Progress in calculating multiparticle amplitudes from lattice QCD

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Dreams of a lattice theorist
Observation of $CP$ violation in charm decays

LHCb collaboration†

Abstract

A search for charge-parity ($CP$) violation in $D^0 \rightarrow K^-K^+$ and $D^0 \rightarrow \pi^-\pi^+$ decays is reported, using $pp$ collision data corresponding to an integrated luminosity of 6 fb$^{-1}$ collected at a center-of-mass energy of 13 TeV with the LHCb detector. The flavor of the charm meson is inferred from the charge of the pion in $D^*(2010)^+ \rightarrow D^0\pi^+$ decays or from the charge of the muon in $\bar{B} \rightarrow D^0\mu^-\bar{\nu}_\mu X$ decays. The difference between the $CP$ asymmetries in $D^0 \rightarrow K^-K^+$ and $D^0 \rightarrow \pi^-\pi^+$ decays is measured to be $\Delta A_{CP} = [-18.2 \pm 3.2\,(\text{stat.}) \pm 0.9\,(\text{syst.})] \times 10^{-4}$ for $\pi$-tagged and $\Delta A_{CP} = [-9 \pm 8\,(\text{stat.}) \pm 5\,(\text{syst.})] \times 10^{-4}$ for $\mu$-tagged $D^0$ mesons. Combining these with previous LHCb results leads to

$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4},$$

where the uncertainty includes both statistical and systematic contributions. The measured value differs from zero by more than five standard deviations. This is the first observation of $CP$ violation in the decay of charm hadrons.
What we want...

Minkowski time

gluons & sea-quark loops implicit

out-state

$\pi^+$

$\pi^-$

in-state

$D^0$

$u$

$\bar{d}$

$\bar{u}$

$H_W$

$c$
What we want...

Minkowski time

out-state

$\pi^+$

$\bar{d}$

$u$

$H_W$

$c$

in-state

$\pi^-$

$d$

$q$

$D^0$

$\bar{u}$

 gluons & sea-quark loops implicit

S. Sharpe, “Progress in multiparticle amplitudes from the lattice,” BAPTS, 10/13/23
What we want...

\[ \pi^+ \quad \bar{d} \quad u \quad \pi^- \]

out-state

\[ \pi^- \quad \bar{u} \quad D^0 \]

in-state

Minkowski time

\[ \mathcal{H}_W \]

gluons & sea-quark loops implicit
What we want...

\[
\pi^- u \xrightarrow{D^0} \pi^+ \quad + \quad \pi^- u \xrightarrow{D^0} \pi^+ \\
\bar{d} \xrightarrow{\mathcal{W}} c \\
\bar{d} \xrightarrow{\mathcal{W}} c \\
\bar{u} \xrightarrow{D^0} \bar{u}
\]

Minkowski time
…what we might achieve

Finite volume
Lattice QCD
(LQCD) calculations

Euclidean time

space
Problems

- No in- and out-states in finite volume
  - Cannot separate final-state particles
- Need to analytically continue from Euclidean to Minkowski momenta
  - Ill-posed problem given discrete momenta in finite volume
Rephrasing

What LQCD can determine are finite-volume matrix elements:

\[ L \langle E_n | \mathcal{H}_W | D_0^0 \rangle_L \]

Physical quantities if choose \( E_n(L) = E_D \)

Discrete spectrum of states with quantum numbers of \( \pi^+ \pi^- \)
Rephrasing

What LQCD can determine are finite-volume matrix elements:

\[ L\langle E_n | \mathcal{H}_W | D^0 \rangle_L \]

Physical quantities if choose \( E_n(L) = E_D \)

LQCD methods could, in the near future, allow the calculation of these quantities

How can they be related to the physical decay amplitudes?
The fundamental issue

\[ L\langle E_n | \mathcal{H}_W | D^0 \rangle_L \rightarrow \text{out} \langle \pi^+ \pi^- | \mathcal{H}_W | D^0 \rangle_{\text{in}} \]

- This is a nontrivial (and so-far unsolved) QFT problem because \( |E_n\rangle_L \) are composed of contributions from \( \pi\pi, 4\pi, K\bar{K}, 6\pi, \ldots \) with \( j = 0, 2, \ldots \)
  - Even if you use a two-pion operator, the strong interactions unavoidably lead to mixing with other states
  - Solution will require amplitudes for \( \pi\pi \rightarrow \pi\pi, 3\pi \rightarrow 3\pi, \pi\pi \rightarrow 4\pi, \ldots \), which will need to be determined from the energies \( E_n(L) \)
The fundamental issue

\[ L \langle E_n | \mathcal{H}_W | D^0 \rangle_L \rightarrow \text{out} \langle \pi^+ \pi^- | \mathcal{H}_W | D^0 \rangle_{\text{in}} \]

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- A solution will require amplitudes for \( \pi\pi \rightarrow \pi\pi, 3\pi \rightarrow 3\pi, \pi\pi \rightarrow 4\pi, \ldots \), which will need to be determined from the energies \( E_n(L) \)

- A side benefit of any solution will be the ability to use LQCD results for \( E_n(L) \) to study resonances with decays into multiple two-, three- and four-particle channels

Over the last 10 years, the corresponding issues for three particles have been solved, and are beginning to be implemented in LQCD simulations

Today I will briefly summarize the status, and describe examples of recent work
Outline

• Further motivation for studying 3-particle resonances & decays
• History and status of finite-volume formalism for 2 & 3 particles
• Examples of recent work
  • Extraction of $\pi^+\pi^+K^+ \rightarrow \pi^+\pi^+K^+$ and $K^+K^+\pi^+ \rightarrow K^+K^+\pi^+$ K-matrices using LQCD with close to physical quark masses
  • NLO Chiral PT calculation of three-particle K matrix
• Summary & Outlook
Motivations for studying three particles using LQCD
Cornucopia of exotics

62 new hadrons at the LHC

[I. Danilkin, talk at INT workshop, March 23]

+ data from Babar, Belle, COMPASS, …

S. Sharpe, "Progress in multiparticle amplitudes from the lattice," BAPTS, 10/13/23
Motivations

- Most resonances have 3 (or more) particle decay channels
  - $\omega(782, I^GJ^{PC} = 0^{-1--}) \rightarrow 3\pi$
  - $N(1440, J^P = \frac{1^+}{2}) \rightarrow N\pi, N\pi\pi$
  - $T_{cc}(3875, I = 0, J^P = 1^+?) \rightarrow D^0D^0\pi^+$

- Determining 3-body “forces”
  - NNN interactions needed as input for EFT treatments of large nuclei, and for the neutron-star equation of state
  - $\pi\pi\pi, \pi K\bar{K}, \ldots$ interactions needed as input to study pion & kaon condensation
History & status of finite-volume formalism for 2 particles
Sketch of history for two particles

- 1961: Discovery of the $\rho$ meson

**EVIDENCE FOR A $\pi-\pi$ RESONANCE IN THE $I=1$, $J=1$ STATE**

A. R. Erwin, R. March, W. D. Walker, and E. West
Brookhaven National Laboratory, Upton, New York and University of Wisconsin, Madison, Wisconsin
(Received May 11, 1961)

S. Sharpe, "Progress in multiparticle amplitudes from the lattice," BAPTS, 10/13/23
Sketch of history for two particles

- 1961: Discovery of the $\rho$ meson
- 1986/91: Lüscher derived “two-particle quantization condition” (QC2)

Discrete energy spectrum
- e.g. $\pi\pi$ with $E_{\text{CM}} < 4M_\pi$

Scattering amplitude

$$\det \left[ F(E, P, L)^{-1} + \mathcal{K}_2(E^*) \right] = 0$$

- Matrices in $\ell, m$
- $\mathcal{K}_2 \sim \tan \delta / q$
- $F$ is a known kinematical “zeta-function”, depending on the box shape
- Valid up to $e^{-ML}$ corrections

S. Sharpe, “Progress in multiparticle amplitudes from the lattice,” BAPTS, 10/13/23
Sketch of history for two particles

• 1961: Discovery of the $\rho$ meson

• 1986/91: Lüscher derived “two-particle quantization condition” (QC2)
  • 2005: Kim, Sachrajda & SRS — alternate derivation, basis for many subsequent generalizations

• 1999: Measurement of $\varepsilon'/\varepsilon = 16.1(2.3)10^{-4}$ by KTeV/NA48 — direct CPV in $K \rightarrow \pi\pi$

NA48 @ CERN: Cern website
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- 2001: Lellouch & Lüscher (LL): relation between $L \langle E_n(\pi\pi) | \mathcal{H}_W | K \rangle_L$ and $A(K \to \pi\pi)$
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• 2001: Lellouch & Lüscher (LL): relation between $L\langle E_n(\pi\pi) | \mathcal{H}_W | K \rangle_L$ and $\mathcal{A}(K \to \pi\pi)$

• 2014 - 2019: LQCD implementation of QC2 for the $\rho$ resonance in $\pi\pi$ scattering (and many other resonances subsequently)

Anderson et al., 1808.05007

\[ M_\pi \approx 200 \text{ MeV} \]

\[ M_\rho \approx 280 \text{ MeV} \]
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- 1961: Discovery of the $\rho$ meson
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- 2014 - 2019: LQCD implementation of QC2 for the $\rho$ resonance in $\pi\pi$ scattering
- 2020: LQCD calculation of $\epsilon'/\epsilon = 21.7(8.4)10^{-4}$ in the standard model using LL method with physical quark masses and (almost) all errors controlled

Direct CP violation and the $\Delta I = 1/2$ rule in $K \to \pi\pi$ decay from the standard model

R. Abbott,¹ T. Blum,²,³ P. A. Boyle,⁴,⁵ M. Bruno,⁶ N. H. Christ,¹ D. Hoying,³,² C. Jung,⁴ C. Kelly,⁴,⁶ C. Lehner,⁷,⁴ R. D. Mawhinney,¹ D. J. Murphy,⁸ C. T. Sachrajda,⁹ A. Soni,⁴ M. Tomii,⁷ and T. Wang¹

(RBC and UKQCD Collaborations)
History & status of finite-volume formalism for 3 particles
Sketch of history for three particles

- [Beane, Detmold, Savage et al. 07-11] studied ground state energies of $N\pi^+, MK^+, N\pi^+ + MK^+$ systems, and determined 3-particle interactions for particles at rest

- [Polejaeva & Rusetsky 12] showed in NREFT that 3 body spectrum determined by $2 \rightarrow 2$ & $3 \rightarrow 3$ infinite-volume scattering amplitudes

- [Hansen & SRS 14, 15] derived quantization condition (QC3) for 3 identical scalars in generic, relativistic EFT, working to all orders in Feynman-diagram expansion, keeping all angular momenta—“RFT approach”

- [Hammer & Rusetsky 17] derived QC3 using NREFT—greatly simplified derivation

- [Mai & Döring 17] obtained QC3 using unitary, relativistic representation of $3 \rightarrow 3$ amplitude—“FVU approach”

- [Blanton & SRS 20] showed equivalence of RFT & FVU approaches

- [Hansen, Romero-López, SRS 21] derived formalism for determining $K \rightarrow 3\pi$ amplitude
Additional issues with 3 particles

- Energy shifts \( \Delta E_n = E_n - E_{n,\text{free}} \sim 1/L^3 \)
- Scattering amplitude in each partial wave, at given \( E_{\text{CM}} \), is a (complex) number

VS

- Dominant contribution from pairwise interactions, \( \Delta E_n \sim 1/L^3 \)
- 3-particle interactions give subleading contributions \( \propto 1/L^6 \)
- Scattering amplitude \( M_3 \) at given \( E_{\text{CM}} \) is a (complex) function of Dalitz-plot variables, and incorporates final-state interactions
- \( M_3 \) has divergences for physical momenta

(For simplicity, assume G-parity-like \( Z_2 \) symmetry, so no 2 ↔ 3 transitions; formalism can be generalized)

S. Sharpe, “Progress in multiparticle amplitudes from the lattice,” BAPTS, 10/13/23
Structure of the result \((\mathbb{Z}_2\text{ symmetry})\)

\[ E_{\text{CM}} < 4m \]

\[ E_{\text{CM}} < 5m \]

\[ E_0(L) \]
\[ E_1(L) \]
\[ E_2(L) \]

\[ \mathcal{M}_2 \]
\[ \mathcal{M}_3 \]

S. Sharpe, "Progress in multiparticle amplitudes from the lattice," BAPTS, 10/13/23
Two-step method

2 & 3 particle Spectra from LQCD

Quantization conditions

QC2: \( \det \left[ F^{-1} + \mathcal{K}_2 \right] = 0 \)

QC3: \( \det \left[ F_3^{-1} + \mathcal{K}_{df,3} \right] = 0 \)

Integral equations in infinite volume

Incorporates initial- and final-state interactions

Scattering amplitude \( M_3 \)

Infinite-volume K matrix:
Obtained from Feynman diagrams using PV prescription for poles;
Real, free of unitary cuts

Intermediate infinite-volume K matrix:
A short-distance, real, three-particle interaction free of unitary cuts, and
with physical divergences subtracted; unphysical since depends on cutoff

[These are the RFT forms, and assume \( \mathbb{Z}_2 \) symmetry]
Further details of QC$_{3}$

\[ \det \left[ F_{3}^{-1} + \mathcal{K}_{df,3} \right] = 0 \]

- Derived by determining power-law volume dependence of finite-volume 3-particle correlation functions to all orders in a skeleton expansion in a generic relativistic EFT

- Volume dependence arises from 3-particle cuts

- $F_{3}$ contains two-particle interactions ($\mathcal{K}_{2}$) and kinematic functions (F & G)

\[
F_{3} = \frac{1}{2\omega L^3} \left[ \frac{F}{3} - F \frac{1}{\mathcal{K}_{2}^{-1} + F + G} \right]
\]
• 3 identical spinless particles [Hansen, SRS; Hammer, Pang, Rusetsky; Mai, Döring]

• Mixing of two- and three-particle channels for identical spinless particles [Briceño, Hansen, SRS]

• 3 degenerate but distinguishable particles, e.g. $3\pi$ with isospin 0, 1, 2, 3 [Hansen, Romero-López, SRS]

• 3 nondegenerate particles, e.g. $D_s^+D^0\pi^-$ [Blanton, SRS]

• (Single-channel) 2+1 systems, e.g. $\pi^+\pi^+K^+$ [Blanton, SRS]

• 3 identical spin-$\frac{1}{2}$ particles, e.g. 3 neutrons [Draper, Hansen, Romero-López, SRS]

Many resonances can now be studied!

<table>
<thead>
<tr>
<th>Resonance</th>
<th>$I_{\pi\pi}$</th>
<th>$J^P$</th>
<th>Decays</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega(782)$</td>
<td>0</td>
<td>1$^-$</td>
<td>$\pi^+\pi^0\pi^-$</td>
</tr>
<tr>
<td>$h_1(1170)$</td>
<td>0</td>
<td>1$^+$</td>
<td>$\rho\pi \to 3\pi$</td>
</tr>
<tr>
<td>$\omega_3(1670)$</td>
<td>0</td>
<td>3$^-$</td>
<td>$3\pi, 5\pi$</td>
</tr>
<tr>
<td>$\pi(1300)$</td>
<td>1</td>
<td>0$^-$</td>
<td>$\rho\pi \to 3\pi$</td>
</tr>
<tr>
<td>$a_1(1260)$</td>
<td>1</td>
<td>1$^+$</td>
<td>$3\pi, K\bar{K}\pi$</td>
</tr>
<tr>
<td>$\pi_1(1400)$</td>
<td>1</td>
<td>1$^-$</td>
<td>$\eta\pi, 3\pi?$</td>
</tr>
<tr>
<td>$\pi_2(1670)$</td>
<td>1</td>
<td>2$^-$</td>
<td>$3\pi, K\bar{K}\pi$</td>
</tr>
<tr>
<td>$a_2(1320)$</td>
<td>1</td>
<td>2$^+$</td>
<td>$3\pi, K\bar{K}, 5\pi, \eta\pi$</td>
</tr>
<tr>
<td>$a_4(1970)$</td>
<td>1</td>
<td>4$^+$</td>
<td>$3\pi, K\bar{K}, 5\pi, \eta\pi$</td>
</tr>
</tbody>
</table>
Status: applications

[References at end of slides]

- $3\pi^+$: determined parameters in threshold expansion of $\mathcal{H}_{df,3}$, including pair interactions in s- and d-waves; integral equations solved for s-wave interactions only

- $3K^+$: determined s- and d-wave parameters in $\mathcal{H}_{df,3}$

- $\phi^4$: extracted $\mathcal{H}_{df,3}$ in single-scalar theory; extracted 3-particle resonance parameters in two-scalar theory, using RFT and FVU approaches

- $3\pi$ with $I = 1$: first study of $a_1(1260)$ with formalism based on 2 levels; solved integral equations in FVU approach

- $\pi^+\pi^+K^+$ & $K^+K^+\pi^+$: determined s- and p-wave parameters in $\mathcal{H}_{df,3}$; found evidence for small discretization effects

- Integral equations solved for complex energies for simple system with near-unitary two-particle interactions and Efimov states (bound or resonant)

- ChPT: LO results for $3\pi^+$, $\pi^+\pi^+K^+$, $K^+K^+\pi^+$, $3K^+$, including $a^2$ effects: agree in rough magnitude but not in detail with results from LQCD calculations

- ChPT: NLO result for $3\pi^+$; greatly improves agreement with LQCD results
\[ \pi^+ \pi^+ K^+ \text{ and } K^+ K^+ \pi^+ \] amplitudes using LQCD

[Draper, Hanlon, Hörz, Morningstar, Romero-López & SRS, 2302.13587 (JHEP)]

A step on the way to \( T_{cc} \rightarrow DD\pi, \text{ etc.} \)
Consider multiparticle system with weakly repulsive interactions—pions and kaons at maximal isospin \((2\pi^+/3\pi^+, 2K^+/3K^+, 2\pi^+/\pi^+K^+/3K^+, 2K^+/\pi^+K^+/3K^+)\)

- No resonances in two-particle subchannels or in three-particle system
- Simultaneously fit to several spectra, for example, to obtain the \(\pi^+\pi^+K^+\) interaction need:
Consider multiparticle system with weakly repulsive interactions—pions and kaons at maximal isospin \((2\pi^+/3\pi^+, 2K^+/3K^+, 2\pi^+/\pi^+K^+/3K^+, 2K^+/\pi^+K^+/3K^+)\)

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\[
\pi^+ \pi^+ + \pi^+ K^+ + \pi^+ K^+ 
\]

- Parametrize \(\mathcal{H}_{df,3}\) (and \(\mathcal{H}_2\)) as the most general smooth function consistent with particle interchange, time-reversal and parity symmetries, using an expansion about threshold
  - Generalization of the effective-range expansion for \(\mathcal{H}_2\); here keep first two terms
  - \(s\)-wave interactions in \(\pi^+\pi^+\) (sub)channel, \(s\)- and \(p\)-wave in \(\pi^+K^+\); 9 or 10 parameters in all
Lattices used in pilot calculation

- Improved Wilson fermions at $a = 0.064$ fm (CLS lattices)

<table>
<thead>
<tr>
<th>$(L/a)^3 \times (T/a)$</th>
<th>$M_\pi$ [MeV]</th>
<th>$M_K$ [MeV]</th>
<th>$N_{cfg}$</th>
<th>$t_{src}/a$</th>
<th>$N_{ev}$</th>
<th>dilution</th>
<th>$N_r(\ell/s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>N203 48(^3) × 128</td>
<td>340</td>
<td>440</td>
<td>771</td>
<td>32, 52</td>
<td>192</td>
<td>(LI12, SF)</td>
<td>6/3</td>
</tr>
<tr>
<td>D200 64(^3) × 128</td>
<td>200</td>
<td>480</td>
<td>2000</td>
<td>35, 92</td>
<td>448</td>
<td>(LI16, SF)</td>
<td>6/3</td>
</tr>
</tbody>
</table>

D200 configurations

- $\pi^+$: $M_\pi L = 4.1$
- $K^+$: $M_K L = 10$

$L = 4.1$ fm
Example of fit

Simultaneous fit to 27 $\pi^+\pi^+$, 19 $\pi^+K^+$, & 36 $\pi^+\pi^+K^+$ levels with 9 parameters

$\chi^2$/DOF = 119/(82 − 9)
Fit is to lab-frame shifts

Simultaneous fit to 28 $K^+K^+$, 16 $\pi^+K^+$, & 29 $K^+K^+\pi^+$ levels with 10 parameters on D200: $\chi^2$/DOF = 162/(73 – 10)

S. Sharpe, “Progress in multiparticle amplitudes from the lattice,” BAPTS, 10/13/23
Results: scattering lengths

- 2-particle s-wave scattering lengths are well determined
- All are repulsive and consistent with ChPT
  - Evidence for small discretization errors

Figure 5. Results for $M_{fi}a_{fi}$, $M_{fi}a_{fi}K$ and $M_{fi}a_{fi}$KK as a function of $(M_{\pi}/F_{\pi})^2$, where $M_{fi}a_{fi}K$ = $(M_{fi} + M_{fi}K)/2$. The LO ChPT result is shown, along with a fit to NLO SU(3) ChPT. The shaded bands show the uncertainties in the fit.

Table 18: Effective range parameters for the combination $M_{fi}X_{fi}X_{fi}$. For the case of identical particles ($X = Y = fi$ or $K$), the LO ChPT prediction from section 3.3 is that this quantity equals 3. For two pions, the results lie 15% and 25% below this prediction on the D200 and N203 ensembles, respectively, which is consistent with being due to an NLO correction. For two kaons, the results lie very far away from the LO prediction. Both findings are qualitatively similar to those obtained in Ref. \cite{49}. For the $fiK$ channel, which is a novel result of this work, the LO ChPT prediction—given in eq. (3.22)—depends on the ensemble:

- $M_{fi}a_{fi}K_{D200} = 1.597$
- $M_{fi}a_{fi}K_{N203} = 2.395$

(5.3)

Our results in table 18 lie $\geq 25\%$ and $\geq 30\%$, respectively, below the LO ChPT prediction. Again we view this as reasonable consistency, given the absence of NLO corrections.
We now turn to the $p$-wave scattering parameter, $\delta^{\pi^+K^+}(s)$, reported in the rightmost column of table 17 through the dimensionless combination $P_{fi} K_0 = \pi a_{fi} K_1$. Note that, in contrast to all the $s$-wave results, the value of $P_{fi} K_0$ corresponds to slightly attractive interactions.

We plot the results for the two ensembles in figure 6, including a fit to the leading chiral behavior given by eq. (3.23), which shows reasonable consistency. We also plot the NLO ChPT prediction given in appendix C. To do so we use values for the requisite LECs determined in Ref. [89] from experimental data (specifically, fit 10 to $O(p^4)$ from that work). As can be seen, the NLO ChPT result has the same sign as our results, but its magnitude is significantly smaller. The failure of NLO ChPT for this quantity was, in fact, expected, based on the observation of Ref. [88] that the NNLO contribution is two orders of magnitude larger than the NLO one at the physical point (see table 2 of that work).

We can also compare to the expectations and results in the literature from experiment and dispersive analyses. The current understanding is summarized in figure 10 of Ref. [98].

Experimental results [99] for the $p$-wave phase shift point to a negative (repulsive) value at high energies. By contrast, the dispersive analysis indicates a change of sign for the phase at around $M_{\pi} K + 3 M_{\pi}$ (physical values of the masses), resulting in an attractive scattering length. The value from analysis of Ref. [98] is also shown in figure 6. As can be seen, our results for the two ensembles of this work are in qualitative agreement with the low-momentum behavior found by the dispersive analysis. We note that our fits only...

- Find evidence for **attractive** $p$-wave scattering length
- Consistent with dispersive analysis

[Pelaez, Rodas, 2010.11222]
s-wave contributions to $H_{df,3}$

- Evidence for nonzero values ($2 - 5\sigma$)
- Overall effect of $H_{df,3}$ is repulsive
- LO ChPT predicts opposite sign (but see later)
\( p\)-wave contributions to \( \mathcal{K}_{df,3} \)

\[ \pi^+ \pi^+ K^+ \]

\[ K^+ K^+ \pi^+ \]

- Evidence for nonzero values in some cases
  - \( \mathcal{K}_E \) is only contribution of \( \mathcal{K}_{df,3} \) to nontrivial irreps
- Appear at NLO in ChPT—prediction not yet available

\begin{align*}
M^2_{\pi}\mathcal{K}(\pi\pi K) &\quad \text{Chiral fit} \\
M^2_{\pi}\mathcal{K}_B(\pi\pi K) &\quad \text{"ChPT-inspired" fit} \\
M^2_{\pi}\mathcal{K}_E(\pi\pi K) &
\end{align*}

\begin{align*}
M^2_K\mathcal{K}(KK\pi) &\quad \text{Chiral fit} \\
M^2_K\mathcal{K}_B(KK\pi) &\quad \text{"ChPT-inspired" fit} \\
M^2_K\mathcal{K}_E(KK\pi) &
\end{align*}
NLO ChPT results for $\mathcal{K}_{df,3}$ for $3\pi^+ \rightarrow 3\pi^+$

[Baeza-Ballesteros, Bijnens, Husek, Romero-López, SRS, Sjö, 2303.13206]
$2\pi/3\pi$ K matrices vs ChPT

$2\pi^+$ scattering length

\[ M_{\pi} a_{0}^{\pi \pi} \]

- LO ChPT
- NLO fit
- This work
- Physical point

\[ \frac{(M_{\pi}/F_{\pi})^2}{(M_{\pi}/F_{\pi})^4} \]

$3\pi^+$ K matrix

\[ M_{K_{\pi}}^{K_{\pi}} \]

- LO ChPT
- Linear fit
- This work
- Physical point

[Results from Blanton, Hanlon, Hörz, Morningstar, Romero-López, SRS, 2106.05590 (JHEP)]

- LO ChPT describes 2-pion sector well
- Large discrepancy in 3-pion sector!
NLO ChPT for $\mathcal{K}^{\text{df,3}}$

• Integral equations simplify to:

$$\mathcal{K}^{\text{NLO}}_{\text{df,3}} = \text{Re } M^{\text{NLO}}_{\text{df,3}}$$

- Integral equations simplify to:

- One-particle-exchange diagrams

- One-particle-irreducible diagrams

- Bull's-head subtraction

NLO 6-pion amplitude computed in
[Bijnens, Husek 2107.06291]
[Bijnens, Husek, Sjö, 2206.14212]
Threshold expansion for $\mathcal{K}_{df,3}$

- $\mathcal{K}_{df,3}$ is a real, smooth function which is Lorentz, P and T invariant

- Expand about threshold in powers of $\Delta = (s - 9M^2_\pi)/9M^2_\pi$, $\tilde{t}_{ij} = (p'_i - p_j)^2/9M^2_\pi$, ...

$$\mathcal{K}_{df,3} = \mathcal{K}_{iso,0}^{df,3} + \mathcal{K}_{iso,1}^{df,3} \Delta + \mathcal{K}_{iso,2}^{df,3} \Delta^2 + \mathcal{K}_A \Delta_A + \mathcal{K}_B \Delta_B + \mathcal{O}(\Delta^3)$$

- Can separate terms in fit based on dependence on energy and rotational properties
  - E.g. only $\mathcal{K}_B$ contributes to nontrivial irreps
NLO ChPT results for $\mathcal{K}_{\text{df,3}}$

$$\kappa = 1/(16\pi^2)$$

$$\mathcal{K}_0 = \left(\frac{M_\pi}{F_\pi}\right)^4 18 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-3\kappa (35 + 12 \log 3) - \mathcal{D}_0 + 111L + \ell_{(0)}^r\right],$$

$$\mathcal{K}_1 = \left(\frac{M_\pi}{F_\pi}\right)^4 27 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-\frac{\kappa}{20} (1999 + 1920 \log 3) - \mathcal{D}_1 + 384L + \ell_{(1)}^r\right],$$

$$\mathcal{K}_2 = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{207\kappa}{1400} (2923 - 420 \log 3) - \mathcal{D}_2 + 360L + \ell_{(2)}^r\right],$$

$$\mathcal{K}_A = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{9\kappa}{560} (21809 - 1050 \log 3) - \mathcal{D}_A - 9L + \ell_{(A)}^r\right],$$

$$\mathcal{K}_B = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{27\kappa}{1400} (6698 - 245 \log 3) - \mathcal{D}_B + 54L + \ell_{(B)}^r\right].$$

Numerical coefficients Depend on cutoff $H(k)$

$\mu$-dependence cancels

$L \equiv \kappa \log \left(\frac{M_\pi^2}{\mu^2}\right)$

LECs
Comparison to LQCD

- (Very) large NLO corrections
- Discrepancy with LO ChPT resolved!
- ChPT not trustworthy for $\mathcal{K}_1$
Comparison to LQCD

- $\mathcal{K}_B$ first appears at NLO in ChPT
- Discrepancy may be resolved by NNLO terms?
Summary & Outlook
Summary

• Two-particle sector is entering precision phase
  
  • Frontier is two nucleons, which are more challenging for LQCD

• Major steps have been taken in the three-particle sector
  
  • Formalism well established & cross checked, and almost complete
  
  • Several applications to three-particle spectra from LQCD
  
  • Initial discrepancy with LO ChPT explained by large NLO contributions
  
  • Integral equations solved in several cases
  
  • Path to a calculation of $K \rightarrow 3\pi$ decay amplitudes is now open
Outlook

- Generalize formalism to broaden applications
  - 3 nucleons with $I = \frac{1}{2}$ (nnp & ppn)
  - $T_{cc}(3875, I = 0, J^P = 1^+) \rightarrow D^0D^0\pi^+, D^+D^0\pi^0, D^+D^+\pi^-$
  - Accessing the WZW term: $K\bar{K} \leftrightarrow \pi^+\pi^0\pi^-(I = 0)$
  - $N(1440, J^P = \frac{1^+}{2}) \rightarrow N\pi, N\pi\pi$
  - $J^{PC}, I^{G} = 1^{-}, 1^{-}: \pi_1(1600) \rightarrow \eta\pi, 3\pi, KK\pi\pi, \eta\pi\pi\pi, 5\pi$

- Extend implementations using LQCD simulations
  - $3\pi^+, 3K^+, \pi^+\pi^+K^+, K^+K^+\pi^+$ at physical quark masses
  - $I = 0, 1$ three-particle resonances ($\omega, a_1, \ldots$)

- Extend applications of integral equations in the presence of three-particle resonances, e.g. $T_{cc}$

- Move on to 4 particles!
The Exo(tic) Had(ron) Collaboration started in 2023 to explore all aspects of exotic hadron physics, from predictions within lattice QCD, through reliable extraction of their existence and properties from experimental data, to descriptions of their structure within phenomenological models.
Thank you!

Questions?
References
RFT 3-particle papers

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“Relativistic, model-independent, three-particle quantization condition,”
arXiv:1408.5933 (PRD) [HS14]

“Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,”
arXiv:1504.04028 (PRD) [HS15]

“Perturbative results for 2- & 3-particle threshold energies in finite volume,”
arXiv:1509.07929 (PRD) [HSPT15]

“Threshold expansion of the 3-particle quantization condition,”
arXiv:1602.00324 (PRD) [HSTH15]

“Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,”
arXiv: 1609.04317 (PRD) [HSBS16]

“Lattice QCD and three-particle decays of Resonances,”
Raúl Briceño, Max Hansen & SRS:
“Relating the finite-volume spectrum and the 2-and-3-particle
S-matrix for relativistic systems of identical scalar particles,”
arXiv:1701.07465 (PRD) [BHS17]
“Numerical study of the relativistic three-body quantization
condition in the isotropic approximation,”
arXiv:1803.04169 (PRD) [BHS18]
“Three-particle systems with resonant sub-processes in a finite
volume,” arXiv:1810.01429 (PRD 19) [BHS19]

SRS
“Testing the threshold expansion for three-particle energies at fourth order in φ⁴ theory,”
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“Implementing the three-particle quantization condition including
higher partial waves,” arXiv:1901.07095 (JHEP) [BRS19]
“I=3 three-pion scattering amplitude from lattice QCD,”
arXiv:1909.02973 (PRL) [BRS-PRL19]
“Implementing the three-particle quantization condition
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Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

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arXiv:1905.11188 (PRD)

Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

“On the Equivalence of Three-Particle Scattering Formalisms,”
arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

“Generalizing the relativistic quantization condition to include all three-pion isospin channels”, arXiv:2003.10974 (JHEP) [HRS20]

“Decay amplitudes to three particles from finite-volume matrix elements,”
arXiv: 2101.10246 (JHEP)
Tyler Blanton & SRS:
“Alternative derivation of the relativistic three-particle quantization condition,”
arXiv:2007.16188 (PRD) [BS20a]
“Equivalence of relativistic three-particle quantization conditions,”
arXiv:2007.16190 (PRD) [BS20b]
“Relativistic three-particle quantization condition for nondegenerate scalars,”
“Three-particle finite-volume formalism for $\pi^+\pi^+K^+$ & related systems,” arXiv:2105.12904 (PRD)

Tyler Blanton, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS
“$3\pi^+$ & $3K^+$ interactions beyond leading order from lattice QCD,” arXiv:2106.05590 (JHEP)

Zack Draper, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS
“Interactions of $\pi K$, $\pi\pi K$ and $KK\pi$ systems at maximal isospin from lattice QCD,” arXiv:2302.13587

Other work

★ Implementing RFT integral equations

- M.T. Hansen et al. (HADSPEC), 2009.04931, PRL [Calculating $3\pi^+$ spectrum and using to determine three-particle scattering amplitude]

- A. Jackura et al., 2010.09820, PRD [Solving s-wave RFT integral equations in presence of bound states]

- S. Dawid, Md. Islam and R. Briceño, 2303.04394 [Analytic continuation of 3-particle amplitudes]

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★ Reviews

- A. Rusetsky, 1911.01253 [LATTICE 2019 plenary]

- M. Mai, M. Döring and A. Rusetsky, 2103.00577 [Review of formalisms and chiral extrapolations]

- F. Romero-López, 2112.05170, [Three-particle scattering amplitudes from lattice QCD]

★ Other numerical simulations

- F. Romero-López, A. Rusetsky, C. Urbach, 1806.02367, JHEP [2- & 3-body interactions in $\phi^4$ theory]

- M. Fischer et al., 2008.03035, Eur.Phys.J.C [2$\pi^+$ & 3$\pi^+$ at physical masses]

- M. Garofolo et al., 2211.05605, JHEP [3-body resonances in $\phi^4$ theory]
Other work

★ NREFT approach

- M. Döring et al., 1802.03362, PRD [Numerical implementation]
- J.-Y. Pang et al., 1902.01111, PRD [large volume expansion for excited levels]
- F. Müller, T. Yu & A. Rusetsky, 2011.14178, PRD [large volume expansion for I=1 three pion ground state]
- J-Y. Pang, M. Ebert, H-W. Hammer, F. Müller, A. Rusetsky, 2204.04807, JHEP, [Spurious poles in a finite volume]
- F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, 2110.09351, JHEP [Relativistic-invariant formulation of the NREFT three-particle quantization condition]
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- R. Bubna, F. Müller, A. Rusetsky, 2304.13635 [Finite-volume energy shift of the three-nucleon ground state]
Alternate 3-particle approaches

★ Finite-volume unitarity (FVU) approach

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- M. Mai et al., 1706.06118, EPJA [unitary parametrization of $M_3$ involving $R$ matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of $R$ matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]
- M. Mai et al., 1909.05749, PRD [applying FVU approach to $3\pi^+$ spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]
- A. Alexandru et al., 2009.12358, PRD [calculating $3K^-$ spectrum and comparing with FVU predictions]
- R. Brett et al., 2101.06144, PRD [determining $3\pi^+$ interaction from LQCD spectrum]
- M. Mai et al., 2107.03973, PRL [three-body dynamics of the $a_1(1260)$ from LQCD]
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★ HALQCD approach

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