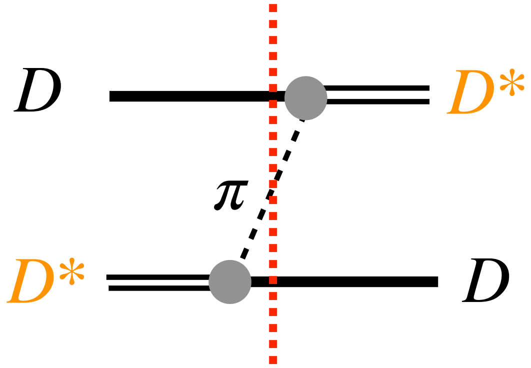
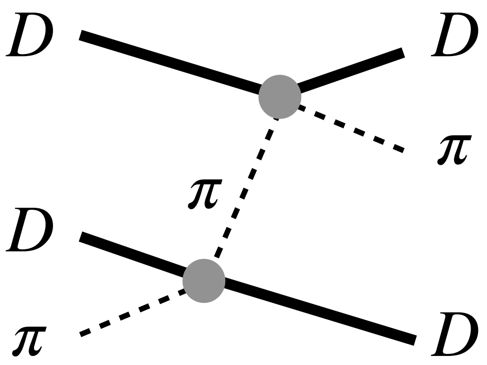


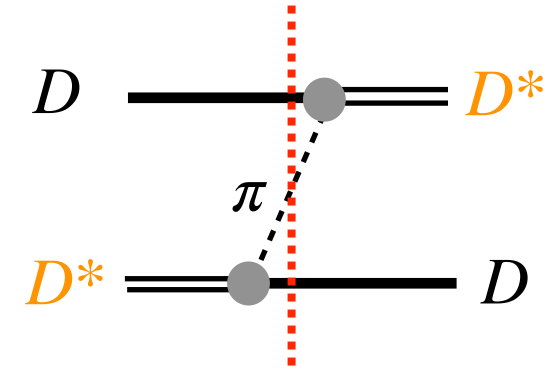
- Use $DD\pi$ QC3, but include D^* as a bound state in p-wave $D\pi$ channel



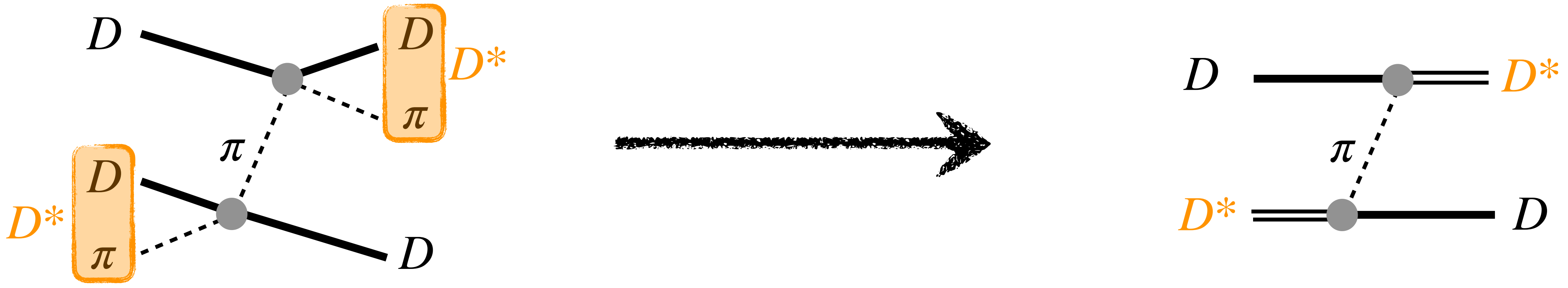
3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

- Left-hand cut is a three-body effect



- Use $DD\pi$ QC3, but include D^* as a bound state in p-wave $D\pi$ channel

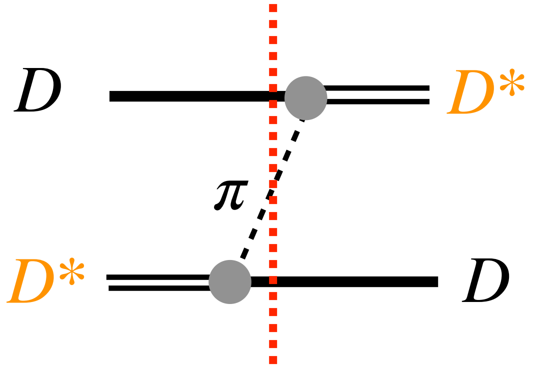


  - Finite-volume effects from u-channel pion exchange naturally incorporated

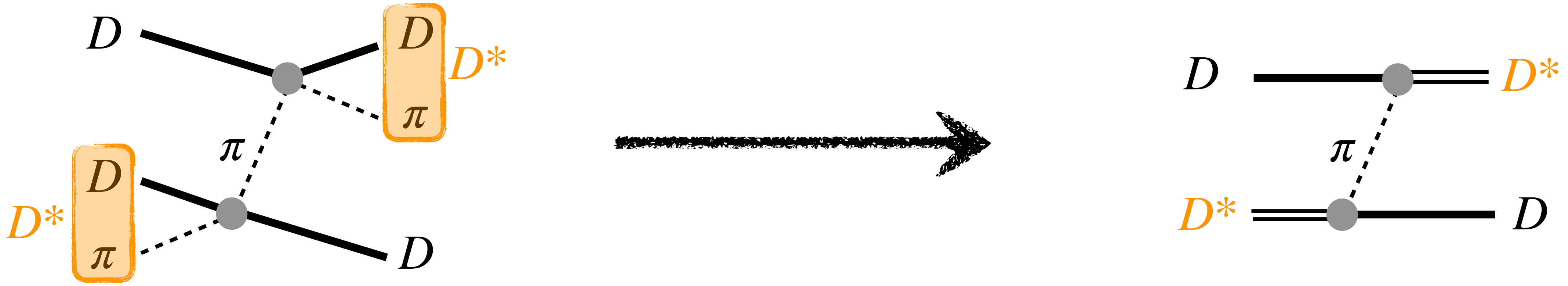
3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

- Left-hand cut is a three-body effect



- Use $DD\pi$ QC3, but include D^* as a bound state in p-wave $D\pi$ channel



  - Finite-volume effects from u-channel pion exchange naturally incorporated

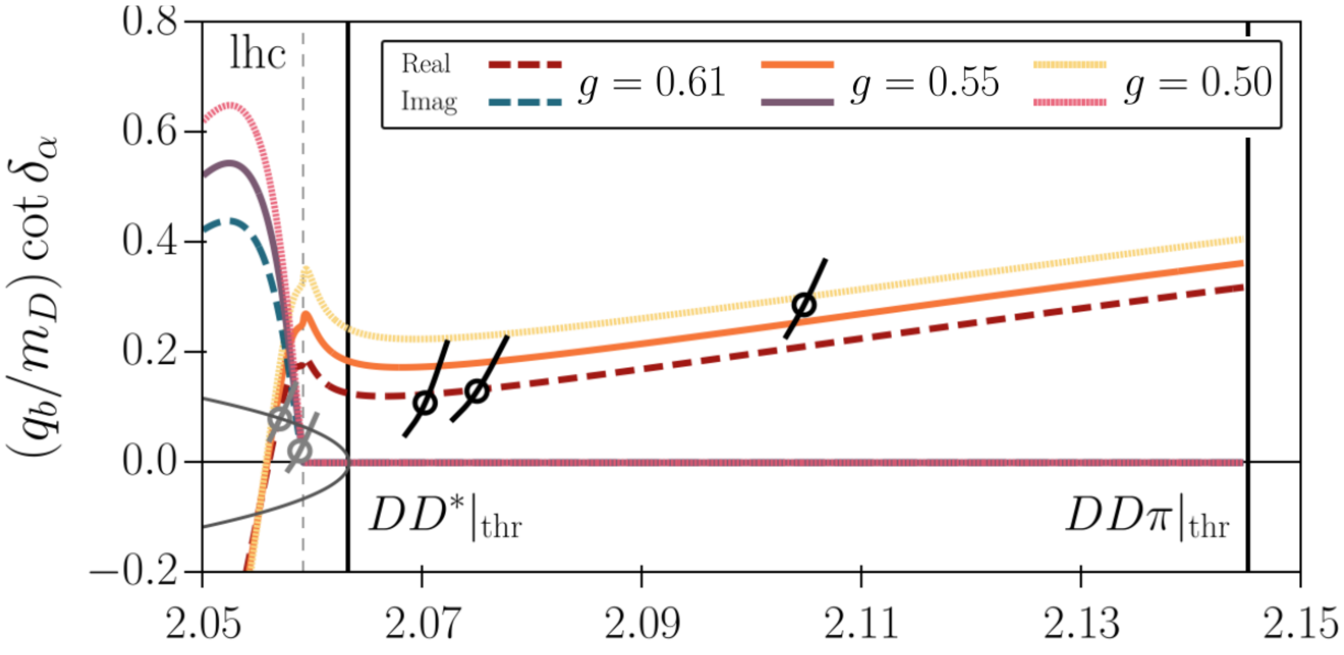
- Another advantage of QC3 approach:
  - Works for all choices of quark masses, including physical case of unbound D^*

- Successfully implemented in [Dawid, Romero-López, SRS, 2409.17059, JHEP & 2505.05466, JHEP; plus work in progress (talk by André Baião Raposo)]

Application to LQCD data

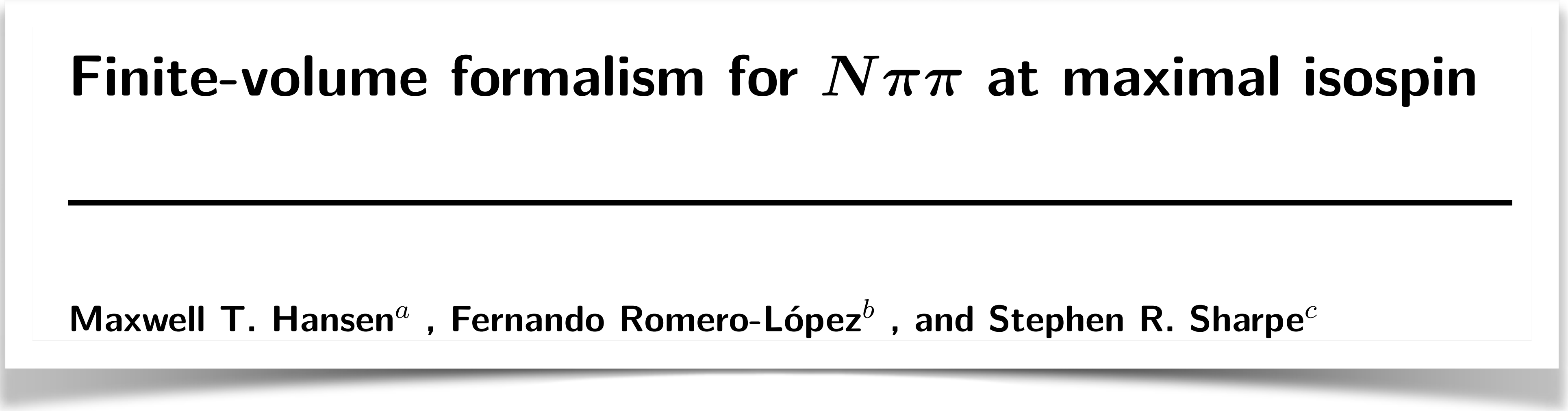
[Dawid, Romero-López, SRS, 2409.17059, JHEP]

- Proof of concept study to (limited) existing LQCD data with bound D^*
  - Choose p-wave $D\pi$ scattering amplitude to obtain D^*, for given choice of $g_{D^*D\pi}$
  - Choose reasonable s-wave $D\pi$ and DD amplitudes
  - Eyeball fit to results of [Padmanath, Prelovsek, arXiv:2202.10110] requires inclusion of $\mathscr{K}_{\text{df},3}$
  - Solve integral equations, analytically continue to D^* pole, use LSZ to obtain $\mathscr{M}(DD^*)$
  - Result shows appropriate behavior at and below left-hand cut



- Virtual bound-state pole moves into complex plane!
- Looking forward: complete analysis requires more extensive LQCD spectra
  - DD, $D\pi$, $DD\pi$ & DD^* levels

Towards a formalism for the Roper



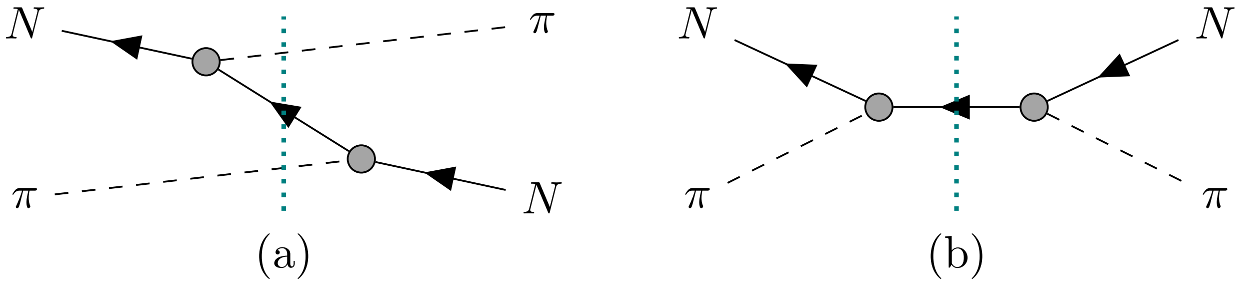
Finite-volume formalism for $N\pi\pi$ at maximal isospin

Maxwell T. Hansen[a] , **Fernando Romero-López**[b] , **and Stephen R. Sharpe**[c]

[2509.24778 (to appear in JHEP)

A failed hope for studying N(1440)

- Need a formalism that accommodates $N\pi + N\pi\pi$
  - Can one include N as a pole in p-wave $N\pi$ scattering, and use the $N\pi\pi$ formalism?
  - No! Nucleon pole lies below the singularity due to u-channel nucleon exchange



$$\sigma_{N\pi} = M_N^2 + 2M_\pi^2 \qquad \sigma_{N\pi} = M_N^2$$

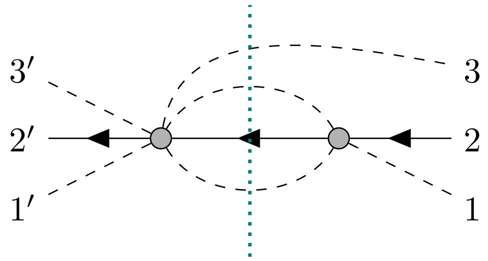
- Formalism for $N\pi\pi$ alone only valid at maximal isospin, which forbids mixing with $N\pi$
  - Interesting dynamics due to $\Delta\pi$ subchannel
  - Provides one 3-particle block of arbitrary isospin $N\pi + N\pi\pi$ formalism

Parameter space in QC3

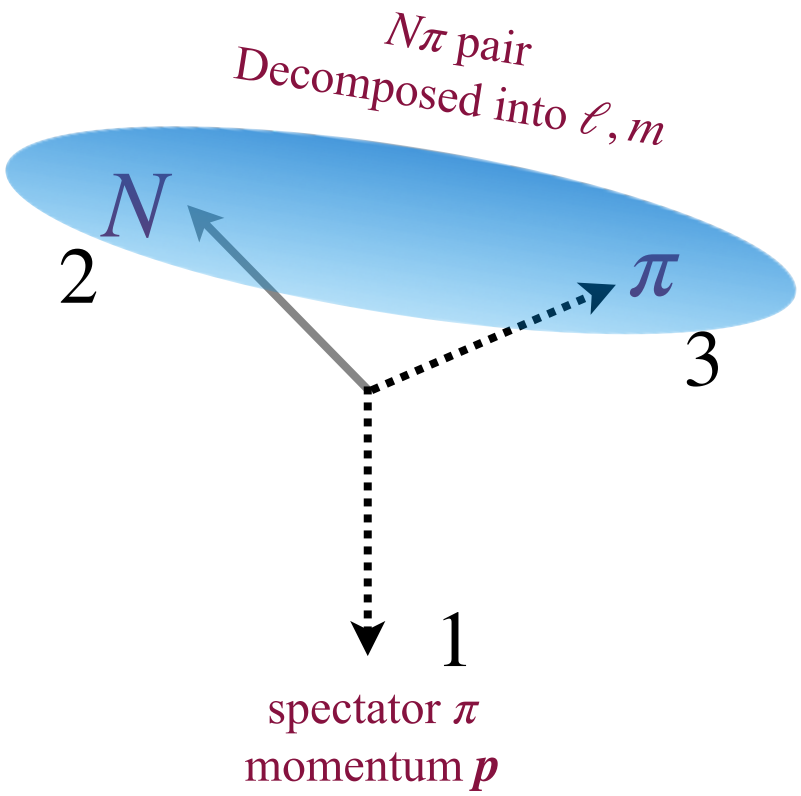
QC3 for $N\pi\pi$: $\det_{i\boldsymbol{p}\ell m m_s}\Big(1+\widehat{\mathcal{K}}_{\mathrm{df},3}(E^\star)\widehat{F}_3(E,\boldsymbol{P},L)\Big)=0$

Matrix indices are:
channel, spectator momentum,
pair angular momentum,
lab frame nucleon spin

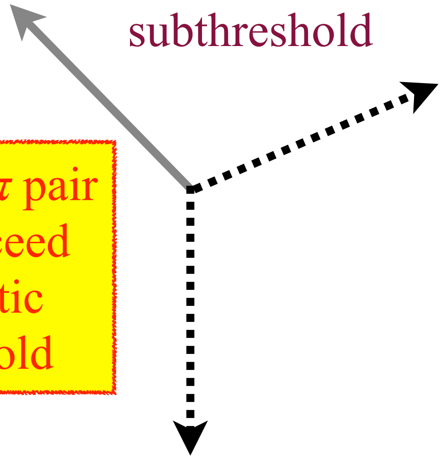
- A key quantity in derivation is B_3, $3 \to 3$ Bethe-Salpeter kernel (3 Particle Irreducible)
  - Must be nonsingular in range of kinematic parameters
  - Otherwise introduce uncontrolled L^{-n} finite-volume effects



- Parameter space (for pion spectator)
  - Energy E, momentum $\boldsymbol{P}$


$N\pi$ pair
Decomposed into ℓ, m
N
2
π
3
1
spectator π
momentum $\boldsymbol{p}$

Subthreshold singularities

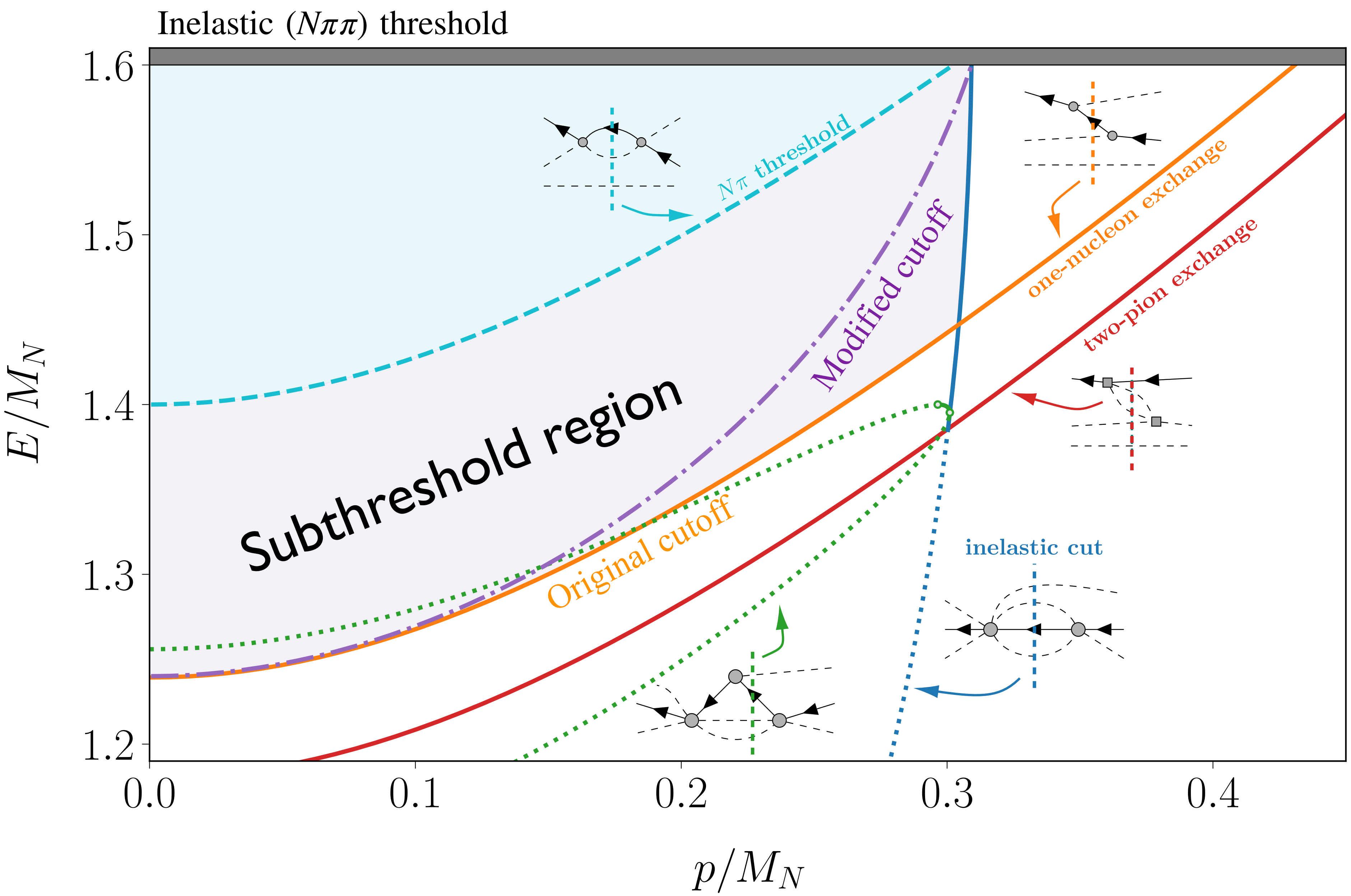


Nπ pair can be subthreshold

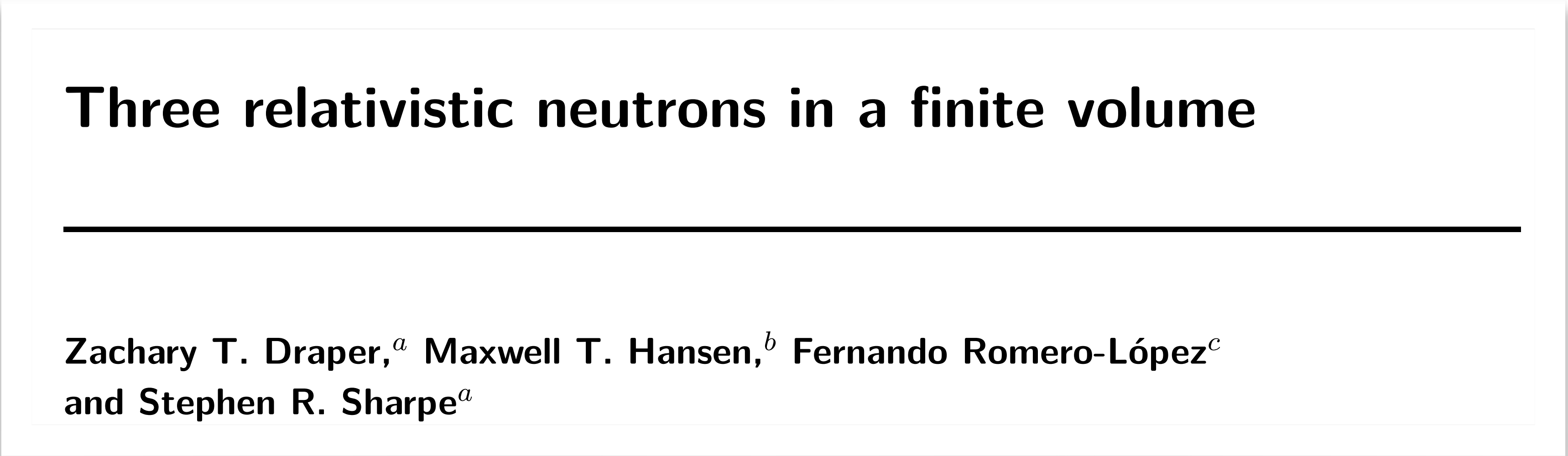
Other *Nπ* pair can exceed inelastic threshold

spectator *π* momentum ***p***

Parameter space for $\boldsymbol{P} = 0,\ M_\pi/M_N = 0.2$



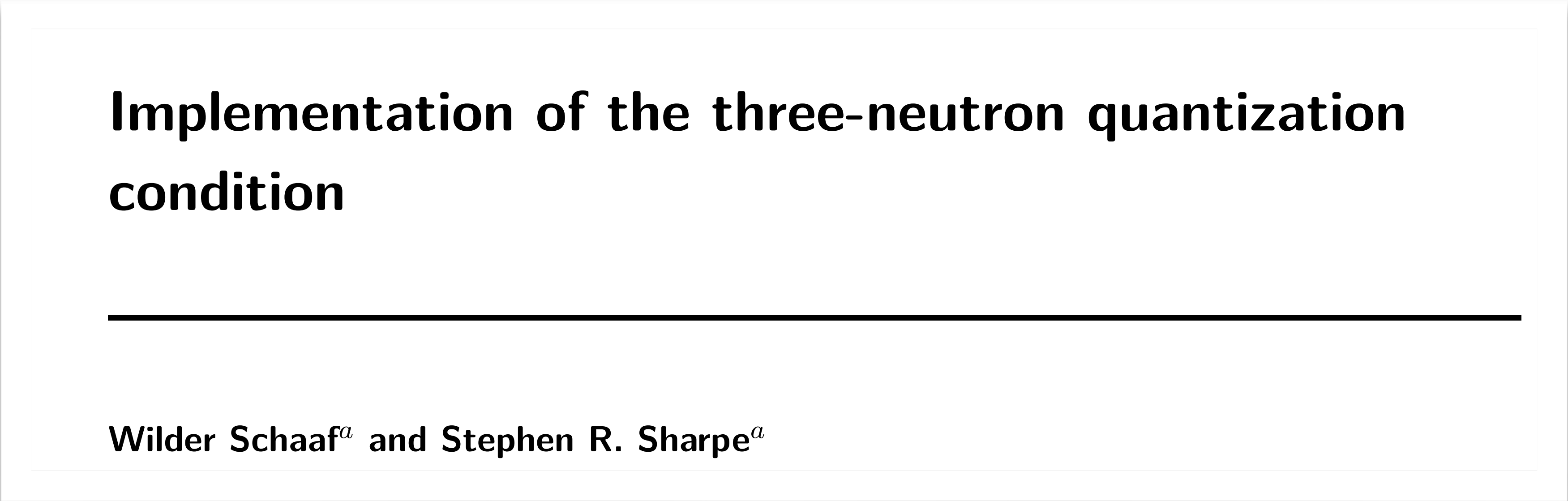
Dreaming of 3 neutrons



Three relativistic neutrons in a finite volume

Zachary T. Draper,[a] Maxwell T. Hansen,[b] Fernando Romero-López[c]
and Stephen R. Sharpe[a]

[2303.10219, JHEP]



Implementation of the three-neutron quantization condition

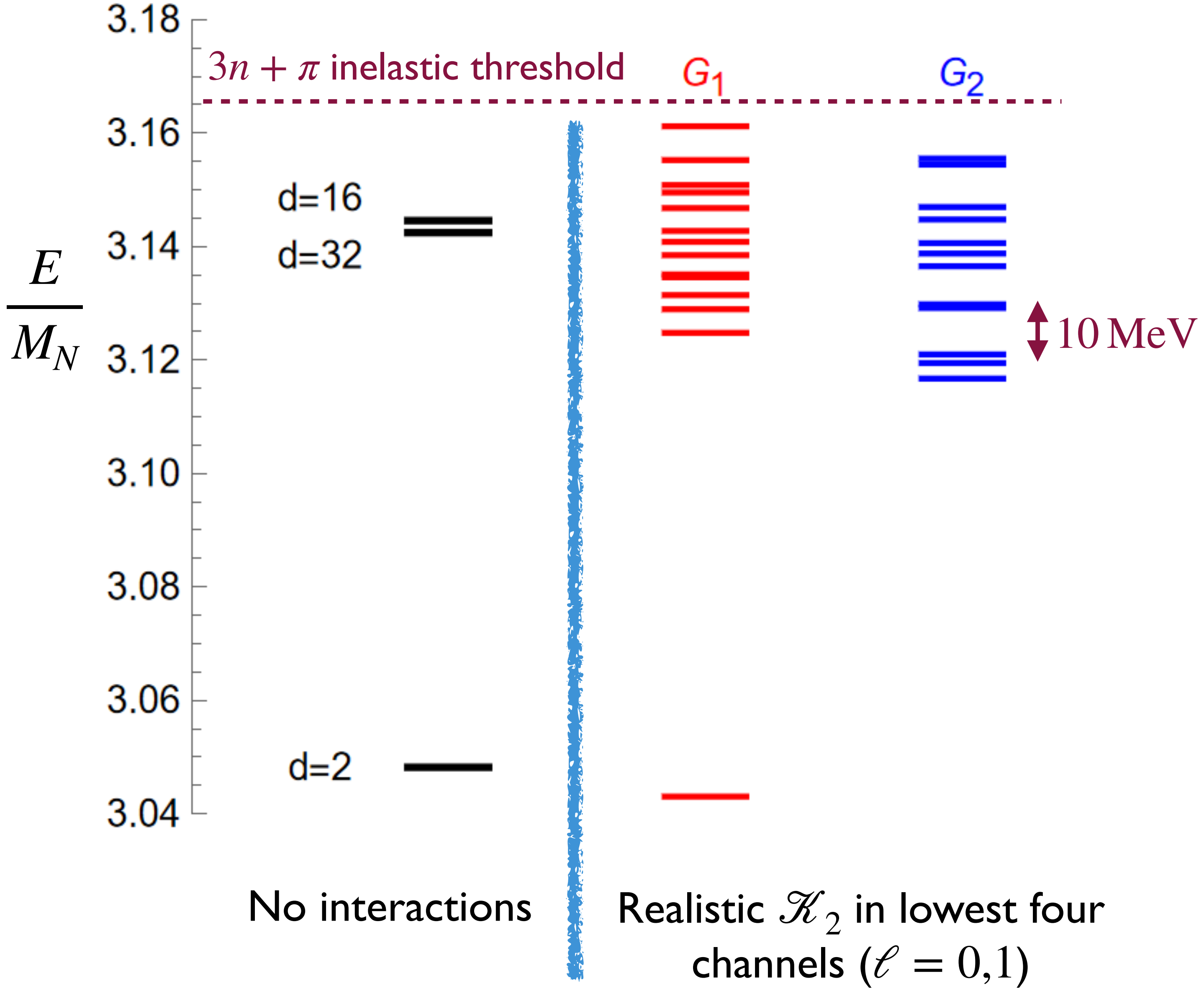
Wilder Schaaf[a] and Stephen R. Sharpe[a]

[2512.24508]

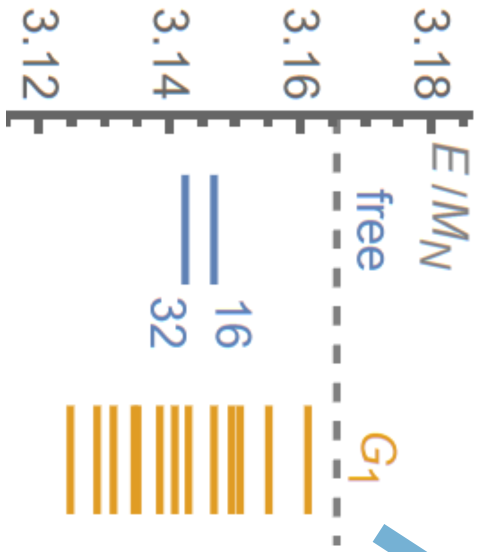
Spectrum in realistic setup

- Physical masses, $L \approx 4.3\ \mathrm{fm}$:
  - $M_N L = 20,\ M_\pi = 0.15 M_N \Rightarrow M_\pi L = 3$

Levels in $\boldsymbol{P} = \frac{2\pi}{L}(0{,}0{,}1)$ frame

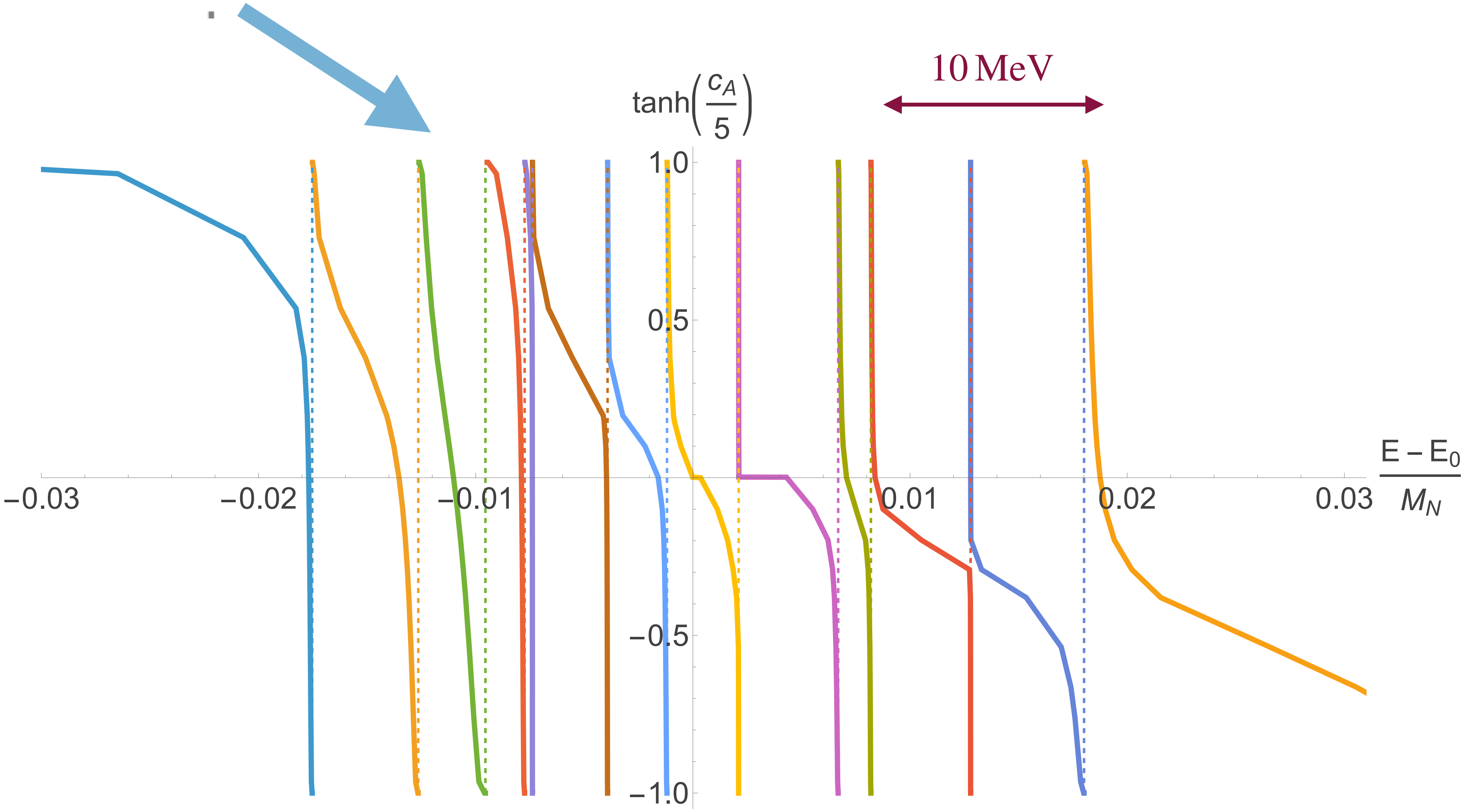


Spectrum in realistic setup



Add leading-order $\mathcal{K}_{\mathrm{df},3}$ in threshold expansion

Keep one term (out of two) with coefficient c_A



Summary & Outlook

Summary & Outlook

- Major steps have been taken in the three-particle sector
    - Formalism well established & cross checked, and almost complete
    - Several pioneering end-to-end applications to three-particle spectra from LQCD
    - Path to a calculation of $K \to 3\pi$ decay amplitudes is now open
- Near-term steps in implementation
    - $T_{cc}^{+} \to D^{*}D \to DD\pi$ for lighter quark masses
    - Extended calculations for $3\pi(I = 0,1,2)$
    - $N\pi\pi \leftrightarrow \Delta\pi(I = 5/2)$
- Final steps in three-particle formalism
    - $3\pi(I = 0) \leftrightarrow K\overline{K}$ (WZW term)
    - $NNN(I = \frac{1}{2}), N\pi\pi + N\pi$ [for Roper], $NN\pi + NN$
- Very recent review—talk by SRS at Lattice 2025 [2601.04147]

Challenges & Future

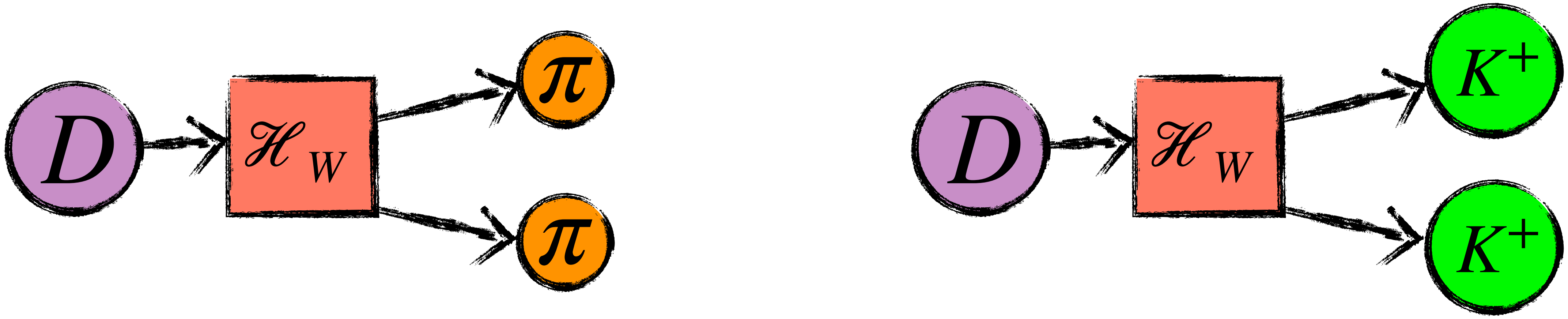
- Reducing model-dependence in fitting & analytic continuation
    - [Salg, Romero-López, Jay, 2506.16161] propose using Bayesian reconstruction + Nevanlinna interpolation
- Controlling all systematic errors (when will results be FLAG-ready?)
    - For which processes is precision important?
    - For three-nucleon interaction even a semiquantitative result would be useful
- Comparing formalisms in detail (e.g. RFT vs FVU)
    - Equivalent up to technical details (e.g. smooth vs hard cutoff functions)
    - Share data for cross-checking analysis? Bootcamp?
- Moving to more particles (needed, e.g., for $D \to \pi\pi, K\overline{K}$ & physical $\pi(1300)$)
    - QC4 (which is not yet known except in threshold expansion) may be practical
    - QCN seems impractical—combine QC2-4 with inclusive methods in some yet-to-developed manner?
        - "Scattering amplitudes from finite-volume spectral functions" [Bulava & Hansen, 1903.11735]
        - Squeeze as much information as we can out of the lattice simulations!
- Investigate alternative approaches
    - "Scattering amplitudes from Euclidean correlators: Haag-Ruelle theory and approximation formulae", [Patella & Tantalo, 2407.02069]
    - HALQCD approach? [Doi et al., 1106.2276: 3 nucleon potentials in NR regime]
- Combine LQCD results with EFT and amplitude-analysis methods (Exohad collab.)

Thank you!
Questions?

Backup slides

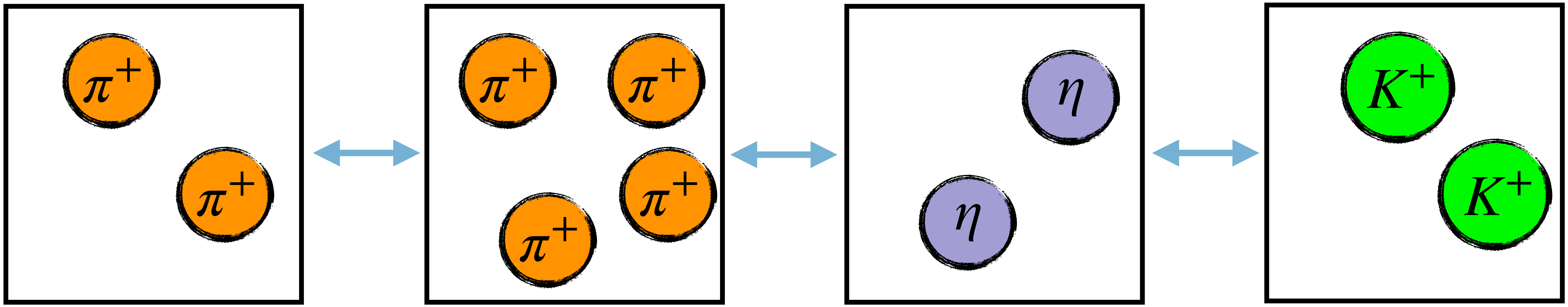
Long-term dream

CP violation in D decays



D
$\mathcal{H}_W$
π
π
D
$\mathcal{H}_W$
K^+
K^+

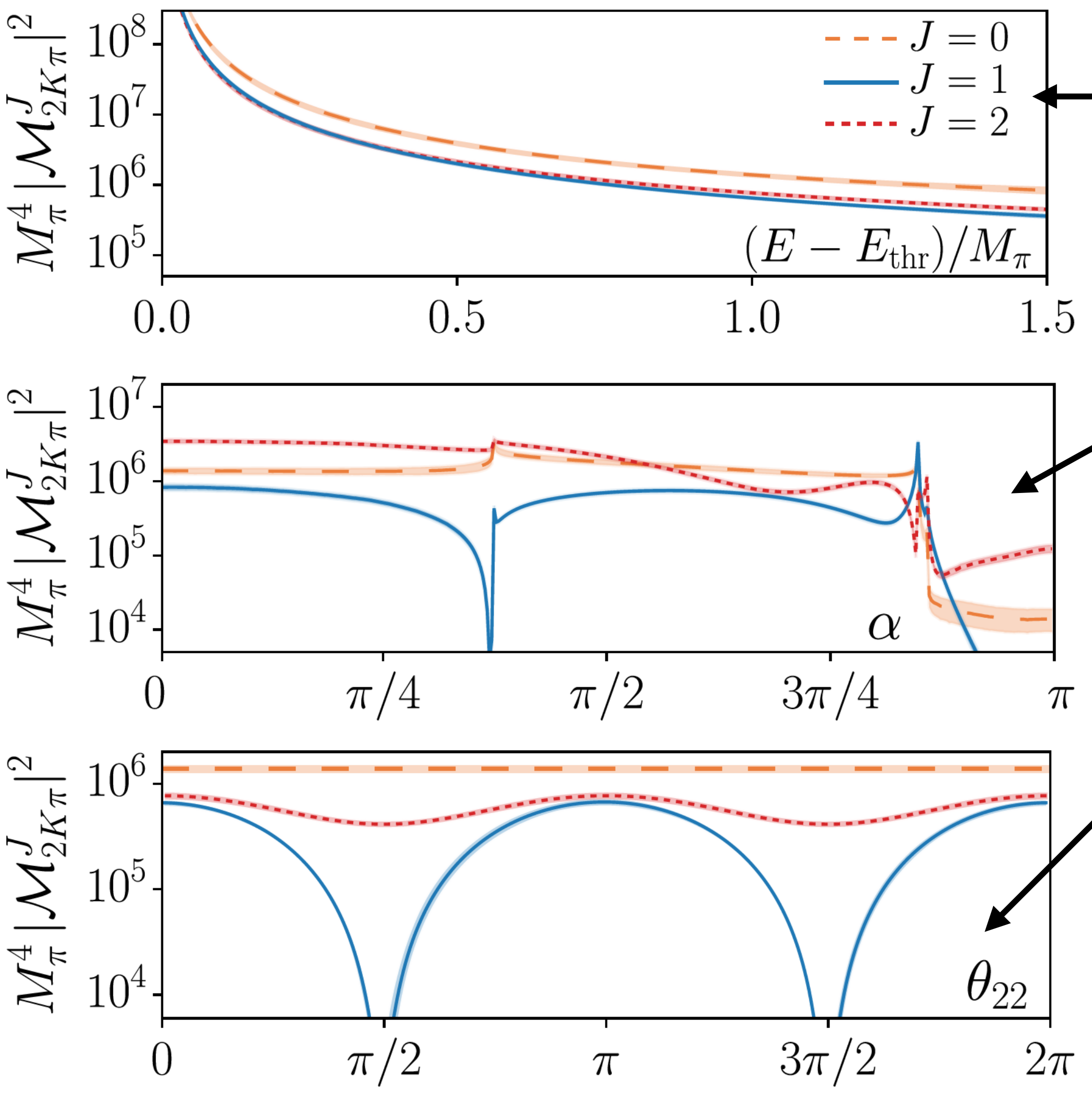
Challenge: finite-volume mixing with $4\pi, 6\pi, \eta\eta, \ldots$



π^+
π^+
π^+
π^+
π^+
π^+
η
η
K^+
K^+

4+-particle formalism not yet developed

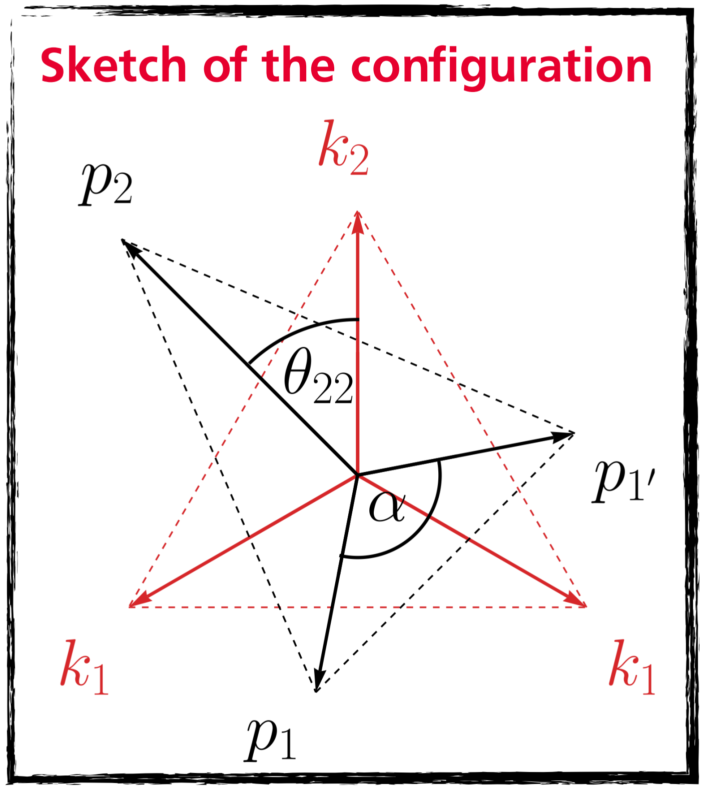
Some angular dependence



Different partial waves (largest is $J = 0$)

**Angle in the triangle:
equilateral ($\alpha = \pi/3$) vs isosceles**

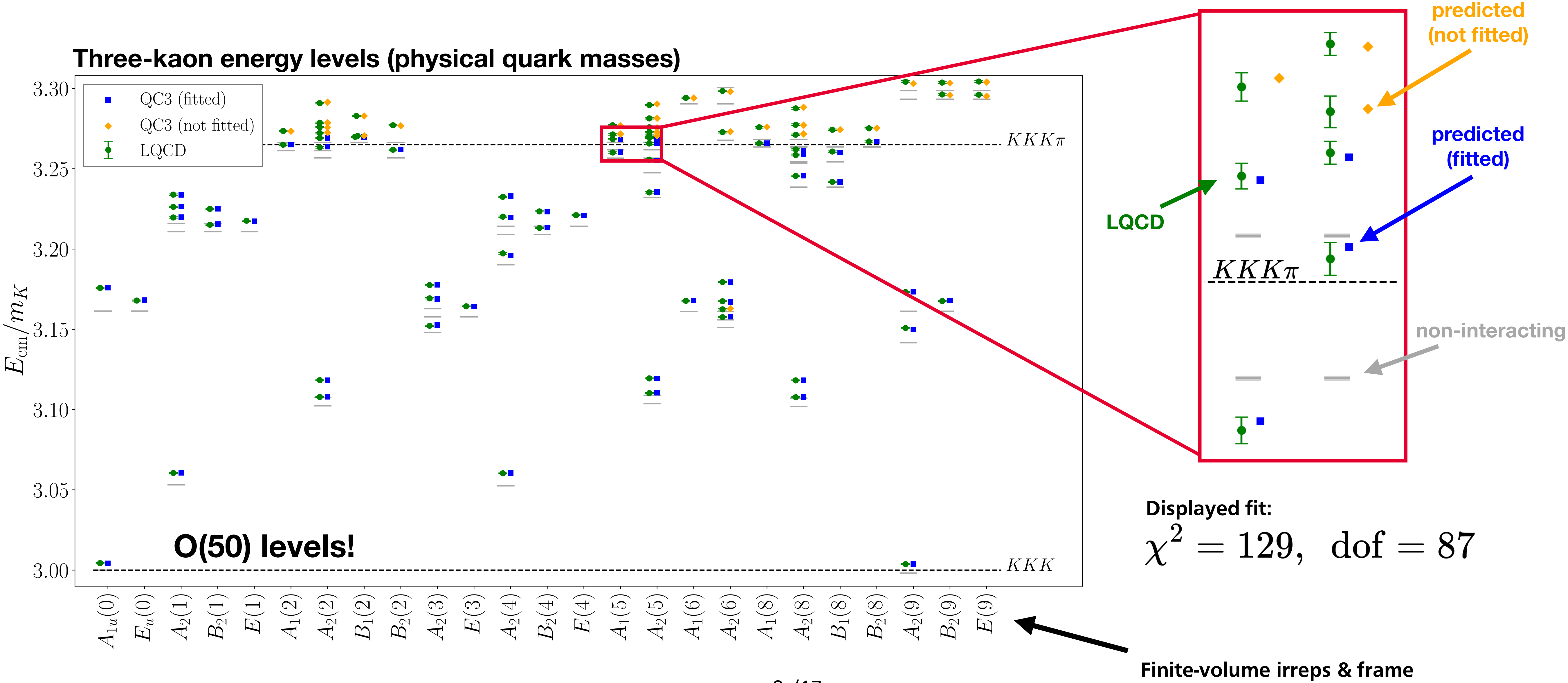
**Angle between
in/out momenta**



Sketch of the configuration

Three-meson spectrum

Three-kaon energy levels (physical quark masses)



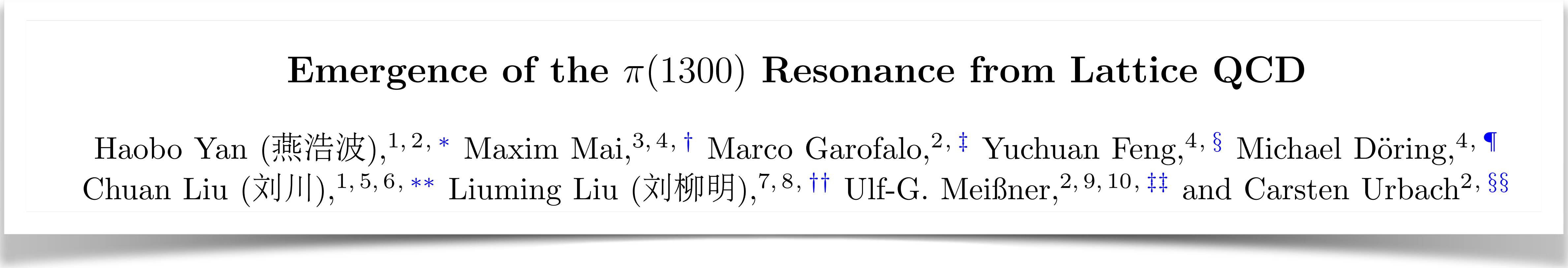
Displayed fit:

$$\chi^2 = 129, \quad \text{dof} = 87$$

Fernando Romero-López, Uni Bern

8 /17

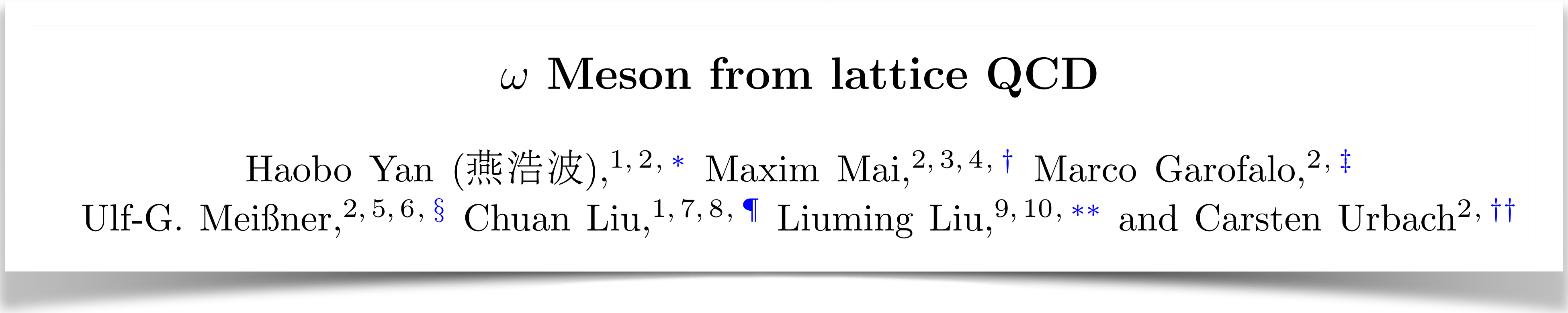
3-particle resonances



Emergence of the $\pi(1300)$ Resonance from Lattice QCD

Haobo Yan (燕浩波),[1,2,*] Maxim Mai,[3,4,†] Marco Garofalo,[2,‡] Yuchuan Feng,[4,§] Michael Döring,[4,¶]
Chuan Liu (刘川),[1,5,6,**] Liuming Liu (刘柳明),[7,8,††] Ulf-G. Meißner,[2,9,10,‡‡] and Carsten Urbach[2,§§]

[2510.09476, talk by Maxim Mai]



ω Meson from lattice QCD

Haobo Yan (燕浩波),[1,2,*] Maxim Mai,[2,3,4,†] Marco Garofalo,[2,‡]
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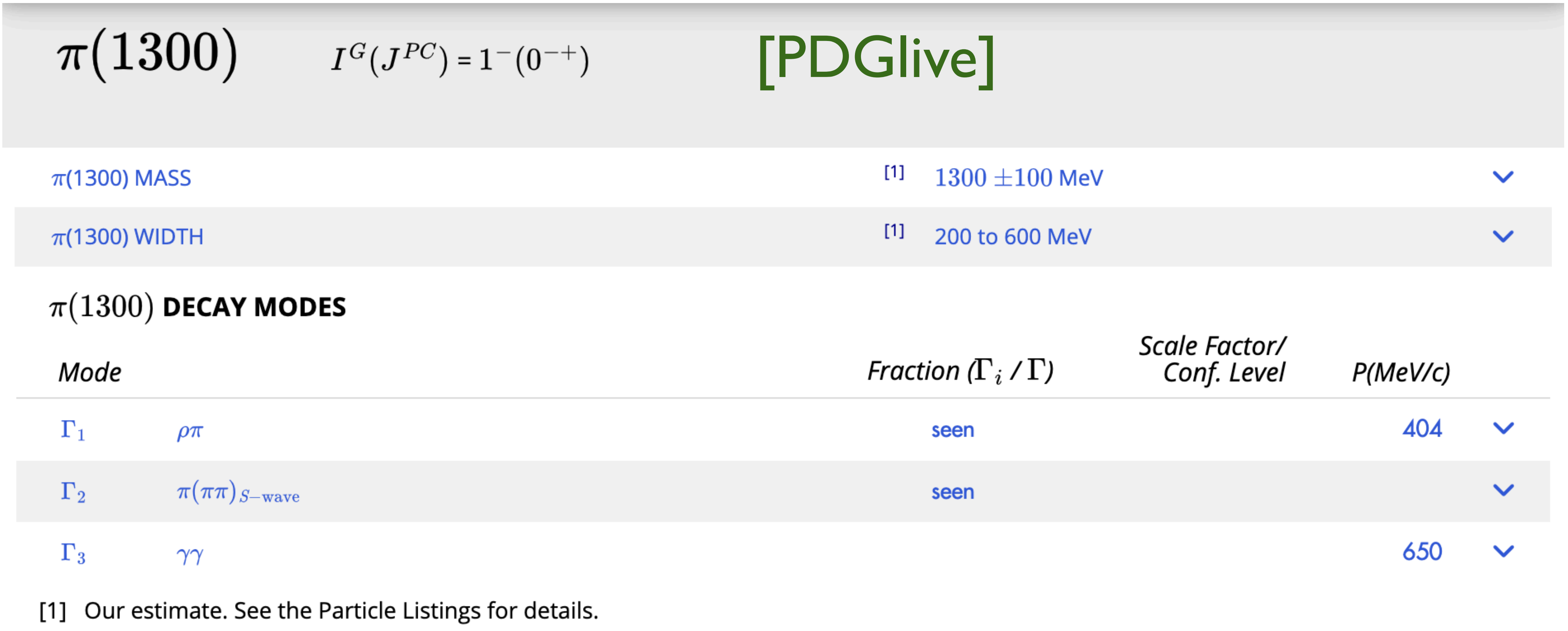
[2407.16659, PRL]

Natural candidate for QC3?



$\pi(1300)$ $I^G(J^{PC}) = 1^-(0^{-+})$ [PDGlive]

$\pi(1300)$ MASS	[1]	1300 ± 100 MeV
$\pi(1300)$ WIDTH	[1]	200 to 600 MeV

$\pi(1300)$ DECAY MODES

Mode		Fraction (Γ_i / Γ)	Scale Factor/ Conf. Level	P(MeV/c)
Γ_1	$\rho\pi$	seen		404
Γ_2	$\pi(\pi\pi)_{S\text{–wave}}$	seen		
Γ_3	$\gamma\gamma$			650

[1] Our estimate. See the Particle Listings for details.

Pro: s-wave decay to 3π implies strong 3-particle coupling

Con: Lies well above the inelastic threshold for QC3

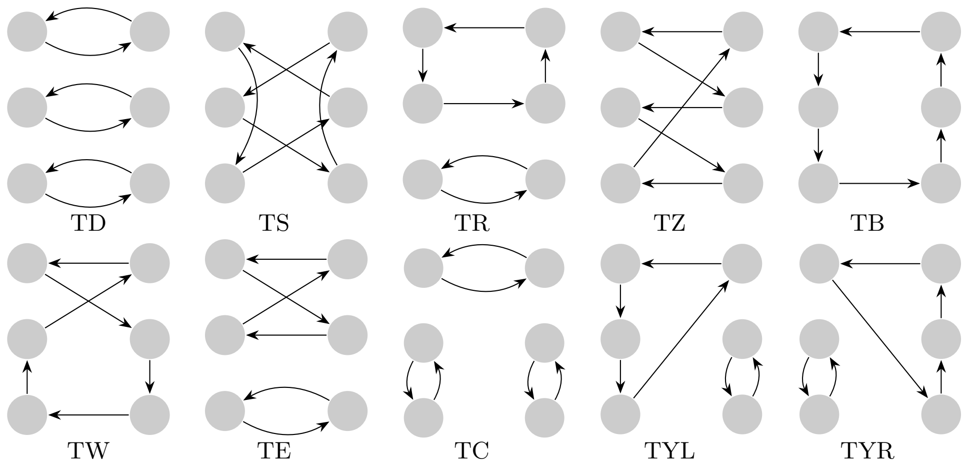
- $M(\pi(1300)) \approx 10 M_\pi \gg 5 M_\pi$
- QC3 will apply for heavier than physical quark masses

Simulation details

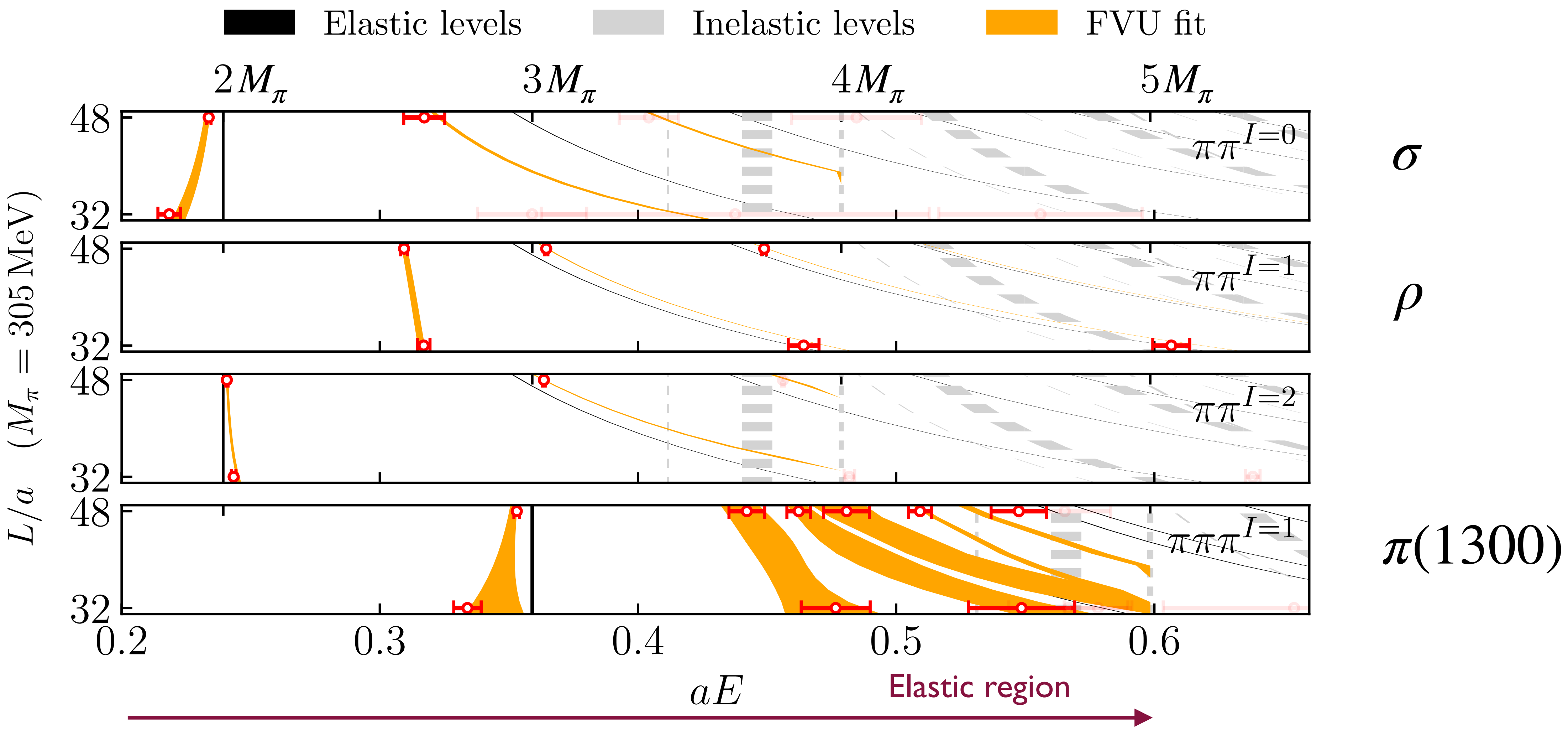
- Use CLQCD ensembles (tadpole improved Clover), fixed m_s, $a = 0.07746(18)\text{fm}$
  - Rest-frame only and single irrep

ens.	size	M_π(MeV)	M_K(MeV)	$M_\pi L$	# 2π levels	# 3π levels
F32P30	$32^3 \times 96$	303	525	3.8	5	3
F48P30	$48^3 \times 96$	303	524	5.7	7	6
F32P21	$32^3 \times 64$	211	492	2.7	5	0
F48P21	$48^3 \times 96$	207	493	3.9	7	3

- Large basis of operators: π, $\rho\pi$, $\sigma\pi$, 3π for three-particle case
  - Large set of contractions, e.g.



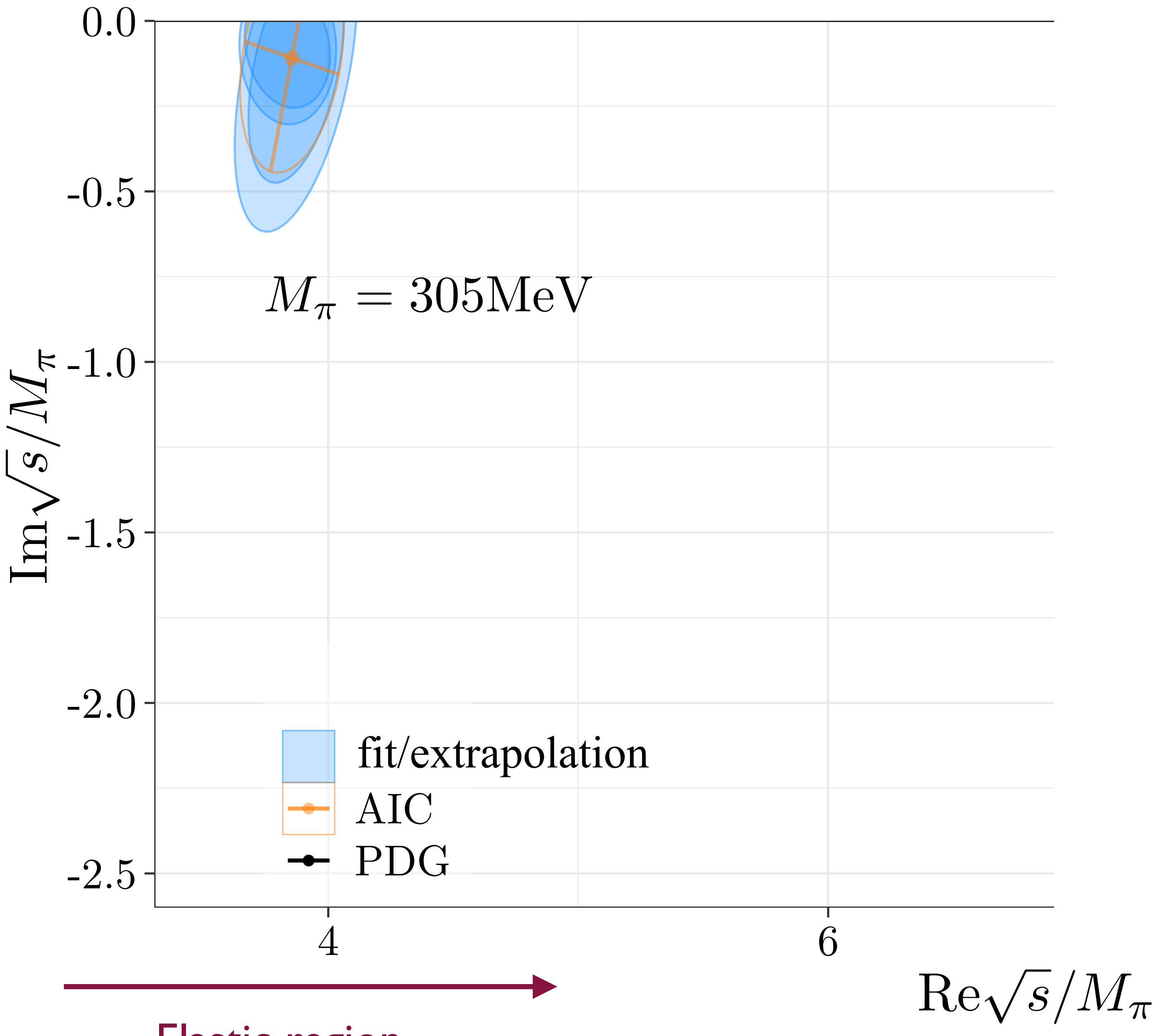
Fits



- Best global fit to both masses and volumes
  - 9 params (3 for 2π, and 6 for 3π) to 36 levels, with $\chi^2/\text{dof} = 1.08$
  - Final results from model averaging with Akaike information criterion

Pole positions

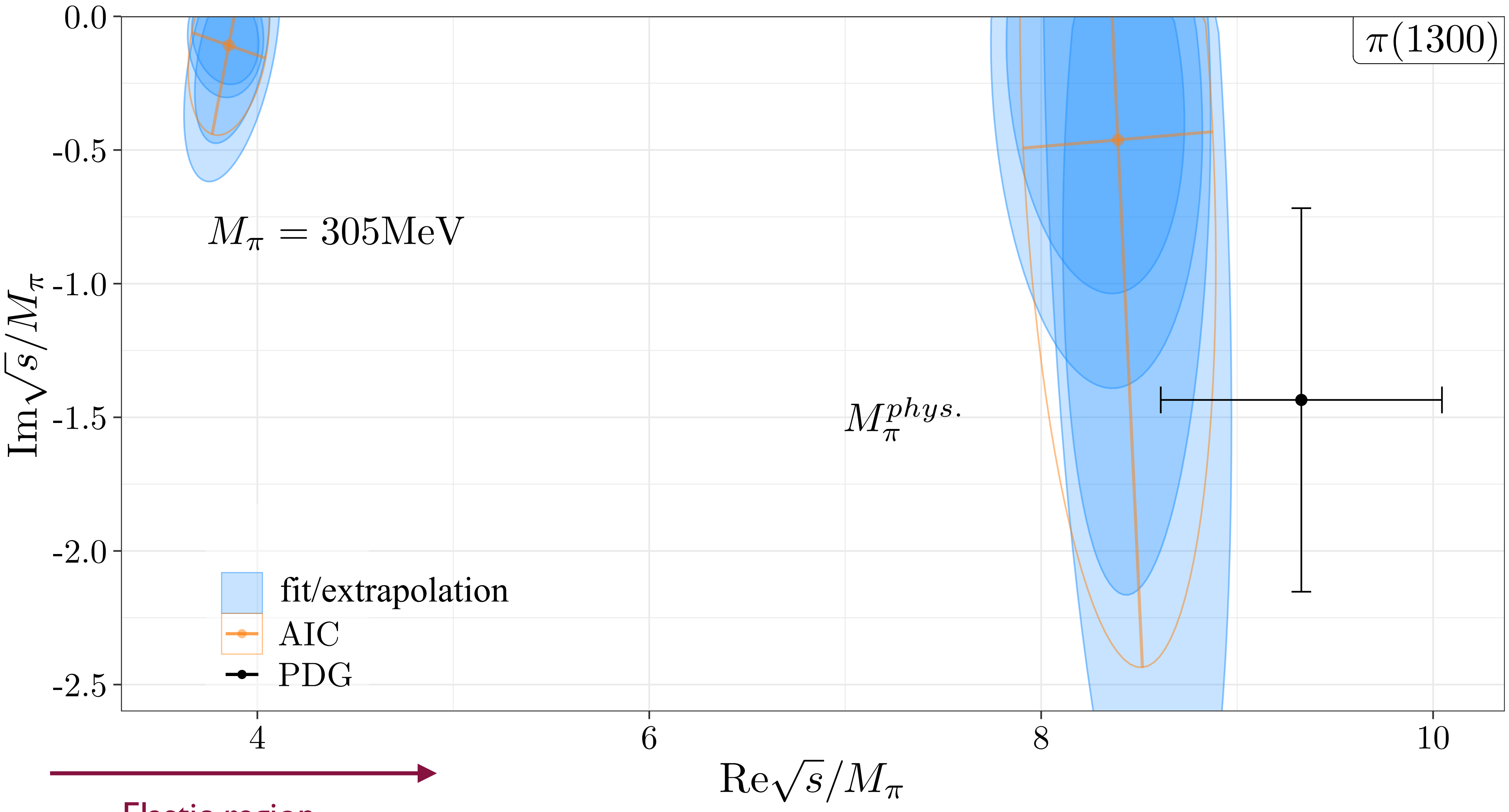
- Solve IVU (=infinite-volume unitarity) form of integral equations
  - Find resonance pole for $M_\pi = 305\,\mathrm{MeV}$!



Elastic region

Pole positions

- Solve IVU (=infinite-volume unitarity) form of integral equations
  - Find resonance pole for $M_\pi = 305\,\mathrm{MeV}$!
- Extrapolate to physical masses using EFT-inspired form



Elastic region